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ON A PROBLEM OF J. P. WILLIAMS

EDWARD KISSIN AND VICTOR S. SHULMAN

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Abstract. Let $B(H)$ be the algebra of all bounded operators on a Hilbert space H. Let g be a continuous function on the closed disk D and let

$$\|g(A)X - Xg(A)\| \leq C\|AX - XA\|,$$

for some $C > 0$, for all $X \in B(H)$ and all $A \in B(H)$ with $\|A\| \leq 1$. Then g is differentiable on D. The paper shows that the function g may have a discontinuous derivative.

1. Introduction

Let $B(H)$ be the algebra of all bounded operators on a Hilbert space H and $\mathbf{B}_1$ be the unit ball of $B(H)$. For $A, B \in B(H)$, we denote by $[A, B]$ their commutator $AB - BA$. Let $D = \{z \in \mathbb{C} : |z| \leq 1\}$ be the closed unit disk. In his paper [7] Williams raised the following problem. If g is a continuous complex-valued function on D, possessing the property

$$\|[g(A), X]\| \leq C\|[A, X]\|, \tag{1}$$

for some $C > 0$, for any $X \in B(H)$ and any normal operator A in $\mathbf{B}_1$, *must g always be continuously differentiable on D?*

It should be noted that Johnson and Williams proved earlier [2, Theorem 4.1] that g must be differentiable on D and therefore analytic in the interior D° of D, and its derivative must be bounded on D.

We will show that the answer to Williams's problem is negative. Moreover, we will show that the function on D may have a discontinuous derivative even if it satisfies (1) for *all* (not necessarily normal) contractions A.

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2. Fully Operator Lipschitz functions

We denote by $\mathcal{A}(D)$ the *disk algebra*: the algebra of all continuous complex-valued functions on D which are analytic on D°. The algebra $\mathcal{A}(D)$ is a closed subalgebra of the algebra $C(D)$ of all continuous complex-valued functions on D with the norm $\|g\| = \sup_{z\in D} |g(z)|$. The subalgebra $P(D)$ of all polynomials on D is dense in $\mathcal{A}(D)$ (see, for example, [4, §3.2.13]).

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By von Neumann's theorem (see [6, Proposition I.8.3]), $\|p(A)\| \le \|p\|$ for any polynomial p and any $A \in \mathbf{B}_1$. Therefore functions from $\mathcal{A}(D)$ act on $\mathbf{B}_1$ and

$$\|g(A)\| \le \|g\|, \quad \text{for any } g \in \mathcal{A}(D) \text{ and } A \in \mathbf{B}_1. \tag{2}$$

We call a function $g \in \mathcal{A}(D)$ *Fully Operator Lipschitzian* if there is $C > 0$ such that

$$\|g(A) - g(B)\| \le C\|A - B\|, \quad \text{for } A, B \in \mathbf{B}_1. \tag{3}$$

The class of Fully Operator Lipschitz functions is contained in the wider class of Operator Lipschitz functions on D which consists of all continuous functions on D satisfying inequality (3) for all normal operators in $\mathbf{B}_1$ (see [3]). The function $g(z) = \bar{z}$, for example, is Operator Lipschitzian on D, since $\|A^* - B^*\| = \|A - B\|$, for all normal $A, B \in \mathbf{B}_1$. However, it is not Fully Operator Lipschitzian. Both classes of functions are important for applications in mathematical physics and have attracted much attention (see, for example, Bibliography in [1]).

Proposition 1. *A function $g \in \mathcal{A}(D)$ is Fully Operator Lipschitzian if and only if there is $C > 0$ such that*

$$\|[g(A), X]\| \le C\|[A, X]\|, \quad \textit{for } A \in \mathbf{B}_1 \textit{ and } X \in B(H). \tag{4}$$

Proof. If $A, B \in \mathbf{B}_1$, the operator $L = \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ on $H \oplus H$ belongs to the unit ball of $B(H \oplus H)$. Let $X = \left(\begin{smallmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{smallmatrix}\right)$. Clearly, (4) holds for operators on $H \oplus H$. Hence $\|[g(L), X]\| \le C\|[L, X]\|$ which implies (3).

Conversely, let $\|A\| < 1$. For $X \in B(H)$, the operators $A(t) = e^{tX}Ae^{-tX}$ belong to $\mathbf{B}_1$ for sufficiently small t. If (3) holds then, taking into account that $g(e^{tX}Ae^{-tX}) = e^{tX}g(A)e^{-tX}$, we obtain

$$\|g(A) - e^{tX}g(A)e^{-tX}\| = \|g(A) - g(e^{tX}Ae^{-tX})\| \le C\|A - e^{tX}Ae^{-tX}\|.$$

Dividing through by t and taking the limit as $t \to 0$, we have that (4) holds.

Let $\|A\| = 1$, $X \in B(H)$. For $r < 1$, $\|[g(rA), X]\| \le C\|[rA, X]\|$. Taking the limit as $r \to 1$, we obtain that (4) holds. □

It follows from Proposition 1 that our aim is to construct a Fully Operator Lipschitz function with discontinuous derivative.

3. FULLY OPERATOR LIPSCHITZ FUNCTIONS WITH DISCONTINUOUS DERIVATIVE

Consider the following function on D:

$$h(1) = 0 \qquad \text{and} \qquad h(z) = (z-1)^2 \exp((z-1)^{-1}), \quad \text{for } z \in D,\ z \ne 1.$$

Since $\frac{x-1}{(x-1)^2+y^2} < 0$, if $z = x + iy \in D\backslash 1$, we have that

$$\begin{aligned} \sup_{z \in D\backslash 1} |\exp((z-1)^{-1})| &= \sup_{z \in D\backslash 1} \left|\exp\left(\frac{(x-1) - iy}{(x-1)^2 + y^2}\right)\right| \\ &= \sup_{z \in D\backslash 1} \left|\exp\left(\frac{x-1}{(x-1)^2 + y^2}\right)\right| \left|\exp\left(\frac{iy}{(x-1)^2 + y^2}\right)\right| \\ &= \sup_{z \in D\backslash 1} \left|\exp\left(\frac{x-1}{(x-1)^2 + y^2}\right)\right| < 1. \end{aligned} \tag{5}$$

The function h is analytic on D° and continuous on D, since, by (5),

$$|h(z)| = |h(x+iy)| = |z-1|^2 |\exp((z-1)^{-1})| \le |z-1|^2 \to 0,$$

as $z \to 1$. Thus $h \in \mathcal{A}(D)$. We obtain similarly that

$$\left|\frac{h(z)-h(1)}{z-1}\right| = |z-1|\,|\exp((z-1)^{-1})| \leq |z-1| \to 0,$$

as $z \to 1$, so $h'(1) = 0$. We also obtain that

$$h'(z) = 2(z-1)\exp((z-1)^{-1}) - \exp((z-1)^{-1}), \quad \text{for } z \in D,\ z \neq 1.$$

We have, as above, that $(z-1)\exp((z-1)^{-1}) \to 0$, as $z \to 1$, while $\exp((z-1)^{-1})$ does not have limit as $z \to 1$. Therefore h' is discontinuous at $z = 1$.

Theorem 2. *The function h is Fully Operator Lipschitzian.*

Proof. By Proposition 1, we only need to prove that (4) holds for h. For $0 < \lambda < 1$, set $h_\lambda(z) = h(\lambda z)$. Every h_λ is analytic in a neighbourhood of D, so it belongs to $\mathcal{A}(D)$, and $\|h - h_\lambda\| \to 0$, as $\lambda \to 1$. Hence it follows from (2) that

$$\begin{aligned} \|[h(A),X] - [h_\lambda(A),X]\| &= \|[(h(A)-h_\lambda(A)),X]\| \\ &\leq 2\|h(A)-h_\lambda(A)\|\,\|X\| \leq 2\|h-h_\lambda\|\,\|A\|\,\|X\| \to 0. \end{aligned} \tag{6}$$

For any $A \in \mathbf{B}_1$ and $X \in B(H)$,

$$\begin{aligned} \|[h_\lambda(A),X]\| &= \|[(\lambda A-\mathbf{1})\exp((\lambda A-\mathbf{1})^{-1})(\lambda A-\mathbf{1}),X]\| \\ &= \|[(\lambda A-\mathbf{1}),X]\exp((\lambda A-\mathbf{1})^{-1})(\lambda A-\mathbf{1}) \\ &\quad + (\lambda A-\mathbf{1})[\exp((\lambda A-\mathbf{1})^{-1}),X](\lambda A-\mathbf{1}) \\ &\quad\quad + (\lambda A-\mathbf{1})\exp((\lambda A-\mathbf{1})^{-1})[(\lambda A-\mathbf{1}),X]\| \\ &\leq 2\lambda\|[A,X]\|\,\|\exp((\lambda A-\mathbf{1})^{-1})\|\,\|\lambda A-\mathbf{1}\| \\ &\quad + \|(\lambda A-\mathbf{1})[\exp((\lambda A-\mathbf{1})^{-1}),X](\lambda A-\mathbf{1})\|. \end{aligned}$$

We have that $\|\lambda A - \mathbf{1}\| < 2$ and that the function $\exp((\lambda z-1)^{-1})$ belongs to $\mathcal{A}(D)$. We obtain from (2) and (5) that

$$\|\exp((\lambda A-\mathbf{1})^{-1})\| \leq \|\exp((\lambda z-1)^{-1})\| \leq \sup_{z \in D\setminus 1} |\exp((z-1)^{-1})| < 1. \tag{7}$$

Therefore

$$\|[h_\lambda(A),X]\| \leq 4\lambda\|[A,X]\| + \|(\lambda A-\mathbf{1})[\exp((\lambda A-\mathbf{1})^{-1}),X](\lambda A-\mathbf{1})\|. \tag{8}$$

It follows from Lemma 2 of [5] that, for any $B \in B(H)$,

$$[\exp(B),X] = \int_0^1 \exp(tB)[B,X]\exp((1-t)B)\,dt.$$

If B is invertible, then $B[B^{-1},X]B = [X,B]$. Hence

$$\begin{aligned} &\|(\lambda A-\mathbf{1})[\exp((\lambda A-\mathbf{1})^{-1}),X](\lambda A-\mathbf{1})\| \\ &\quad = \left\|\int_0^1 \exp(t(\lambda A-\mathbf{1})^{-1})(\lambda A-\mathbf{1})[(\lambda A-\mathbf{1})^{-1},X](\lambda A-\mathbf{1}) \right. \\ &\qquad\qquad\qquad\qquad \left. \exp((1-t)(\lambda A-\mathbf{1})^{-1})dt\right\| \\ &\quad \leq \|[X,\lambda A-\mathbf{1}]\| \int_0^1 \|\exp(t(\lambda A-\mathbf{1})^{-1})\|\,\|\exp((1-t)(\lambda A-\mathbf{1})^{-1})\|\,dt. \end{aligned}$$

As in (7), we have that

$$\|\exp(t(\lambda A-\mathbf{1})^{-1})\|<1 \quad\text{and}\quad \|\exp((1-t)(\lambda A-\mathbf{1})^{-1})\|<1.$$

Therefore

$$\|(\lambda A-\mathbf{1})[\exp((\lambda A-\mathbf{1})^{-1}),X](\lambda A-\mathbf{1})\|\le\lambda\|[A,X]\|.$$

Hence we obtain from (8) that $\|[h_\lambda(A),X]\|\le 5\lambda\|[A,X]\|$. Combining this with (6), we conclude that $\|[h(A),X]\|\le 5\|[A,X]\|$. ☐

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School of Communications Technology and Mathematical Sciences, University of North London, Holloway, London N7 8DB, Great Britain

E-mail address: `e.kissin@unl.ac.uk`

School of Communications Technology and Mathematical Sciences, University of North London, Holloway, London N7 8DB, Great Britain – and – Department of Mathematics, Vologda State Technical University, Vologda, Russia

E-mail address: `shulman_v@yahoo.com`