

HAHN-BANACH EXTENSION OPERATORS AND SPACES OF OPERATORS

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ABSTRACT. Let $X \subseteq Y$ be Banach spaces and let $\mathcal{A} \subseteq \mathcal{B}$ be closed operator ideals. Let Z be a Banach space having the Radon-Nikodým property. The main results are as follows. If $\Phi : \mathcal{A}(Z, X)^* \rightarrow \mathcal{B}(Z, Y)^*$ is a Hahn-Banach extension operator, then there exists a set of Hahn-Banach extension operators $\phi_i : X^* \rightarrow Y^*$, $i \in I$, such that $Z = \sum_{i \in I} \oplus_1 Z_{\Phi\phi_i}$, where $Z_{\Phi\phi_i} = \{z \in Z : \Phi(x^* \otimes z) = (\phi_i x^*) \otimes z, \forall x^* \in X^*\}$. If $\mathcal{A}(\hat{Z}, X)$ is an ideal in $\mathcal{B}(\hat{Z}, Y)$ for all equivalently renormed versions \hat{Z} of Z , then there exist Hahn-Banach extension operators $\Phi : \mathcal{A}(Z, X)^* \rightarrow \mathcal{B}(Z, Y)^*$ and $\phi : X^* \rightarrow Y^*$ such that $Z = Z_{\Phi\phi}$.

INTRODUCTION

Let X be a closed subspace of a Banach space Y . A linear operator $\phi : X^* \rightarrow Y^*$ is called a *Hahn-Banach extension operator* if $(\phi x^*)(x) = x^*(x)$ and $\|\phi x^*\| = \|x^*\|$ for all $x \in X$ and all $x^* \in X^*$. Let us denote the set of all Hahn-Banach extension operators $\phi : X^* \rightarrow Y^*$ by $HB(X, Y)$.

It is well known (and straightforward to verify) that $HB(X, Y) \neq \emptyset$ if and only if X is an ideal in Y . Let us recall that X is an *ideal* in Y if X^\perp , the annihilator of X in Y^* , is the kernel of a norm one projection on Y^* . The notion of an ideal was introduced and studied by Godefroy, Kalton, and Saphar in [7].

For a Banach space $Z \neq \{0\}$, let \mathcal{A} be a closed subspace of the Banach space $\mathcal{L}(Z, X)$ of bounded linear operators and let \mathcal{B} be a closed subspace of $\mathcal{L}(Z, Y)$. We assume that the subspaces of finite rank operators $\mathcal{F}(Z, X)$ and $\mathcal{F}(Z, Y)$ are respectively contained in \mathcal{A} and \mathcal{B} . We also assume that $\mathcal{A} \subseteq \mathcal{B}$.

If $\Phi \in HB(\mathcal{A}, \mathcal{B})$, then it is easy to see that any pair of elements $z \in Z$, $z^* \in Z^*$ satisfying $\|z\| = \|z^*\| = z^*(z) = 1$ defines $\phi \in HB(X, Y)$ by

$$(\phi x^*)(y) = (\Phi(x^* \otimes z))(z^* \otimes y), \quad \forall x^* \in X^*, \forall y \in Y.$$

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This means that

$$(\phi x^*)(Tz) = (\Phi(x^* \otimes z))(T)$$

for all $x^* \in X^*$ and all elementary operators $T \in \mathcal{B}$ of the form $T = z^* \otimes y$, $y \in Y$.

For given $\Phi \in HB(\mathcal{A}, \mathcal{B})$ and $\phi \in HB(X, Y)$, let us introduce

$$Z_{\Phi\phi} = \{z \in Z : (\phi x^*)(Tz) = (\Phi(x^* \otimes z))(T), \forall x^* \in X^*, \forall T \in \mathcal{B}\}.$$

Then $Z_{\Phi\phi}$ is a closed subspace of Z and we may have $Z_{\Phi\phi} = \{0\}$. One of the main results of [9] asserts that if X^* has the compact approximation property with conjugate operators (i.e. there exists a net (K_α) in $\mathcal{K}(X, X)$ such that (K_α^*) converges to the identity I_{X^*} uniformly on compact subsets of X^*), then for any reflexive Banach space Z and any $\phi \in HB(X, Y)$ there exists $\Phi \in HB(\mathcal{K}(Z, X), \mathcal{W}(Z, Y))$ so that $Z_{\Phi\phi} = Z$.

In this paper, we shall look at the general situation: we shall not assume any approximation property. We shall prove (see Theorems 1.3 and 2.3) that if a Banach space Z has the Radon-Nikodým property and does not contain any isometric copy of ℓ_1 , then for every $\Phi \in HB(\mathcal{A}, \mathcal{B})$ there exists a finite set $\{\phi_1, \dots, \phi_n\} \subseteq HB(X, Y)$ so that

$$Z = Z_{\Phi\phi_1} \oplus_1 \dots \oplus_1 Z_{\Phi\phi_n},$$

and, under some additional hypotheses, there exist $\phi \in HB(X, Y)$ and $\Phi \in HB(\mathcal{A}, \mathcal{B})$ so that $Z = Z_{\Phi\phi}$.

The notation we use is standard. We consider Banach spaces over the real field \mathbb{R} . The closure of a set A is denoted by \overline{A} and its convex hull by $\text{conv } A$. For a Banach space Z and $r > 0$, we denote $B_Z(0, r) = \{z \in Z : \|z\| \leq r\}$, $S_Z(0, r) = \{z \in Z : \|z\| = r\}$, and $B_Z = B_Z(0, 1)$. The set of denting points of $B_Z(0, r)$ is denoted by $\text{dent } B_Z(0, r)$. The identity operator on Z is denoted by I_Z . By $\overline{\mathcal{F}(Z, Y)}$, $\mathcal{K}(Z, Y)$, and $\mathcal{W}(Z, Y)$ we denote the subspaces of $\mathcal{L}(Z, Y)$ of approximable operators (i.e. norm limits of finite rank operators), compact operators, and weakly compact operators.

1. HAHN-BANACH EXTENSION OPERATORS AND SPACES OF OPERATORS

Let X be a closed subspace of a Banach space Y and let Z be a Banach space. Let $\mathcal{A} \subseteq \mathcal{L}(Z, X)$ and $\mathcal{B} \subseteq \mathcal{L}(Z, Y)$ be closed subspaces such that $\mathcal{F}(Z, X) \subseteq \mathcal{A}$, $\mathcal{F}(Z, Y) \subseteq \mathcal{B}$, and $\mathcal{A} \subseteq \mathcal{B}$.

For given $\phi \in HB(X, Y)$ and $\Phi \in HB(\mathcal{A}, \mathcal{B})$, we have mappings

$$\phi \otimes I_Z : X^* \otimes Z \rightarrow Y^* \otimes Z \subseteq \mathcal{B}^*$$

and

$$\Phi|_{X^* \otimes Z} : X^* \otimes Z \rightarrow \mathcal{B}^*.$$

By definition, $Z_{\Phi\phi}$ is the maximal subspace of Z for which

$$\phi \otimes I_{Z_{\Phi\phi}} = \Phi|_{X^* \otimes Z_{\Phi\phi}}.$$

Thus

$$Z_{\Phi\phi} = \{z \in Z : \Phi(x^* \otimes z) = (\phi x^*) \otimes z, \forall x^* \in X^*\}.$$

Lemma 1.1. *Let X be a closed subspace of a Banach space Y and let Z be a Banach space. Let $\mathcal{A} \subseteq \mathcal{L}(Z, X)$ and $\mathcal{B} \subseteq \mathcal{L}(Z, Y)$ be closed subspaces such that $\mathcal{F}(Z, X) \subseteq \mathcal{A}$, $\mathcal{F}(Z, Y) \subseteq \mathcal{B}$, and $\mathcal{A} \subseteq \mathcal{B}$. Assume that $\Phi \in HB(\mathcal{A}, \mathcal{B})$. If ϕ_1, ϕ_2, \dots are in $HB(X, Y)$, all different, and if $z_i \in Z_{\Phi\phi_i}$ are such that $\sum_{i=1}^{\infty} z_i = 0$ and $\sum_{i=1}^{\infty} \|z_i\| < \infty$, then $z_i = 0$ for all $i \in \mathbb{N}$.*

Proof. Assume for contradiction that some z_i , say $z_1 \neq 0$. For $x^* \in X^*$ we get that

$$0 = \Phi(x^* \otimes 0) = \Phi(x^* \otimes \sum_{i=1}^{\infty} z_i) = \sum_{i=1}^{\infty} \Phi(x^* \otimes z_i) = \sum_{i=1}^{\infty} (\phi_i x^*) \otimes z_i.$$

Let $z^* \in Z^*$ and $y \in Y$. Evaluating at $z^* \otimes y$ gives

$$\begin{aligned} 0 &= \sum_{i=1}^{\infty} ((\phi_i x^*) \otimes z_i)(z^* \otimes y) \\ &= \sum_{i=1}^{\infty} (\phi_i x^*)(y) z^*(z_i) \\ &= z^* \left(\sum_{i=1}^{\infty} (\phi_i x^*)(y) z_i \right). \end{aligned}$$

Since this is true for all $z^* \in Z^*$, we get $\sum_{i=1}^{\infty} (\phi_i x^*)(y) z_i = 0$.

We can repeat this argument, and then we get

$$\sum_{i=1}^{\infty} ((\phi_i x^*)(y))^m z_i = 0,$$

for all $m \in \mathbb{N}$ and all $x^* \in X^*$, $y \in Y$. But then, for every polynomial p and all $x^* \in X^*$, $y \in Y$,

$$\sum_{i=1}^{\infty} p((\phi_i x^*)(y)) z_i = 0.$$

Using the Weierstrass theorem, for every continuous function $f \in \mathcal{C}[-1, 1]$ and all $x^* \in B_{X^*}$, $y \in B_Y$, we get

$$\sum_{i=1}^{\infty} f((\phi_i x^*)(y)) z_i = 0.$$

Since the operators ϕ_i are all different, we can choose $x^* \in B_{X^*}$ and $y \in B_Y$ such that $(\phi_1 x^*)(y) \neq (\phi_2 x^*)(y)$. We now choose $f \in \mathcal{C}[-1, 1]$ such that $f((\phi_1 x^*)(y)) = 1$, $f((\phi_2 x^*)(y)) = 0$, and $\sup_{t \in [-1, 1]} |f(t)| = 1$. Then writing $a_{2i} = f((\phi_i x^*)(y))$, we get $|a_{2i}| \leq 1$ and

$$z_1 + \sum_{i=3}^{\infty} a_{2i} z_i = 0.$$

Note that $z_1 \in Z_{\Phi\phi_1}$, $a_{2i} z_i \in Z_{\Phi\phi_i}$, and $\|z_1\| + \sum_{i=3}^{\infty} \|a_{2i} z_i\| < \infty$. Therefore, by the argument above, for all $f \in \mathcal{C}[-1, 1]$ and all $x^* \in B_{X^*}$, $y \in B_Y$, we have

$$f((\phi_1 x^*)(y)) z_1 + \sum_{i=3}^{\infty} a_{2i} f((\phi_i x^*)(y)) z_i = 0.$$

Consequently, we can find numbers $a_{ji} \in [-1, 1]$ such that for every $m \in \mathbb{N}$

$$z_1 + \sum_{i=m+1}^{\infty} a_{2i} \cdots a_{mi} z_i = 0.$$

This implies that $\|z_1\| \leq \sum_{i=m+1}^{\infty} \|z_i\| \rightarrow 0$ when $m \rightarrow \infty$. Hence $z_1 = 0$, and we have a contradiction. \square

Lemma 1.2. *Let X, Y, Z, \mathcal{A} , and \mathcal{B} be as in Lemma 1.1. If $\Phi \in HB(\mathcal{A}, \mathcal{B})$ and $r > 0$, then*

$$\text{dent } B_Z(0, r) \subseteq S_Z(0, r) \cap \bigcup_{\phi \in HB(X, Y)} Z_{\Phi\phi}.$$

Proof. Since $\text{dent } B_Z(0, r) = r \cdot \text{dent } B_Z$ and $Z_{\Phi\phi}$ is a subspace, it suffices to consider $\text{dent } B_Z$. Let $z \in \text{dent } B_Z$. The proof of [9, Lemma 3.1] shows that then, for every $x^* \in X^*$, all norm-preserving extensions of $x^* \otimes z \in \mathcal{F}(Z, X)^*$ to $\mathcal{L}(Z, Y)$ have the form $y^* \otimes z$, where $y^* \in Y^*$ is a norm-preserving extension of x^* . Recalling that $\Phi(x^* \otimes z) \in \mathcal{B}^*$ is a norm-preserving extension of $x^* \otimes z \in \mathcal{A}^*$ and using the linearity of Φ , we see that the map $\phi : X^* \rightarrow Y^*$, which assigns to every $x^* \in X^*$ its norm-preserving extension $y^* = \phi x^* \in Y^*$ satisfying

$$\Phi(x^* \otimes z) = (\phi x^*) \otimes z,$$

is linear. Therefore $\phi \in HB(X, Y)$ and $z \in Z_{\Phi\phi}$. \square

Let I be an arbitrary index set. Let Z be a Banach space and let $(Z_i)_{i \in I}$ be a family of closed subspaces of Z . We shall call Z an ℓ_1 -direct sum of subspaces Z_i , $i \in I$, and write

$$Z = \sum_{i \in I} \oplus_1 Z_i$$

if every $z \in Z$ admits a decomposition $z = \sum_{i \in I} z_i$ with $z_i \in Z_i$ (meaning that the directed system of finite partial sums converges to z) such that $\|z\| = \sum_{i \in I} \|z_i\|$ and $0 = \sum_{i \in I} z_i$ implies $z_i = 0$ for all $i \in I$. Let us recall that the absolute summability of $(z_i)_{i \in I}$ (i.e. $\sum_{i \in I} \|z_i\| < \infty$) guarantees that the family $(z_i)_{i \in I}$ cannot contain more than countably many non-zero elements. This easily implies that every $z \in \sum_{i \in I} \oplus_1 Z_i$ has, in fact, a unique representation $z = \sum_{i \in I} z_i$, where $z_i \in Z_i$. It is clear that if the index set is finite, then Z is the ℓ_1 -direct sum in the usual sense: $Z = Z_1 \oplus \dots \oplus Z_n$ and $\|z\| = \sum_{i=1}^n \|z_i\|$ for all $z = \sum_{i=1}^n z_i$ in Z .

Theorem 1.3. *Let X, Y, Z, \mathcal{A} , and \mathcal{B} be as in Lemma 1.1. If Z has the Radon-Nikodým property and $\Phi \in HB(\mathcal{A}, \mathcal{B})$, then there exists $\{\phi_i : i \in I\} \subseteq HB(X, Y)$ such that*

$$Z = \sum_{i \in I} \oplus_1 Z_{\Phi\phi_i}.$$

If, moreover, Z does not contain any isometrically isomorphic copy of ℓ_1 , then there exists a finite set $\{\phi_1, \dots, \phi_n\} \subseteq HB(X, Y)$ such that

$$Z = Z_{\Phi\phi_1} \oplus_1 \cdots \oplus_1 Z_{\Phi\phi_n}.$$

Proof. Let $\{\phi \in HB(X, Y) : Z_{\Phi\phi} \neq \{0\}\} = \{\phi_i : i \in I\}$ and let W be the Cartesian ℓ_1 -product of the spaces $Z_{\Phi\phi_i}$, $i \in I$. Define an operator $U : W \rightarrow Z$ by $U((z_i)_{i \in I}) = \sum_{i \in I} z_i$. Clearly U is a continuous linear operator with $\|U\| \leq 1$. That U is one-to-one follows from Lemma 1.1.

Recall that $B_Z(0, r) = \overline{\text{conv dent}} B_Z(0, r)$ for all $r > 0$, because Z has the Radon-Nikodým property.

Let $z \in Z$, $z \neq 0$, and let $\epsilon > 0$. We can choose $u_1 \in \text{conv dent } B_Z(0, \|z\|)$ with $\|z - u_1\| \leq \epsilon/2$. Using Lemma 1.2, we can write

$$u_1 = \sum_{j=1}^{j(1)} \lambda_j z_j, \lambda_j > 0, \sum_{j=1}^{j(1)} \lambda_j = 1, z_j \in S_Z(0, \|z\|) \cap \left(\bigcup_{\phi \in HB(X, Y)} Z_{\Phi\phi} \right).$$

Next we choose $u_2 \in \text{conv dent } B_Z(0, \epsilon/2)$ with $\|z - u_1 - u_2\| \leq \epsilon/2^2$. Thus an induction procedure yields a sequence $u_n \in \text{conv dent } B_Z(0, \epsilon/2^{n-1})$ such that $z = \sum_{n=1}^{\infty} u_n$. Let for $n \geq 2$

$$u_n = \sum_{j=j(n-1)+1}^{j(n)} \lambda_j z_j, \lambda_j > 0, \sum_{j=j(n-1)+1}^{j(n)} \lambda_j = 1,$$

$$z_j \in S_Z(0, \epsilon/2^{n-1}) \cap \left(\bigcup_{\phi \in HB(X, Y)} Z_{\Phi\phi} \right).$$

Notice that $\sum_{j=1}^{\infty} \|\lambda_j z_j\| \leq \|z\| + \epsilon$ and $z = \sum_{j=1}^{\infty} \lambda_j z_j$. Therefore, by a rearrangement, we can write $z = \sum_{j=1}^{\infty} w_j$ with $w_j \in Z_{\Phi\phi_j}$ so that all ϕ_j are different. But then $U((w_j)_{j=1}^{\infty}) = z$ and $\|(w_j)_{j=1}^{\infty}\| = \sum_{j=1}^{\infty} \|w_j\| \leq \|z\| + \epsilon$. Since U is one-to-one and $\|U\| \leq 1$, this implies that $\|(w_j)_{j=1}^{\infty}\| = \|z\|$. Thus U is a linear isometry from W onto Z and therefore

$$Z = \sum_{i \in I} \oplus_1 Z_{\Phi\phi_i}$$

as desired.

For the “moreover” part, let us just note that if I were infinite, then Z should contain an isometrically isomorphic copy of ℓ_1 . □

The Radon-Nikodým property is studied in detail e.g. in [6]. The Radon-Nikodým property for a dual space W^* is equivalent to the requirement that separable subspaces of W have separable duals. Thus all reflexive and all separable dual spaces have the Radon-Nikodým property.

By a well-known distortion result of James [8] (see e.g [10, p. 97]), if Z contains a subspace isomorphic to ℓ_1 , then Z contains “nearly isometric” copies of ℓ_1 . If Z happens to be separable (e.g. $Z = \ell_1$), then, by a well-known result of Clarkson [4] (see e.g. [5, pp. 101 and 144]), Z admits an equivalent strictly convex norm. The renormed version of Z contains “nearly isometric” copies of ℓ_1 , but it does not contain any isometric copy of ℓ_1 .

Theorem 1.3 shows, in particular, that $Z = Z_{\Phi\phi_1} \oplus_1 \cdots \oplus_1 Z_{\Phi\phi_n}$ whenever Z is a reflexive Banach space and $HB(\mathcal{A}, \mathcal{B}) \neq \emptyset$, where \mathcal{A} and \mathcal{B} are arbitrary operator spaces (satisfying some natural requirements like $\mathcal{F}(Z, X) \subseteq \mathcal{A}$). This is to compare with one of the main results in [9, Theorem 4.6] showing that under the hypothesis that X^* has the compact approximation property with conjugate operators, for any reflexive Banach space Z , one has $HB(\mathcal{K}(Z, X), \mathcal{W}(Z, Y)) \neq \emptyset$ and $Z = Z_{\Phi\phi}$.

2. HAHN-BANACH EXTENSION OPERATORS AND SPACES OF OPERATORS
GENERATED BY OPERATOR IDEALS

Concerning the notation \mathcal{A} and \mathcal{B} from the previous section, it is convenient to change our notation as follows. In this section \mathcal{A} and \mathcal{B} will denote closed operator ideals in the sense of Pietsch [11]. This guarantees, in particular, that $\mathcal{F}(Z, X) \subseteq \mathcal{A}(Z, X)$ and $\mathcal{A}(Z, X)$ is a closed subspace of $\mathcal{L}(Z, X)$. This also guarantees that if $T \in \mathcal{A}(Z, X)$ and $U \in \mathcal{L}(Z_0, Z)$, $V \in \mathcal{L}(X, X_0)$, then $V \circ T \circ U \in \mathcal{A}(Z_0, X_0)$. Let us recall that, for instance, the classes of all approximable operators, all compact operators, and all weakly compact operators are closed operator ideals.

Let us begin with an easy result which applies to arbitrary Z and \mathcal{A} .

Proposition 2.1. *Let X be a closed subspace of a Banach space Y , let Z be a Banach space and let \mathcal{A} be a closed operator ideal. If X is the range of a norm one projection in X^{**} and if X is an ideal in Y (that is $HB(X, Y) \neq \emptyset$), then there exist $\phi \in HB(X, Y)$ and $\Phi \in HB(\mathcal{A}(Z, X), \mathcal{A}(Z, Y))$ such that $Z = Z_{\Phi\phi}$.*

Proof. Let $\psi \in HB(X, Y)$ and let $P : X^{**} \rightarrow X$ be a norm one projection onto X . Define $\Phi : \mathcal{A}(Z, X)^* \rightarrow \mathcal{A}(Z, Y)^*$ by

$$(\Phi F)(T) = F(P \circ \psi_{|Y}^* \circ T), \quad F \in \mathcal{A}(Z, X)^*, \quad T \in \mathcal{A}(Z, Y).$$

Then Φ is linear and $\|\Phi F\| \leq \|F\|$. Since $\psi^*x = x$ for all x , that is $\psi_{|X}^* = I_X$, we have, for all $S \in \mathcal{A}(Z, X)$,

$$(\Phi F)(S) = F(P \circ I_X \circ S) = F(S),$$

meaning that ΦF is an extension of F . Hence Φ is a Hahn-Banach extension operator.

Moreover, if $x^* \in X^*$, $T \in \mathcal{A}(Z, Y)$, and $z \in Z$, then

$$\begin{aligned} (\Phi(x^* \otimes z))(T) &= (x^* \otimes z)(P \circ \psi_{|Y}^* \circ T) = x^*(P\psi_{|Y}^*Tz) \\ &= ((\psi_{|Y}^*)^*P^*x^*)(Tz) = (\phi x^*)(Tz), \end{aligned}$$

where $\phi = (\psi_{|Y}^*)^* \circ P^*$. Noting that $\phi \in HB(X, Y)$, we have $Z = Z_{\Phi\phi}$ as desired. \square

Let us recall that X is always an ideal in X^{**} because the canonical embedding $J_{X^*} : X^* \rightarrow X^{***}$ is a Hahn-Banach extension operator. But if we have a norm one projection P from X^{**} onto X as in Proposition 2.1, then clearly $P^* \in HB(X, X^{**})$.

Corollary 2.2. *Let X and Z be Banach spaces and let \mathcal{A} be a closed operator ideal. If X is the range of a norm one projection P in X^{**} , then $P^* \in HB(X, X^{**})$ and there exist $\Phi \in HB(\mathcal{A}(Z, X), \mathcal{A}(Z, X^{**}))$ such that $Z = Z_{\Phi P^*}$.*

Proof. It suffices to note that if we put $\psi = P^*$, then $\psi_{|X^{**}}^* = P$ and the proof of Proposition 2.1 will yield $\Phi \in HB(\mathcal{A}(Z, X), \mathcal{A}(Z, X^{**}))$ defined by

$$(\Phi F)(T) = F(P \circ T), \quad F \in \mathcal{A}(Z, X)^*, \quad T \in \mathcal{A}(Z, X^{**}),$$

and also the equality $Z = Z_{\Phi P^*}$. \square

The following theorem is the main result of this paper. Its proof relies on Theorem 1.3.

Theorem 2.3. *Let X be a closed subspace of a Banach space Y and let Z be a Banach space having the Radon-Nikodým property. Let \mathcal{A} and \mathcal{B} be closed operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$. If $\mathcal{A}(\hat{Z}, X)$ is an ideal in $\mathcal{B}(\hat{Z}, Y)$ for all equivalently renormed versions \hat{Z} of Z , then there exist*

$$\phi \in HB(X, Y) \quad \text{and} \quad \Phi \in HB(\mathcal{A}(Z, X), \mathcal{B}(Z, Y))$$

such that $Z = Z_{\Phi\phi}$. Moreover, the (non-empty) subset

$$\mathcal{F}_Z = \{\phi \in HB(X, Y) : \exists \Phi \in HB(\mathcal{A}(Z, X), \mathcal{B}(Z, Y)) \text{ such that } Z = Z_{\Phi\phi}\}$$

of $\mathcal{L}(X^*, Y^*) = (Y \hat{\otimes}_{\pi} X^*)^*$ is weak* compact.

Proof. Let $\Phi \in HB(\mathcal{A}(Z, X), \mathcal{B}(Z, Y))$. By Theorem 1.3, there exists $\{\phi_i : i \in I\} \subseteq HB(X, Y)$ such that

$$Z = \sum_{i \in I} \oplus_1 Z_{\Phi\phi_i}, \quad Z_{\Phi\phi_i} \neq \{0\}, \quad i \in I.$$

If I consists of only one element, then we have nothing to prove. If I contains more than just one element, then denoting $W = Z_{\Phi\phi_i}$ and $V = \sum_{j \in I \setminus \{i\}} \oplus_1 Z_{\Phi\phi_j}$ for some fixed $i \in I$, we can write $Z = W \oplus_1 V$.

For every $\epsilon \in (0, 1]$, let $Z_{\epsilon} = (Z, \|\cdot\|_{\epsilon})$ be the equivalently renormed version of Z with $\|z\|_{\epsilon} = (\|w\|^{1+\epsilon} + \|v\|^{1+\epsilon})^{1/1+\epsilon}$, $w \in W$, $v \in V$, and let $\Phi_{\epsilon} \in HB(\mathcal{A}(Z_{\epsilon}, X), \mathcal{B}(Z_{\epsilon}, Y))$. By a theorem due to Behrends (see [1], [2, p. 8], or [3]), a Banach space (except $(\mathbb{R}^2, \|\cdot\|_{\infty})$ which does not contain any nontrivial L^p -summand for $1 < p < \infty$) cannot contain nontrivial L^p -summands and nontrivial L^r -summands simultaneously if $p \neq r$. Since Z_{ϵ} contains nontrivial $L^{1+\epsilon}$ -summands (W and V), it does not contain any nontrivial L^1 -summand. Therefore, by Theorem 1.3, there exists $\phi_{\epsilon} \in HB(X, Y)$ such that $Z_{\epsilon} = (Z_{\epsilon})_{\Phi_{\epsilon}\phi_{\epsilon}}$.

Let us denote by $I_{\epsilon} : Z_{\epsilon} \rightarrow Z$ the identity mapping. Then $\|I_{\epsilon}^{-1}\| = 1$ and $\|I_{\epsilon}\| \rightarrow 1$ as $\epsilon \rightarrow 0$. Define $\Psi_{\epsilon} \in \mathcal{L}(\mathcal{A}(Z, X)^*, \mathcal{B}(Z, Y)^*)$ by

$$(\Psi_{\epsilon}F)(T) = (\Phi_{\epsilon}F_{\epsilon})(T \circ I_{\epsilon}), \quad F \in \mathcal{A}(Z, X)^*, \quad T \in \mathcal{B}(Z, Y),$$

where $F_{\epsilon} \in \mathcal{A}(Z_{\epsilon}, X)^*$ is defined by

$$F_{\epsilon}(S) = F(S \circ I_{\epsilon}^{-1}), \quad S \in \mathcal{A}(Z_{\epsilon}, X).$$

Then $\|F_{\epsilon}\| \leq \|F\|$ and $\|\Psi_{\epsilon}\| \leq \|I_{\epsilon}\| \leq \sqrt{2}$ for $\epsilon \in (0, 1]$. Since $(\Psi_{\epsilon})_{\epsilon \in (0, 1]}$ is a bounded net in $\mathcal{L}(\mathcal{A}(Z, X)^*, \mathcal{B}(Z, Y)^*) = (\mathcal{B}(Z, Y) \hat{\otimes}_{\pi} \mathcal{A}(Z, X)^*)^*$, it has a subnet $(\Psi_{\epsilon(\nu)})$ that converges weak* to some $\Psi \in \mathcal{L}(\mathcal{A}(Z, X)^*, \mathcal{B}(Z, Y)^*)$. Noting that $\phi_{\epsilon(\nu)} \in B_{\mathcal{L}(X^*, Y^*)} = B_{(Y \hat{\otimes}_{\pi} X^*)^*}$ and by passing to a subnet, we may assume that $(\phi_{\epsilon(\nu)})$ converges weak* to some $\phi \in B_{\mathcal{L}(X^*, Y^*)}$. It is clear that $\phi \in HB(X, Y)$ because $\phi_{\epsilon(\nu)} \in HB(X, Y)$ and

$$(\phi x^*)(y) = \phi(y \otimes x^*) = \lim_{\nu} \phi_{\epsilon(\nu)}(y \otimes x^*) = \lim_{\nu} (\phi_{\epsilon(\nu)} x^*)(y)$$

for all $x^* \in X^*$ and $y \in Y$. It remains to show that $\Psi \in HB(\mathcal{A}(Z, X), \mathcal{B}(Z, Y))$ and $Z = Z_{\Psi\phi}$.

Consider any $F \in \mathcal{A}(Z, X)^*$. Since, for all $T \in \mathcal{B}(Z, Y)$,

$$(\Psi F)(T) = \Psi(T \otimes F) = \lim_{\nu} \Psi_{\epsilon(\nu)}(T \otimes F) = \lim_{\nu} (\Psi_{\epsilon(\nu)} F)(T),$$

we get that

$$\|\Psi F\| \leq \limsup_{\nu} \|\Psi_{\epsilon(\nu)}\| \|F\| \leq \lim_{\nu} \|I_{\epsilon(\nu)}\| \|F\| = \|F\|.$$

Since for all $S \in \mathcal{A}(Z, X)$ we also have

$$\begin{aligned} (\Psi F)(S) &= \lim_{\nu} (\Phi_{\epsilon(\nu)} F_{\epsilon(\nu)})(S \circ I_{\epsilon(\nu)}) \\ &= \lim_{\nu} F_{\epsilon(\nu)}(S \circ I_{\epsilon(\nu)}) = \lim_{\nu} F(S \circ I_{\epsilon(\nu)} \circ I_{\epsilon(\nu)}^{-1}) = F(S), \end{aligned}$$

we conclude that $\Psi \in HB(\mathcal{A}(Z, X), \mathcal{B}(Z, Y))$.

Finally, for arbitrary $x^* \in X^*$, $T \in \mathcal{B}(Z, Y)$, and $z \in Z$, we have

$$\begin{aligned} (\Psi(x^* \otimes z))(T) &= \lim_{\nu} (\Phi_{\epsilon(\nu)}(x^* \otimes I_{\epsilon(\nu)}^{-1} z))(T \circ I_{\epsilon(\nu)}) \\ &= \lim_{\nu} (\phi_{\epsilon(\nu)} x^*)(T I_{\epsilon(\nu)} I_{\epsilon(\nu)}^{-1} z) \\ &= \lim_{\nu} (\phi_{\epsilon(\nu)} x^*)(Tz) = (\phi x^*)(Tz), \end{aligned}$$

meaning that $Z = Z_{\Psi\phi}$.

For the “moreover” part, let us note that the above proof shows, in particular, that $HB(X, Y)$ is a weak* compact subset of $\mathcal{L}(X^*, Y^*) = (Y \hat{\otimes}_{\pi} X^*)^*$. Therefore if we consider a net (ϕ_{α}) in \mathcal{F}_Z together with a net (Φ_{α}) in $HB(\mathcal{A}(Z, X), \mathcal{B}(Z, Y))$ satisfying $Z = Z_{\Phi_{\alpha}\phi_{\alpha}}$ for all α , then by passing to a subnet, we may assume that $\phi_{\alpha} \rightarrow \phi \in HB(X, Y)$ and $\Phi_{\alpha} \rightarrow \Phi \in HB(\mathcal{A}(Z, X), \mathcal{B}(Z, Y))$ with respect to the corresponding weak* topologies. We also have that $Z = Z_{\Phi\phi}$ because

$$\begin{aligned} (\Phi(x^* \otimes z))(T) &= \lim_{\alpha} (\Phi_{\alpha}(x^* \otimes z))(T) \\ &= \lim_{\alpha} (\phi_{\alpha} x^*)(Tz) = (\phi x^*)(Tz) \end{aligned}$$

for all $x^* \in X^*$, $T \in \mathcal{B}(Z, Y)$, and $z \in Z$. □

Theorem 2.4. *Let X be a closed subspace of a Banach space Y . Let \mathcal{A} and \mathcal{B} be closed operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$. If $\mathcal{A}(W, X)$ is an ideal in $\mathcal{B}(W, Y)$ for all Banach spaces W having the Radon-Nikodým property, then there exists $\phi \in HB(X, Y)$ such that, for every Banach space Z having the Radon-Nikodým property, there exists $\Phi \in HB(\mathcal{A}(Z, X), \mathcal{B}(Z, Y))$ such that $Z = Z_{\Phi\phi}$.*

Proof. Let \mathcal{Z} be the collection of all Banach spaces Z having the Radon-Nikodým property. We need to show that $\bigcap_{Z \in \mathcal{Z}} \mathcal{F}_Z$ is non-empty. Since $B_{\mathcal{L}(X^*, Y^*)} = B_{(Y \hat{\otimes}_{\pi} X^*)^*}$ is compact with respect to the weak* topology and, by Theorem 2.3, its subsets \mathcal{F}_Z , $Z \in \mathcal{Z}$, are closed, the claim follows whenever each finite subfamily of $\{\mathcal{F}_Z : Z \in \mathcal{Z}\}$ has non-empty intersection.

Consider $Z_1, \dots, Z_n \in \mathcal{Z}$ and denote $Z = \sum_{i=1}^n \oplus_1 Z_i$. Let $I_i : Z_i \rightarrow Z$ and $P_i : Z \rightarrow Z_i$ be the natural embedding and projection. Since $Z \in \mathcal{Z}$, by Theorem 2.3, there exist $\phi \in HB(X, Y)$ and $\Phi \in HB(\mathcal{A}(Z, X), \mathcal{B}(Z, Y))$ such that $Z = Z_{\Phi\phi}$. We shall prove that

$$\phi \in \bigcap_{i=1}^n \mathcal{F}_{Z_i}.$$

For this, let us fix $i \in \{1, \dots, n\}$ and define $\Psi \in \mathcal{L}(\mathcal{A}(Z_i, X)^*, \mathcal{B}(Z_i, Y)^*)$ by

$$(\Psi F)(T) = (\Phi G)(T \circ P_i), \quad F \in \mathcal{A}(Z_i, X)^*, \quad T \in \mathcal{B}(Z_i, Y),$$

where $G \in \mathcal{A}(Z, X)^*$ is defined by

$$G(S) = F(S \circ I_i), \quad S \in \mathcal{A}(Z, X).$$

Then

$$\|\Psi F\| \leq \|\Phi G\| = \|G\| \leq \|F\|$$

and

$$(\Psi F)(T) = (\Phi G)(T \circ P_i) = G(T \circ P_i) = F(T \circ P_i \circ I_i) = F(T)$$

for all $F \in \mathcal{A}(Z_i, X)^*$ and $T \in \mathcal{A}(Z_i, X)$. Hence $\Psi \in HB(\mathcal{A}(Z_i, X), \mathcal{B}(Z_i, Y))$. Moreover, for all $x^* \in X^*$, $T \in \mathcal{B}(Z_i, Y)$, and $w \in Z_i$, we have (since $Z = Z_{\Phi\phi}$)

$$\begin{aligned} (\Psi(x^* \otimes w))(T) &= (\Phi(x^* \otimes I_i w))(T \circ P_i) \\ &= (\phi x^*)(TP_i I_i w) \\ &= (\phi x^*)(Tw), \end{aligned}$$

meaning that $Z_i = (Z_i)_{\Psi\phi}$. Consequently, $\phi \in \mathcal{F}_{Z_i}$. □

We conclude with some applications of Theorem 2.4. They also use results from the recent paper [9] by the authors.

Corollary 2.5. *Let X be an ideal in a Banach space Y . Then there exists $\phi \in HB(X, Y)$ such that, for every Banach space Z having the Radon-Nikodým property, there exists $\Phi \in HB(\overline{\mathcal{F}(Z, X)}, \overline{\mathcal{F}(Z, Y)})$ such that $Z = Z_{\Phi\phi}$.*

Proof. This is an immediate consequence of Theorem 2.4 because $\overline{\mathcal{F}(W, X)}$ is an ideal in $\overline{\mathcal{F}(W, Y)}$ for all Banach spaces W whenever X is an ideal in Y (see [9, Corollary 2.3]). □

Corollary 2.6. *Let X be a closed subspace of a Banach space Y . If $\mathcal{K}(W, X)$ is an ideal in $\mathcal{K}(W, Y)$ for all separable reflexive Banach spaces W , then there exists $\phi \in HB(X, Y)$ such that, for every Banach space Z having the Radon-Nikodým property, there exists $\Phi \in HB(\mathcal{K}(Z, X), \mathcal{K}(Z, Y))$ such that $Z = Z_{\Phi\phi}$.*

Proof. By [9, Theorem 4.2], if $\mathcal{K}(W, X)$ is an ideal in $\mathcal{K}(W, Y)$ for all separable reflexive W , then it is also an ideal for all Banach spaces W . And Theorem 2.4 applies. □

Corollary 2.7. *Let X be a closed subspace of a Banach space Y . If $\mathcal{K}(W, X)$ is an ideal in $\mathcal{W}(W, Y)$ for all reflexive Banach spaces W , then there exists $\phi \in HB(X, Y)$ such that, for every Banach space Z having the Radon-Nikodým property, there exists $\Phi \in HB(\mathcal{K}(Z, X), \mathcal{W}(Z, Y))$ such that $Z = Z_{\Phi\phi}$.*

Proof. In this case, by [9, Theorem 4.4], $\mathcal{K}(W, X)$ is an ideal in $\mathcal{W}(W, Y)$ for all Banach spaces W , and Theorem 2.4 applies. □

Corollary 2.8. *Let X be an ideal in a Banach space Y . If X^* has the compact approximation property with conjugate operators, then there exists $\phi \in HB(X, Y)$ such that, for every Banach space Z having the Radon-Nikodým property, there exists $\Phi \in HB(\mathcal{K}(Z, X), \mathcal{W}(Z, Y))$ such that $Z = Z_{\Phi\phi}$.*

Proof. In this case, again, Theorem 2.4 applies because $\mathcal{K}(W, X)$ is an ideal in $\mathcal{W}(W, Y)$ for all Banach spaces W (see [9, Corollary 4.7]). □

In fact (through [9, Corollary 4.7]), Corollary 2.8 is also an application of Theorem 4.6 of [9]. Let us compare these two results. Their assumptions concerning X and Y are the same. Corollary 2.8 applies to Z having the Radon-Nikodým property, whereas in Theorem 4.6 of [9], Z is supposed to be reflexive. On the

other hand, Theorem 4.6 of [9] claims more: for any $\phi \in HB(X, Y)$, there exists $\Phi \in HB(\mathcal{K}(Z, X), \mathcal{W}(Z, Y))$ such that $Z = Z_{\Phi\phi}$.

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