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ALGEBRAIC OBSTRUCTIONS AND A COMPLETE SOLUTION OF A RATIONAL RETRACTION PROBLEM

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Abstract. For each compact smooth manifold W containing at least two points we prove the existence of a compact nonsingular algebraic set Z and a smooth map $g : Z \longrightarrow W$ such that, for every rational diffeomorphism $r : Z' \longrightarrow Z$ and for every diffeomorphism $s : W' \longrightarrow W$ where Z' and W' are compact nonsingular algebraic sets, we may fix a neighborhood $\mathcal{U}$ of $s^{-1} \circ g \circ r$ in $C^\infty(Z', W')$ which does not contain any regular rational map. Furthermore $s^{-1} \circ g \circ r$ is not homotopic to any regular rational map. Bearing in mind the case in which W is a compact nonsingular algebraic set with totally algebraic homology, the previous result establishes a clear distinction between the property of a smooth map f to represent an algebraic unoriented bordism class and the property of f to be homotopic to a regular rational map. Furthermore we have: every compact Nash submanifold of $\mathbb{R}^n$ containing at least two points has not any tubular neighborhood with rational retraction.

1. Introduction

This paper deals with algebraic obstructions. First, it is necessary to recall part of the classical problem of making smooth objects algebraic.

Let M be a smooth manifold. A nonsingular real algebraic set V diffeomorphic to M is called an *algebraic model* of M. In [28] Tognoli proved that any compact smooth manifold has an algebraic model. By virtue of this result it is natural to wonder if a geometric object of M as a homology class $\alpha \in H_*(M, \mathbb{Z}/2\mathbb{Z})$ can be realized algebraically in some algebraic model of M. Such a question may be subdivided in two complementary problems: 1-*total obstruction problem*) find smooth compact manifolds M and homology classes $\alpha \in H_*(M, \mathbb{Z}/2\mathbb{Z})$ such that, in every algebraic model M' of M, α is not represented by any real algebraic subset of M'; 2-*constructive problem*) given a subgroup G of $H_k(M, \mathbb{Z}/2\mathbb{Z})$ for some integer k, find an algebraic model M' such that G corresponds to the subgroup $H_k^{alg}(M', \mathbb{Z}/2\mathbb{Z})$ of $H_k(M', \mathbb{Z}/2\mathbb{Z})$ generated by the homology classes represented by all k-dimensional real algebraic subsets of M'. The latter problem has been investigated by Bochnak and Kucharz in [11] and [12] (see also section 11.3 of [8]). With regard to the first problem Benedetti and Dedò [5] proved the following theorem which, in the final analysis, even today remains the unique result regarding algebraic obstructions up to homeomorphism.

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Benedetti-Dedò theorem. *For each $d \geq 11$ there exists a d-dimensional compact connected smooth manifold M and a homology class $\alpha \in H_{d-2}(M, \mathbb{Z}/2\mathbb{Z})$ such that, for every homeomorphism $h : M \longrightarrow M'$ where M' is an algebraic manifold, the homology class $h_*(\alpha)$ of M' is not contained in $H^{alg}_{d-2}(M', \mathbb{Z}/2\mathbb{Z})$.*

Later, Teichner [26] showed that such a result is true for each $d \geq 6$ which is the best possible (see Theorem 11.3.12 of [8]). The Benedetti-Dedò-Teichner theorem has the following corollary (see page 144 of [5]).

Corollary. *For each $d \geq 6$ there exist a d-dimensional compact connected smooth manifold M, a $(d-2)$-dimensional compact connected smooth manifold N and a smooth map $f : N \longrightarrow M$ such that, for every algebraic model N' of N and M' of M, the map f viewed as a smooth map from N' to M' is not approximable by regular rational maps.*

In this note, we make progress in the direction of the above obstruction corollary. Before stating our main theorem (Theorem 1) and one of its consequences (Theorem 2), we shall specify the meaning of some notions mentioned above and shall give other definitions.

Let V be a subset of $\mathbb{R}^n$, let W be a subset of $\mathbb{R}^m$ and let $f : V \longrightarrow W$ be a map. f is a *regular rational map* if, for each $x \in V$, there are a Zariski neighborhood U of x in $\mathbb{R}^n$, a polynomial $p : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and a polynomial $q : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $q^{-1}(0) \cap U = \emptyset$ and $f = p/q$ on $V \cap U$. A (*real*) *algebraic set* V is a real algebraic subset of some $\mathbb{R}^n$. V shall always be considered equipped with the Euclidean topology inherited from $\mathbb{R}^n$. Now let V and W be two nonsingular algebraic sets and let $f : V \longrightarrow W$ be a regular rational map. If f is a diffeomorphism when V and W are considered as smooth manifolds, then f is said to be a *rational diffeomorphism*. Let $M \subset \mathbb{R}^n$ and let $\overline{M}$ be the Zariski closure of M in $\mathbb{R}^n$. We say that M is a *compact strictly Nash submanifold of* $\mathbb{R}^n$ if $\overline{M}$ is compact and M is a union of certain nonsingular connected components of $\overline{M}$. Obviously a compact nonsingular algebraic subset of $\mathbb{R}^n$ is a compact strictly Nash submanifold of $\mathbb{R}^n$.

Let N and M be two smooth manifolds. We denote by $C^\infty(N, M)$ the set of all smooth maps from N to M equipped with the C^∞-topology.

Let $g : N \longrightarrow M$ and $h : N \longrightarrow M$ be two smooth maps. We say that g is *homotopic* to h if there exists a *smooth* map $H : N \times [0,1] \longrightarrow M$ such that $H(x,0) = g(x)$ and $H(x,1) = h(x)$ for each $x \in N$.

Theorem 1. *For each compact smooth manifold W (respectively compact real analytic or Nash submanifold W of $\mathbb{R}^m$) containing at least two points there exist a compact nonsingular algebraic set Z and a smooth map (resp. real analytic or Nash map) $g : Z \longrightarrow W$ such that, for every rational diffeomorphism $r : Z' \longrightarrow Z$ where Z' is a compact nonsingular algebraic set and for every diffeomorphism $s : W' \longrightarrow W$ where W' is a compact Nash submanifold of some $\mathbb{R}^k$, we may fix a neighborhood $\mathcal{U}$ of $s^{-1} \circ g \circ r$ in $C^\infty(Z', W')$ which does not contain any regular rational map. Furthermore if W' is a compact strictly Nash submanifold of some $\mathbb{R}^k$, then $s^{-1} \circ g \circ r$ is not homotopic to any regular rational map.*

Let V be a nonsingular algebraic set, let W be a subset of V and let U be an open neighborhood of W in V. A regular rational map $\varrho : U \longrightarrow W$ is a *rational retraction of U on W* if $\varrho(x) = x$ for each $x \in W$. We say that W has an *algebraic tubular neighborhood in V* if there is a rational retraction of some open neighborhood of W in V on W.

Piecing together the Weierstrass approximation theorem with Theorem 1 we obtain, at once, the following result.

Theorem 2. *Let W be a compact Nash submanifold of $\mathbb{R}^m$ containing at least two points. Then W does not have any algebraic tubular neighborhood in $\mathbb{R}^m$. In particular if U is a tubular neighborhood of W with submersive smooth retraction $\varrho : U \longrightarrow W$ (for example the closest point map), then ϱ is never a regular rational map.*

This result was expected. Let us explain the reasons.

Let W be a compact nonsingular algebraic set of $\mathbb{R}^m$ containing at least two points. Thanks to Theorem 1 there exists a compact nonsingular algebraic set Z such that the set $\mathcal{R}(Z, W)$ of all regular rational maps from Z to W is not dense in $C^\infty(Z, W)$, in other words a Weierstrass-type approximation theorem for smooth maps from Z to W is false. Notwithstanding the many existing generalizations of the Weierstrass approximation theorem (see [9], [10], [11], [13], [15], [16], [18], [20] and section 13.3 in [8]), in general, there are well-known obstructions to the possibility of approximating a smooth map from a compact nonsingular algebraic set to W by regular rational maps. In view of the classical Weierstrass approximation theorem, these algebraic obstructions prevent the existence of algebraic tubular neighborhoods of W in $\mathbb{R}^m$.

Let us recall the main above-mentioned obstructions. We suppose that there exists an algebraic tubular neighborhood of W in $\mathbb{R}^m$. Using the Steenrod representability theorem [27], Tognoli's theorem (see Teorema 3 of [28] or Theorem 14.1.10 of [8]) and the existence of an algebraic tubular neighborhood of W in $\mathbb{R}^m$, it is easy to see that W has *totally algebraic homology*, namely $H_k(W, \mathbb{Z}/2\mathbb{Z}) = H_k^{alg}(W, \mathbb{Z}/2\mathbb{Z})$ for every integer k. However in literature there exists a large quantity of examples of compact nonsingular algebraic subsets W of some $\mathbb{R}^m$ without totally algebraic homology and hence without algebraic tubular neighborhoods in $\mathbb{R}^m$ (see [4], [5], [7], [11], [14], [17], [19], [22], [23], [24], [26] and section 11.3 in [8]). When W is not connected, the obstruction to the existence of algebraic tubular neighborhoods of W in $\mathbb{R}^m$ is obtained in Théorème 2 of [29] making use of a property of the degree of certain regular rational maps (see Lemma 1.2 of [6], see also section 5 of [4]). Finally, Theorem 2 says that the unique compact Nash submanifold W of some $\mathbb{R}^m$, which has an algebraic tubular neighborhood in $\mathbb{R}^m$, is the single point.

In Section 4 we will present two applications of our results and in Section 5 we will briefly investigate the problem of when there exists a "local algebraic tubular neighborhood".

2. Variants of some remarkable results

Let P be a compact smooth manifold, let M be a compact nonsingular algebraic set and let $q : P \longrightarrow M$ be a smooth map. The unoriented bordism class β of q is *algebraic* if β is also represented by a regular rational map $q' : P' \longrightarrow M$ where P' is a compact nonsingular algebraic set.

We are interested in the following version of the Benedetti-Dedò-Teichner theorem.

Theorem 3. *For each $d \geq 6$ there exist a d-dimensional compact connected nonsingular algebraic set M, a $(d-2)$-dimensional compact connected nonsingular algebraic set N and a smooth map $f : N \longrightarrow M$ such that, for every diffeomorphism*

$a : M' \longrightarrow M$ where M' is a nonsingular algebraic set, the unoriented bordism class of $a^{-1} \circ f$ is not algebraic. In particular the following assertion is false: there exist diffeomorphisms $b : N' \longrightarrow N$ and $c : M' \longrightarrow M$ where N' and M' are nonsingular algebraic sets and a regular rational map $f' : N' \longrightarrow M'$ such that f' is arbitrarily close to $c^{-1} \circ f \circ b$ in $C^\infty(N', M')$.

Proof. It suffices to repeat the proof of Theorem 2 in [5] (page 150) also using Tognoli's theorem mentioned above. □

In the paper [2], Akbulut and King give a complete topological classification of all real algebraic sets with isolated singularities. One of the basic tools used to obtain this classification is a result which allows us to approximate a smooth map f between compact nonsingular algebraic sets by regular rational maps after slightly modifying the source of f. More precisely we are referring to Lemma 2.4 of [2] (see also Proposition 2 of [3]). We need such a lemma in the following form.

Lemma 4. *Let N be a compact Nash submanifold of $\mathbb{R}^n$, let M be a compact Nash submanifold of $\mathbb{R}^m$ and let $f : N \longrightarrow M$ be a smooth map. Then there are a compact Nash submanifold Z of $N \times \mathbb{R}^m$, an open subset Z_0 of Z and a regular rational map $P : Z \longrightarrow M$ such that: if $\pi : N \times \mathbb{R}^m \longrightarrow N$ is the natural projection, then $\pi|_{Z_0} : Z_0 \longrightarrow N$ is a Nash isomorphism and $P|_{Z_0}$ is arbitrarily close to $f \circ \pi|_{Z_0}$ in $C^\infty(Z_0, M)$. Moreover if both N and M are compact nonsingular algebraic sets, then Z is a compact nonsingular algebraic set also.*

Proof. We give a proof. By the Weierstrass approximation theorem, we may fix a polynomial map $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that $g|_N$ is arbitrarily close to f in $C^\infty(N, \mathbb{R}^m)$. Let U be a tubular neighborhood of M in $\mathbb{R}^m$ and let $\varrho : U \longrightarrow M$ be the closest point map. We may suppose g so close to f on N that $g(N) \subset U$ and the distance from each point of $g(N)$ to M is less than μ for some $\mu > 0$.

Let $\mathcal{M} := \{(y, v) \in M \times \mathbb{R}^m \,|\, v \text{ is orthogonal to } T_y(M)\}$ be the embedded normal bundle of M in $\mathbb{R}^m$, let $\gamma : N \times \mathbb{R}^m \longrightarrow \mathbb{R}^m \times \mathbb{R}^m$ be the regular rational map defined by $\gamma(x, v) := (g(x) + v, v)$ and let $Z := \gamma^{-1}(\mathcal{M})$. Let us briefly study the set Z. Obviously Z is a Nash subset of $N \times \mathbb{R}^m$. Z is also compact, in fact it is bounded in $N \times \mathbb{R}^m$: N and M are compact so $g(N)$ and M are contained in a ball of $\mathbb{R}^m$ centered in the origin with a certain radious R, in particular if $(x, v) \in Z$, then $|v| \leq |g(x) + v| + |g(x)| < 2R$ where $|v|$ is the usual norm of v in $\mathbb{R}^m$. Now we show that γ is transverse to $\mathcal{M}$ so Z will be a compact Nash submanifold of $N \times \mathbb{R}^m$ having dimension equal to $(\dim N + m) + m - 2m = \dim N$. Let $(x, v) \in Z$ and let $y := g(x) + v \in M$. By definition of $\mathcal{M}$ we have that $T_{(y,v)}(\mathcal{M}) = T_y(M) \times T_y^\perp(M) \subset \mathbb{R}^m \times \mathbb{R}^m$ where $T_y^\perp(M)$ is the vectorial subspace of $\mathbb{R}^m$ orthogonal to $T_y(M)$. It is easy to verify that $d\gamma_{(x,v)}(T_{(x,v)}(N \times \mathbb{R}^m))$ contains the diagonal Δ_m of $\mathbb{R}^m \times \mathbb{R}^m$ and $\Delta_m \cap \big(T_y(M) \times T_y^\perp(M)\big) = \{(0, 0)\}$ so $d\gamma_{(x,v)}(T_{(x,v)}(N \times \mathbb{R}^m)) + T_{(y,v)}(\mathcal{M}) = \mathbb{R}^m \times \mathbb{R}^m$ as desired. Now let $w : N \longrightarrow M$ be the smooth map defined by $\varrho \circ g$ and let $P : Z \longrightarrow M$ be the regular rational map defined by $P(x, v) := g(x) + v$. The open subset $Z_0 := \{(x, v) \in Z \,|\, |v| < \mu\}$ of Z is the graph of the smooth map $(w - g) : N \longrightarrow \mathbb{R}^m$, hence $\pi|_{Z_0} : Z_0 \longrightarrow N$ is a Nash isomorphism. We observe that $P \circ (\pi|_{Z_0})^{-1}(x) = P(x, w(x) - g(x)) = w(x)$ for each $x \in N$ so $P|_{Z_0} = w \circ \pi|_{Z_0}$. If g is close to f, then w is close to g and so w is close to f. In particular $P|_{Z_0} = w \circ \pi|_{Z_0}$ is close to $f \circ \pi|_{Z_0}$ in $C^\infty(Z_0, M)$ as we want. □

We shall also need a Nash version of Theorem 4 of [3].

Theorem 5. *Let N be a compact nonsingular algebraic set, let M be a compact strictly Nash submanifold of $\mathbb{R}^m$ and let $f : N \longrightarrow M$ be a smooth map which is homotopic to a regular rational map. Then there are a compact nonsingular algebraic set N', a regular rational map $R : N' \longrightarrow M$ and a rational diffeomorphism $\varphi : N' \longrightarrow N$ such that R is arbitrarily close to $f \circ \varphi$ in $C^\infty(N', M)$.*

Proof. Let $\overline{M}$ be the Zariski closure of M in $\mathbb{R}^m$. By definition, $\overline{M}$ is compact and M is a union of some nonsingular components of $\overline{M}$. Using Hironaka's resolution theorem, we may suppose $\overline{M}$ to be nonsingular and hence we may apply Theorem 4 of [3] to $f : N \longrightarrow \overline{M}$, completing the proof. □

3. Proof of the theorems

Proof of Theorem 1.

First step. Let W be a compact smooth manifold containing at least two distinct points w and y. Let $f : N \longrightarrow M$ be as in Theorem 3. Suppose $N \subset \mathbb{R}^n$ and $M \subset \mathbb{R}^m$. By Tognoli's theorem there exist a compact nonsingular algebraic set $\overline{W}$ and a diffeomorphism $\psi : \overline{W} \longrightarrow W$. Let $\pi_N : N \times \overline{W} \longrightarrow N$ and $\pi_{\overline{W}} : N \times \overline{W} \longrightarrow \overline{W}$ be the natural projections and let $F : N \times \overline{W} \longrightarrow M$ be the smooth map defined by $f \circ \pi_N$. Applying Lemma 4 to F we obtain a compact nonsingular algebraic subset Z of $(N \times \overline{W}) \times \mathbb{R}^m$, an open subset Z_0 of Z and a regular rational map $P : Z \longrightarrow M$ such that if $\pi : (N \times \overline{W}) \times \mathbb{R}^m \longrightarrow N \times \overline{W}$ is the natural projection, then $\pi|_{Z_0} : Z_0 \longrightarrow N \times \overline{W}$ is a Nash isomorphism and $P|_{Z_0}$ is arbitrarily close to $F \circ \pi|_{Z_0}$ in $C^\infty(Z_0, M)$. In particular Z is the disjoint union of Z_0 and a compact subset Q of $(N \times \overline{W}) \times \mathbb{R}^m$. Let us define the map $g : Z \longrightarrow W$ as follows: $g := \psi \circ \pi_{\overline{W}} \circ \pi|_{Z_0}$ on Z_0 and $g(z) := y$ if $z \in Q$. The map g is a smooth map and, since $\pi_{\overline{W}}$ is transverse to the point $\{\psi^{-1}(w)\}$, g is transverse to $\{w\}$ also.

Second step. Let $r : Z' \longrightarrow Z$ and $s : W' \longrightarrow W$ be as in the statement of the theorem. Let $w' := s^{-1}(w)$ and define $g' : Z' \longrightarrow W'$ by $s^{-1} \circ g \circ r$. By [21] we may choose $w \in W$ in such a way that w' is a nonsingular point of the Zariski closure of W'. The map g' is transverse to $\{w'\}$ so

$$N' := (g')^{-1}(w') = r^{-1}(\pi|_{Z_0})^{-1}(N \times \{w\})$$

is a smooth compact submanifold of Z'. Suppose g' is approximable by regular rational maps in $C^\infty(Z', W')$ and pick a regular rational map $g'' : Z' \longrightarrow W'$ close to g'. If g'' is sufficiently close to g', then g'' is again transverse to w', $N'' := (g'')^{-1}(w')$ is a compact nonsingular algebraic subset of Z' and there exists a diffeomorphism $H : Z' \longrightarrow Z'$ arbitrarily close to the identity such that $H(N'') = N'$ (see Theorem 20.2 of [1]). Let $h : N'' \longrightarrow N'$ be the diffeomorphism defined by $h := H|_{N''}$, let $b : N'' \longrightarrow N$ be the diffeomorphism defined by $b := \pi_N \circ \pi|_{Z_0} \circ r \circ h$ and let $f'' : N'' \longrightarrow M$ be the regular rational map $P \circ r|_{N''}$. Since P can be chosen arbitrarily close to $F \circ \pi$ on Z_0, it follows that f'' is as close to $F \circ \pi \circ r \circ h = f \circ \pi_N \circ \pi \circ r \circ h = f \circ b$ as we want. This is impossible thanks to the last part of Theorem 3. It follows that g' cannot be approximated by regular rational maps.

Third step. Let us prove that if W' is strictly Nash, then g' is not homotopic to any regular rational map. Suppose on the contrary that g' is homotopic to a regular rational map. By Theorem 5 applied to g' we have a compact nonsingular algebraic set Z'', a regular rational map $R : Z'' \longrightarrow W'$ and a rational diffeomorphism $\varphi : Z'' \longrightarrow Z'$ such that R is arbitrarily close to $g' \circ \varphi$ in $C^\infty(Z'', W')$. If R is sufficiently close to $g' \circ \varphi$, then R is transverse to w' (because $g' \circ \varphi$ is), $N''' := R^{-1}(w')$ is a compact nonsingular algebraic subset of Z'' and there is a diffeomorphism $H' : Z'' \longrightarrow Z''$ arbitrarily close to the identity such that $H'(N''') = \varphi^{-1}(N')$ (again use Theorem 20.2 of [1]). Let $h' : N''' \longrightarrow \varphi^{-1}(N')$ be the diffeomorphism defined by $h' := H'|_{N'''}$, let $b' : N''' \longrightarrow N$ be the diffeomorphism defined by $b' := \pi_N \circ \pi|_{Z_0} \circ r \circ \varphi \circ h'$ and let $f''' : N''' \longrightarrow M$ be the regular rational map defined as the composition $P \circ r \circ \varphi|_{N'''}$. We have that f''' is arbitrarily close to $F \circ \pi \circ r \circ \varphi \circ h' = f \circ b'$ and so the last part of Theorem 3 is again contradicted.

Final remark. We conclude the proof by making an observation about the first step. If W is a compact real analytic (Nash) submanifold of some $\mathbb{R}^m$, then g may be replaced by a real analytic (Nash) map sufficiently close to g (this is always possible because the set of all real analytic (Nash) maps from Z to W is dense in $C^\infty(Z, W)$). With few changes the rest of the proof still works. □

Remark 6. Theorem 1 cannot be improved. Let us explain in which sense. First, we state Proposition 2.8 of [2] in a particular case.

Proposition 2.8 ([2])**.** *Let N and M be two compact nonsingular algebraic sets and let $f : N \longrightarrow M$ be a smooth map. Suppose M has totally algebraic homology, namely $H_k^{alg}(M, \mathbb{Z}/2\mathbb{Z}) = H_k(M, \mathbb{Z}/2\mathbb{Z})$ for each integer k. Then there are a compact nonsingular algebraic set N', a diffeomorphism $\psi : N' \longrightarrow N$ and a regular rational map $P : N' \longrightarrow M$ such that P is arbitrarily close to $f \circ \psi$ in $C^\infty(N', M)$.*

Let W be a compact nonsingular algebraic set with totally algebraic homology containing at least two points and let $g : Z \longrightarrow W$ be as in Theorem 1. Applying the previous version of Proposition 2.8 of [2] to g we obtain a compact nonsingular algebraic set Z', a diffeomorphism $r : Z' \longrightarrow Z$ and a regular rational map $f' : Z' \longrightarrow W$ such that f' is arbitrarily close to $g \circ r$ in $C^\infty(Z', W)$. This fact tells us that in the statement of Theorem 1 we cannot replace "*for every rational diffeomorphism $r : Z' \longrightarrow Z$*" with "*for every diffeomorphism $r : Z' \longrightarrow Z$*". Furthermore, it is well-known that with such choice of W the unoriented bordism group of W is algebraic. In this way, Theorem 1 establishes a clear distinction between the property of a smooth map f to represent an algebraic unoriented bordism class and the property of the map f to be homotopic to a regular rational map.

This remark may also be seen from the point of view of Proposition 2.8 of [2] saying that in such a proposition the diffeomorphism $\psi : N' \longrightarrow N$ in general is not a regular rational map.

Proof of Theorem 2. Let W be a compact Nash submanifold of $\mathbb{R}^m$ containing at least two points and let $g : Z \longrightarrow W$ be as in the statement of Theorem 1. Consider g as a Nash map from Z to $\mathbb{R}^m$. Let us suppose that there exist an open neighborhood U of W in $\mathbb{R}^m$ and a rational retraction $\varrho : U \longrightarrow W$ on W. By the Weierstrass approximation theorem we may fix a polynomial map $P : Z \longrightarrow \mathbb{R}^m$ arbitrarily close to g in $C^\infty(Z, \mathbb{R}^m)$. If P is sufficiently close to g, then $P(Z) \subset U$ and the regular rational map $\varrho \circ P : Z \longrightarrow W$ is close to g in $C^\infty(Z, W)$ as we want. This contradicts Theorem 1. □

4. Some applications

In this section we will present two applications of our results. The first concerns an obstruction to improve the Artin-Mazur Theorem and the second regards an obstruction to embed birationally in $\mathbb{R}^e$ certain open subsets of an ample class of e-dimensional algebraic fiber bundles.

Let us recall the Artin-Mazur Theorem (see Theorem 8.4.4 of [8]).

Artin-Mazur Theorem. *Let Y be a d-dimensional connected Nash submanifold of $\mathbb{R}^p$ and let $f : Y \longrightarrow \mathbb{R}^m$ be a Nash map. There exists a d-dimensional nonsingular irreducible algebraic subset V of $\mathbb{R}^p \times \mathbb{R}^k$ for some integer k, an open semialgebraic subset Y' of V and a polynomial map $P : V \longrightarrow \mathbb{R}^m$ such that if $\pi : \mathbb{R}^p \times \mathbb{R}^k \longrightarrow \mathbb{R}^p$ is the natural projection, then $\pi|_{Y'} : Y' \longrightarrow Y$ is a Nash isomorphism and $P = f \circ \pi$ on Y'.*

A question arises: Can we take $Y' = V$?

The next proposition gives a negative answer. In particular for each m there are Nash maps f from compact connected Nash submanifolds Y to $\mathbb{R}^m$ such that if Y' and V satisfy the Artin-Mazur Theorem applied to f, then Y' is different from V.

Proposition 7. *For each non-void compact connected Nash submanifold W of $\mathbb{R}^m$ there are a compact connected Nash submanifold Y of some $\mathbb{R}^p$ and a Nash map $\eta : Y \longrightarrow W$ such that the following assertion is not true: there exist a compact nonsingular algebraic set Y', a rational diffeomorphism $\varphi : Y' \longrightarrow Y$ and a regular rational map $R : Y' \longrightarrow W$ such that $R = \eta \circ \varphi$. In particular if V, Y' and P satisfy the conclusion of the Artin-Mazur Theorem applied to η, then $Y' \neq V$.*

Proof. The proof is similar to the one of Theorem 1. Let $f : N \longrightarrow M$ be as in Theorem 3. Repeat the first step of the proof of Theorem 1 with $\overline{W} := W$. We obtain a compact Nash submanifold Z_0, a Nash isomorphism $\pi : Z_0 \longrightarrow N \times W$ and a regular rational map $P : Z_0 \longrightarrow M$ such that P is arbitrarily close to $f \circ \pi_N \circ \pi$ in $C^\infty(Z_0, M)$. Define $Y := Z_0$ and $\eta : Y \longrightarrow W$ by $\pi_W \circ \pi$. Since N and W are connected, it follows that Y is also connected. Suppose that there exist a compact nonsingular algebraic set Y', a rational diffeomorphism $\varphi : Y' \longrightarrow Y$ and a regular rational map $R : Y' \longrightarrow W$ such that $R = \eta \circ \varphi$. Fix $w \in W$ as in the second step of the proof of Theorem 1 with s equal to the identity on W. Observe that η is transverse to $\{w\}$ so R is transverse to $\{w\}$ also. Let $N' := R^{-1}(w)$. N' is a compact nonsingular algebraic subset of Y'. Let $b : N' \longrightarrow N$ be the diffeomorphism $\pi_N \circ \pi \circ \varphi|_{N'}$ and let $f' : N' \longrightarrow M$ be the regular rational map $P \circ \varphi|_{N'}$. Since P may be chosen arbitrarily close to $f \circ \pi_N \circ \pi$ in $C^\infty(Z_0, M)$, we have that f' is close to $f \circ b$ in $C^\infty(N', M)$ as we want. This contradicts Theorem 3. □

If W consists of a single point, then the above proof gives the following result: *for each $d \geq 4$, there is a d-dimensional compact connected Nash submanifold Y of some $\mathbb{R}^p$ such that if Y' is an algebraic model of Y and $\psi : Y' \longrightarrow Y$ is a diffeomorphism, then ψ is never a regular rational map.*

Let us proceed with another application.

Let E be an e-dimensional abstract algebraic manifold and let U be an open subset of E. We say that U can be *birationally embedded in* $\mathbb{R}^e$ if there exist an open subset U' of $\mathbb{R}^e$ and a *regular birational isomorphism* from U to U', namely a regular rational map $\varphi : U \longrightarrow U'$ having a regular rational inverse $\varphi^{-1} : U' \longrightarrow U$.

Let W be a nonsingular algebraic set. A topological fiber bundle $\pi : E \longrightarrow W$ over W is *algebraic* if E is an algebraic abstract manifold and π is a submersive regular rational map.

Proposition 8. *Let W be a compact nonsingular algebraic subset of $\mathbb{R}^m$ containing at least two points, let $\pi : E \longrightarrow W$ be an algebraic fiber bundle over W, let $\sigma : W \longrightarrow E$ be a regular rational section of π and let U be an open neighborhood of $\sigma(W)$ in E. If $\dim(E) = e$, then U cannot be birationally embedded in $\mathbb{R}^e$.*

In particular the following is true.

Let $E := \{(x, v) \in W \times \mathbb{R}^m \mid v$ is orthogonal to $T_x(W)\}$, let $\pi : E \longrightarrow W$ be the normal bundle of W in $\mathbb{R}^m$ and let $\theta : E \longrightarrow \mathbb{R}^m$ be the polynomial map defined by $\theta(x, v) := x + v$. By the Tubular Neighborhood Theorem there are an open neighborhood U_0 of the zero section of π in E and an open neighborhood U of W in $\mathbb{R}^m$ such that $\theta|_{U_0} : U_0 \longrightarrow U$ is a rational diffeomorphism. Then $(\theta|_{U_0})^{-1}$ is never a regular rational map.

Proof. If there would exist a birational map $\varphi : U \longrightarrow U'$ where U' is an open subset of $\mathbb{R}^e$, then the map $\varrho : U' \longrightarrow \varphi \circ \sigma(W)$ defined by $\varphi \circ \sigma \circ \pi \circ \varphi^{-1}$ would be a rational retraction from U' on the compact nonsingular algebraic subset $\varphi \circ \sigma(W)$ of $\mathbb{R}^e$. Now $\varphi \circ \sigma(W)$ contains at least two points, so Theorem 2 is contradicted. □

For other algebraic obstructions of the same kind we refer to [6] and [7].

5. Final remarks about rational retractions

In Theorem 2 we proved that the problem regarding the existence of algebraic tubular neighborhoods of a compact nonsingular algebraic subset W of $\mathbb{R}^m$ is solvable only when W is a single point. In this section we will study briefly the existence of local rational retractions on W.

First, let us give a definition. Let V be a nonsingular algebraic set, let W be a subset of V and let p be a point of W. We say that W has a *rational retraction locally at p in V* if there exist an open neighborhood U of p in V and a rational retraction of U on $U \cap W$. The following is a simple result.

Proposition 9. *Let R be a polynomial of $\mathbb{R}[x_1, \dots, x_n]$ of degree less than or equal to 2, let $V := R^{-1}(0)$ and let $p \in V$ such that the differential dR_p of R at p is not null. Then V has a rational retraction locally at p in $\mathbb{R}^n$.*

Furthermore the following is true. Let q be a point of V. Suppose the degree of R to be exactly 2 and write $R(x) = R_1(x - q) + R_2(x - q)$ where each R_k is a homogeneous polynomial of $\mathbb{R}[x_1, \dots, x_n]$ of degree k. Let C_q be the cone of $\mathbb{R}^n$ with vertex q defined by $C_q := \{x \in \mathbb{R}^n \mid R_1(x - q) \cdot R_2(x - q) = 0\}$. If V is not contained in C_q, then the map $\varrho_p : \mathbb{R}^n \setminus C_q \longrightarrow V \setminus C_q$ defined by

$$\varrho_p(x) := q - \frac{R_1(x - q)}{R_2(x - q)} \cdot (x - q)$$

is a rational retraction of $\mathbb{R}^n \setminus C_q$ on $V \setminus C_q$.

Proof. Using an affinity if needed we may suppose that p is equal to the origin 0 of $\mathbb{R}^n$ and the Zariski tangent space of V at 0 is $\{x \in \mathbb{R}^n \mid x_n = 0\}$ which we identify with $\mathbb{R}^{n-1}$. If V coincides with $\mathbb{R}^{n-1}$ locally at 0, then the natural projection $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$, suitably restricted locally at 0, is a desired rational retraction. Now suppose V to be different from $\mathbb{R}^{n-1}$ locally at 0. By this hypothesis we may pick a point $q \in V \setminus \{0\}$ arbitrarily close to 0 such that the straight line r containing q and 0 does not lie completely in V (otherwise $V = \mathbb{R}^{n-1}$ locally at 0). We write $R(x) = R_1(x - q) + R_2(x - q)$ and define C_q and ϱ_q as in the second part of the statement. The map ϱ_q is a rational retraction of $\mathbb{R}^n \setminus C_q$ on $V \setminus C_q$. In fact, $R(\varrho_q(x)) = 0$ and $\varrho_q(x) \notin C_q$ for each $x \in \mathbb{R}^n \setminus C_q$; moreover $\varrho_q(x) = x$ for each $x \in V \setminus C_q$. It remains to be proved that $0 \notin C_q$ so ϱ_q will turn out to be a desired rational retraction. The straight line r is the image of $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^n$ defined by $\gamma(t) := q - tq$. By assumption $r \not\subset V$ so the polynomial equation $0 = R(\gamma(t)) = t^2 R_2(-q) + tR_1(-q)$ must have a finite number of solutions in t which are $t = 0$ and $t = 1$. It follows that $R_2(-q) = -R_1(-q) \neq 0$ and so $0 \notin C_q$. □

In the previous proposition if V is a $(n-1)$-sphere S of $\mathbb{R}^n$ and R is the standard quadratic polynomial equation of S in $\mathbb{R}^n$, then it is easy to verify that, for each $q \in S$, the cone C_q is exactly the tangent space of S at q. In particular $S \cap C_q = \{q\}$ and so $S \setminus \{q\}$ has an algebraic tubular neighborhood in $\mathbb{R}^n$.

Let us make another observation about Proposition 9. Let V be the vanishing set of a polynomial $R \in \mathbb{R}[x_1, \ldots, x_n]$ of degree $d \geq 2$ and let $q \in V$ such that R can be written as follows: $R(x) = R_{d-1}(x - q) + R_d(x - q)$ where each R_k is a homogeneous polynomial of $\mathbb{R}[x_1, \ldots, x_n]$ of degree k. Under these hypotheses the second part of Proposition 9 remains true with C_q and ϱ_q defined as above replacing R_1 with R_{d-1} and R_2 with R_d.

Let us conclude with an obstruction result.

A 1-dimensional nonsingular real algebraic subset of $\mathbb{R}^n$ (respectively $\mathbb{RP}^n$) is called an *affine* (*projective*) *real algebraic curve of* $\mathbb{R}^n$ (resp. $\mathbb{RP}^n$). Let C be an affine (projective) real irreducible algebraic curve of $\mathbb{R}^n$ (resp. $\mathbb{RP}^n$). By Hironaka's resolution of singularities theorem there exists a projective complex irreducible curve $C_\mathbb{C}$ defined over $\mathbb{R}$ such that C is regular birational isomorphic to a Zariski open subset of the set of all real points $C_\mathbb{C}(\mathbb{R})$ of $C_\mathbb{C}$ where $C_\mathbb{C}(\mathbb{R})$ is viewed as a projective real algebraic curve. Since $C_\mathbb{C}$ is unique up to birational isomorphism over $\mathbb{R}$, we may define the *genus* $g(C)$ *of* C as the genus of $C_\mathbb{C}$. Moreover we recall that every regular rational map $r : C \longrightarrow D$ between affine (projective) real irreducible algebraic curves of $\mathbb{R}^n$ (resp. $\mathbb{RP}^n$) admits a unique complex regular rational extension $r_\mathbb{C}$ from $C_\mathbb{C}$ to $D_\mathbb{C}$.

Proposition 10. *Let C be an affine* (*projective*) *real irreducible algebraic curve of $\mathbb{R}^n$* (*resp. $\mathbb{RP}^n$*) *such that $g(C) > 0$. Fix a point p of C. Then C does not have any rational retraction locally at p in $\mathbb{R}^n$* (*resp. $\mathbb{RP}^n$*).

Proof. Since the problem is local, it suffices to consider the affine case. Assume that there is a rational retraction ϱ from an open neighborhood U of p on C. Observe that, by hypotheses, it follows that $n \geq 2$. After an affinity of $\mathbb{R}^n$, we may suppose that p is equal to the origin 0 of $\mathbb{R}^n$ and the Zariski tangent space of C at 0 is $\mathbb{R}_{x_1} := \{x \in \mathbb{R}^n \,|\, x_2 = \ldots = x_n = 0\}$. Let $\mathbb{R}_{x_1x_2} := \{x \in \mathbb{R}^n \,|\, x_3 = \ldots = x_n = 0\}$ and let $\pi : \mathbb{R}_{x_1x_2} \longrightarrow \mathbb{R}_{x_1}$ be the natural projection. Since $\varrho(x) = x$ for each $x \in U \cap C$ and $\mathbb{R}_{x_1}$ is the Zariski tangent space of C at 0, it is easy to establish that $\varrho|_{U \cap \mathbb{R}_{x_1}} : U \cap \mathbb{R}_{x_1} \longrightarrow C$ is a local diffeomorphism at 0. Let $\varepsilon > 0$ such that the image under π of the circle $S^1_\varepsilon := \{(x_1, x_2) \in \mathbb{R}_{x_1x_2} \,|\, x_1^2 + x_2^2 = \varepsilon^2\}$ is contained in $U \cap \mathbb{R}_{x_1}$. The regular rational map $r : S^1_\varepsilon \longrightarrow C$ defined by $r := \varrho \circ \pi|_{S^1_\varepsilon}$ turns out to be well-defined and nonconstant. Let $r_\mathbb{C} : (S^1_\varepsilon)_\mathbb{C} \longrightarrow C_\mathbb{C}$ be the complex extension of r. Since $r_\mathbb{C}$ is not constant, by the Riemann-Hurwitz formula we have that $g(C) \leq g(S^1_\varepsilon) = 0$, which contradicts our hypotheses. □

We refer to [15] and [16] for other results derived from techniques similar to the one used in the previous proof. For the problem of the existence of "local Nash tubular neighborhoods" in the complex case we refer to [25].

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