

GLOBAL EXISTENCE AND NONEXISTENCE FOR DEGENERATE PARABOLIC SYSTEMS

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ABSTRACT. The initial-boundary value problems are considered for the strongly coupled degenerate parabolic system

$$\begin{aligned}u_t &= v^p(\Delta u + au), \\v_t &= u^q(\Delta v + bv)\end{aligned}$$

in the cylinder $\Omega \times (0, \infty)$, where $\Omega \subset R^N$ is bounded and p, q, a, b are positive constants. We are concerned with the global existence and nonexistence of the positive solutions. Denote by λ_1 the first Dirichlet eigenvalue for the Laplacian on Ω . We prove that there exists a global solution iff $\lambda_1 \geq \min\{a, b\}$.

1. INTRODUCTION

This paper investigates the global existence and nonexistence of the positive solutions of the strongly coupled degenerate parabolic system

$$(1.1) \quad \begin{aligned}u_t &= v^p(\Delta u + au), \\v_t &= u^q(\Delta v + bv), \quad \text{in } \Omega_T \equiv \Omega \times (0, T)\end{aligned}$$

with the first initial-boundary condition

$$(1.2) \quad \begin{aligned}u|_{\partial\Omega} &= 0, \quad v|_{\partial\Omega} = 0, \\u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad \text{in } \Omega,\end{aligned}$$

where $\Omega \subset R^N$ is bounded and p, q, a, b are positive constants.

In the past twenty years, the degenerate parabolic equation

$$(1.3) \quad \begin{aligned}u_t &= u^p(\Delta u + u), \quad \text{in } \Omega_T, \\u|_{\partial\Omega} &= 0, \\u(x, 0) &= u_0(x), \quad \text{in } \Omega\end{aligned}$$

has been studied intensively by many authors. Friedman and McLeod [7] investigated the blow-up property of the positive solutions of (1.3) with $p = 2$. They showed that:

- (1) If $\lambda_1 > 1$, there exist global nontrivial solutions;
- (2) If $\lambda_1 < 1$, no solution is global (if $u_0 \not\equiv 0$);
- (3) In case (2), the blowup set S has positive measure.

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Recently, (1) and (2) have been extended to the problem (1.3) with $p > 1$ (see [20]). Moreover, it is shown that

(4) $\lambda_1 = 1$ belongs to the global existence case.

If $p < 1$, (1.3) corresponds to the porous media equation. Galaktionov [9] pointed out that this problem has a critical property. Later Pucci and Serrin [17] proved that (1) and (2) hold for its positive solutions. Furthermore, Sacks [18] showed that (4) also holds. Passo and Luckhaus [15] studied (1.3) with $p = 1$. They showed that there is no uniqueness of its weak solutions. All these results reflect the statement, emphasized in [12], that large domains ($\lambda_1 < 1$) are more unstable than small domains ($\lambda_1 \geq 1$).

Early in the 1980s, Galaktionov et al. [10], [11] studied the porous media system

$$(1.4) \quad \begin{aligned} u_t &= \Delta u^{1+\nu} + v^\alpha, \\ v_t &= \Delta v^{1+\mu} + u^\beta, \quad \text{in } \Omega_T \end{aligned}$$

with the condition (1.2). They proved that $\alpha\beta = (1+\nu)(1+\mu)$ is a critical exponent (see also [12]). But this situation is not as clear as it is for (1.3). However, if $\alpha = 1 + \mu$ and $\beta = 1 + \nu$, (1) and (2) hold. Duan and Zhou [5] studied (1.1), (1.2) of one dimension. It is proved that if $\lambda_1 > \max\{a, b\}$, then there exists a global positive solution, and if $\lambda_1 < \min\{a, b\}$, then no positive solution is global and in this case (3) holds.

In this paper we will also go along the thread to investigate the critical exponents of (1.1), (1.2). It might be expected that the critical exponents are combinations of a and b . We are particularly concerned with the dependence of the critical exponents upon these parameters.

Throughout this paper, $\partial\Omega \in C^{2+\alpha}$ is assumed to satisfy the outside strong sphere property (see [8, p. 69]) and

$$(H1) \quad \begin{aligned} u_0(x), v_0(x) &\in C^1(\bar{\Omega}), \quad u_0(x), v_0(x) > 0, \quad \text{in } \Omega, \\ u_0(x) = v_0(x) &= 0, \quad \partial u_0/\partial n, \partial v_0/\partial n < 0, \quad \text{on } \partial\Omega. \end{aligned}$$

Definition 1.1. A positive solution of (1.1), (1.2) is a vector function $(u, v) \in C(\bar{\Omega}_T) \cap C^{2,1}(\Omega_T)$, positive in Ω_T and satisfying (1.1), (1.2).

A crucial role is played here by the first eigenvalue of the Dirichlet problem

$$(1.5) \quad \begin{aligned} -\Delta\phi(x) &= \lambda\phi(x), \quad \text{in } \Omega, \\ \phi(x) &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

Denote by λ_1 the first eigenvalue and by $\phi(x)$ the corresponding eigenfunction with the normalization $\phi(x) > 0$ in Ω and $\max_{\bar{\Omega}} \phi(x) = 1$. A simple argument with the Rayleigh quotient shows that λ_1 increases as the size of the domain decreases (see [2, Chap. 6]).

Our main result is:

Theorem 1.2. *Assume that (H1) holds. Then there exists a global positive solution of (1.1), (1.2) iff $\lambda_1 \geq \min\{a, b\}$.*

To our knowledge, the local existence theory of such a strongly coupled degenerate parabolic system is not established. Since the comparison principle (see [16]) no longer holds generally for the strongly coupled system, we have no monotonicity as to n of the positive solutions (u_n, v_n) of the regularized systems (2.2) in Section 2, which is the main difficulty. For the proof of the local existence with monotonicity,

see [1], [7] and [20]. The outline of proving the local existence is the following. By a modified comparison principle, the positive solutions of the regularized problems can be shown to have uniform upper and lower bounds. Such uniform bounds enable us to have an interior uniform C^α -estimate (see [13, Theorem 3.1, p. 582]), which can be used to obtain an interior uniform $C^{2+\alpha}$ -estimate (see [8, Theorem 5, p. 64]). Now, appealing to the Ascoli-Arzelà theorem, we have a vector function $(u, v) \in C^{2,1}(\overline{Q'})$, $Q' \subset\subset \Omega_T$, which satisfies (1.1) in $\overline{Q'}$. Next, we extend such obtained (u, v) from Q' to Ω_T step by step and $(u, v) \in C^{2,1}(\Omega_T)$ satisfies (1.1) in Ω_T . A diagonal procedure implies that there exists a subsequence, also denoted by $\{n\}$, such that $\lim_{n \rightarrow \infty} (u_n, v_n) = (u, v)$ in $C^{2,1}(\Omega_T)$. Independently we prove that (u, v) also satisfies the initial-boundary condition (1.2). Thus, (u, v) is a positive solution of (1.1), (1.2).

This paper is organized as follows. Section 2 establishes the local existence theory, Section 3 investigates the global nonexistence and in Section 4 we prove the global existence.

2. LOCAL EXISTENCE

In proving the local existence for degenerate parabolic equations, there are three different approaches (see [1, p. 113]). We will modify the boundary conditions rather than the differential equations. Introduce, for $n = 1, 2, \dots$, the following regularized system:

$$\begin{aligned}
 (2.1) \quad & u_{nt} = f_n(v_n)(\Delta u_n + au_n), \\
 & v_{nt} = g_n(u_n)(\Delta v_n + bv_n), \quad \text{in } \Omega_T, \\
 & u_n|_{\partial\Omega} = 1/n, \quad v_n|_{\partial\Omega} = 1/n, \\
 & u_n(x, 0) = u_0(x) + 1/n, \quad v_n(x, 0) = v_0(x) + 1/n, \quad \text{in } \Omega,
 \end{aligned}$$

where

$$f_n(v_n) = \begin{cases} v_n^p, & v_n \geq 1/n, \\ (1/n)^p, & v_n < 1/n, \end{cases} \quad g_n(u_n) = \begin{cases} u_n^q, & u_n \geq 1/n, \\ (1/n)^q, & u_n < 1/n. \end{cases}$$

By the classical parabolic theory (see [13] or [21]), under (H1), (2.1) has a unique positive solution $(u_n, v_n) \in C(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ for $0 < T < T_n^*$, where T_n^* is the maximal existence time. By a direct computation and the classical maximal principle (see [16, Theorem 13, p. 190]), $u_n, v_n \geq 1/n$. Hence, (u_n, v_n) satisfies

$$\begin{aligned}
 (2.2) \quad & u_{nt} = v_n^p(\Delta u_n + au_n), \\
 & v_{nt} = u_n^q(\Delta v_n + bv_n), \quad \text{in } \Omega_T
 \end{aligned}$$

with the corresponding initial and boundary condition.

We give two comparison principles frequently used in the following.

Lemma 2.1. *Assume that $a_i(x, t), b_i(x, t), c_i(x, t)$ and $d_i(x, t)$ are bounded continuous functions and $a_i(x, t), b_i(x, t) \geq 0, i = 1, 2$. If a bounded vector function $(w, s) \in C(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ satisfies*

$$\begin{aligned}
 (2.3) \quad & w_t \geq a_1\Delta w + b_1s + c_1w + d_1ws, \\
 & s_t \geq a_2\Delta s + b_2w + c_2s + d_2ws, \quad \text{in } \Omega_T, \\
 & w|_{\partial\Omega} \geq 0, \quad s|_{\partial\Omega} \geq 0, \\
 & w(x, 0) \geq 0, \quad s(x, 0) \geq 0, \quad \text{in } \Omega.
 \end{aligned}$$

Then $w, s \geq 0$ in Ω_T .

Proof. Since (w, s) is bounded, a careful examination of the proof of the classical maximum principle (see [16, Theorem 13, p. 190]) shows that this lemma is true in spite of the degeneracy of (2.3). \square

Proposition 2.2. *Assume that (\bar{u}, \bar{v}) and $(\underline{u}, \underline{v}) \in C(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ satisfy (1.1) and*

$$\begin{aligned} \bar{u}|_{\partial\Omega} &\geq \underline{u}|_{\partial\Omega}, \quad \bar{v}|_{\partial\Omega} \geq \underline{v}|_{\partial\Omega}, \\ \bar{u}(x, 0) &\geq \underline{u}(x, 0), \quad \bar{v}(x, 0) \geq \underline{v}(x, 0), \quad \text{in } \Omega. \end{aligned}$$

If either of them is nondecreasing in t , then

$$(\bar{u}, \bar{v}) \geq (\underline{u}, \underline{v}), \quad \text{in } \Omega_T.$$

Proof. This proposition is a direct consequence of Lemma 2.1. \square

Now we construct a uniform upper bound for (u_n, v_n) . Consider the system

$$\begin{aligned} \tilde{u}_t &= \tilde{v}^p(\Delta\tilde{u} + a\tilde{u}), \\ \tilde{v}_t &= \tilde{u}^q(\Delta\tilde{v} + b\tilde{v}), \quad \text{in } \Omega_T, \\ \tilde{u}|_{\partial\Omega} &= f(t), \quad \tilde{v}|_{\partial\Omega} = g(t), \\ \tilde{u}(x, 0) &= \theta, \quad \tilde{v}(x, 0) = \theta, \quad \text{in } \Omega. \end{aligned} \tag{2.4}$$

We choose $\theta = \max_{\overline{\Omega}} u_0 + \max_{\overline{\Omega}} v_0 + 1$ and assume

$$\begin{aligned} f(t), g(t) &\in C^\infty([0, \infty)), \quad f'(t), g'(t) > 0, \\ f(0) &= g(0) = \theta, \\ f'(0) &= a\theta^{p+1}, g'(0) = b\theta^{q+1}. \end{aligned} \tag{H2}$$

By the classical parabolic theory, under (H2), (2.4) has a unique classical solution $(\tilde{u}, \tilde{v}) \in C^{2+\beta, 1+\beta/2}(\overline{\Omega_T})$, where $0 < \beta < 1$ and $0 < T < \tilde{T}^*$, \tilde{T}^* is its maximal existence time. We also have $\tilde{u}, \tilde{v} \geq \theta$. From [8, Theorem 12, p. 75], $(\tilde{u}_{tt}, \tilde{v}_{tt})$ exists in Ω_T . Differentiating (2.4) with respect to t and combining with (H2), we obtain a system like (2.3). Lemma 2.1 implies that $\tilde{u}_t \geq 0, \tilde{v}_t \geq 0$ in Ω_T . Hence by Proposition 2.2 and the choice of θ , we have

$$(u_n, v_n) \leq (\tilde{u}, \tilde{v}), \quad \text{in } \Omega_T \tag{2.5}$$

for all $n \geq 1$ and $0 < T < \tilde{T}^*$. Clearly $T_n^* \geq \tilde{T}^*$ for all n .

Next we give a uniform lower bound. As in [5], let $\hat{u} = \hat{v} = k\phi(x)e^{-\rho t}$, where $\phi(x)$ is the first eigenfunction of (1.5), $k > 0$ is small such that $u_0, v_0 \geq k\phi(x)$. Fix $T \in (0, \tilde{T}^*)$. Choose $\rho \geq \lambda_1 \sup_{\overline{\Omega_T}}(\tilde{u}^q + \tilde{v}^p)$. A direct computation yields that $w = u_n - \hat{u}, s = v_n - \hat{v}$ satisfy

$$\begin{aligned} w_t &= v_n^p(\Delta w + aw) + k\phi e^{-\rho t}(\rho - (\lambda_1 - a)v_n^p) \\ &\geq v_n^p(\Delta w + aw), \\ s_t &\geq u_n^q(\Delta s + bs), \quad \text{in } \Omega_T. \end{aligned}$$

Lemma 2.1 implies that

$$u_n, v_n \geq k\phi(x)e^{-\rho t}, \quad \text{in } \Omega_T. \tag{2.6}$$

Combining (2.5) and (2.6), we have for fixed $T \in (0, \tilde{T}^*)$,

$$(k\phi e^{-\rho t}, k\phi e^{-\rho t}) \leq (u_n, v_n) \leq (\tilde{u}, \tilde{v}), \quad \text{in } \Omega_T.$$

In the following, we employ the so-called extension method outlined in Introduction to prove that there exists a vector function $(u, v) \in C^{2,1}(\Omega_T)$, $0 < T < \tilde{T}^*$, satisfying (1.1) in Ω_T . In proving this we need the following regularity. Denote by c_i different constants depending on T and M , but not on n , where $M = \max_{\Omega_T}(\tilde{u} + \tilde{v})$. Denote by α_i different constants in $(0, 1)$.

Lemma 2.3. $u_n, v_n \in V_2^{1,0}(\Omega_T)$ (see [13, p. 6]).

Proof. We only give the details for u_n . Multiplying the first differential equation of (2.2) by u_{nt}/v_n^p and integrating over Ω_t , $0 < t < T$, we get

$$\begin{aligned} \int_{\Omega_t} \frac{u_{nt}^2}{v_n^p} dx d\tau &= \int_{\Omega_t} u_{nt}(\Delta u_n + a u_n) dx d\tau \\ &\leq \frac{1}{2} \int_{\Omega} (|\nabla u_0|^2 - |\nabla u_n|^2) dx + \frac{a}{2} \int_{\Omega} u_n^2 dx. \end{aligned}$$

Hence,

$$\int_{\Omega} |\nabla u_n|^2 dx \leq c_1.$$

Integrating over $[0, T]$ again, we have

$$\int_{\Omega_T} |\nabla u_n|^2 dx d\tau \leq c_2.$$

By the definition of $V_2^{1,0}(\Omega_T)$, we have

$$\|u_n\|_{V_2^{1,0}(\Omega_T)} = \max_{0 \leq t \leq T} \|u_n(\cdot, t)\|_{2,\Omega} + \|\nabla u_n\|_{2,\Omega_T} \leq c_3.$$

This lemma is proved. □

Let

$$\begin{aligned} \Omega_l &= \{x \in \Omega : \text{dist}(x, \partial\Omega) > 1/l\}, \\ Q_l &= \Omega_l \times (1/l, T - 1/l). \end{aligned}$$

Without loss of generality, we choose $l = 1, 2, \dots$. Hence, we have

$$\Omega_l \subset\subset \Omega_{l+1}, \Omega_\infty = \Omega; \quad Q_l \subset\subset Q_{l+1}, Q_\infty = \Omega_T.$$

Due to (2.6), we have

$$u_n, v_n \geq k\phi(x)e^{-\rho t} \geq \sigma, \quad \text{in } Q_l.$$

Clearly, σ depends on T and l , but not on n . From Lemma 2.3 and applying the interior C^α -estimate on the compact subset \overline{Q}_{l+1} of Q_{l+2} (see [13, Theorem 3.1, p. 582]) we obtain

$$|u_n|_{\overline{Q}_{l+1}}^{(\alpha_1)}, |v_n|_{\overline{Q}_{l+1}}^{(\alpha_1)} \leq c_4.$$

Also applying the interior $C^{2+\alpha}$ -estimate on the compact subset \overline{Q}_l of Q_{l+1} (see [8, Theorem 5, p. 64]), we have

$$|u_n|_{\overline{Q}_l}^{(2+\alpha_2)}, |v_n|_{\overline{Q}_l}^{(2+\alpha_2)} \leq c_5.$$

In the above procedure let $l = 1$, then we have

$$|u_n|_{\overline{Q_1}}^{(2+\alpha_3)}, |v_n|_{\overline{Q_1}}^{(2+\alpha_3)} \leq c_6.$$

Then the Ascoli-Arzelà theorem implies that there exist a subsequence $\{n_{1,k}\}$ and a vector function $(u^1, v^1) \in C^{2,1}(\overline{Q_1})$, such that

$$(2.7) \quad \lim_{k \rightarrow \infty} (u_{n_{1,k}}, v_{n_{1,k}}) = (u^1, v^1), \quad \text{in } C^{2,1}(\overline{Q_1})$$

and (u^1, v^1) satisfies (1.1) in $\overline{Q_1}$.

Similarly for the case $l = 2$, there also exist a subsequence $\{n_{2,k}\} \subset \{n_{1,k}\}$ and a vector function $(u^2, v^2) \in C^{2,1}(\overline{Q_2})$, such that

$$\lim_{k \rightarrow \infty} (u_{n_{2,k}}, v_{n_{2,k}}) = (u^2, v^2), \quad \text{in } C^{2,1}(\overline{Q_2})$$

and (u^2, v^2) satisfies (1.1) in $\overline{Q_2}$.

Since $\{n_{2,k}\} \subset \{n_{1,k}\}$, the uniqueness of the limit (2.7) in $\overline{Q_1}$ implies

$$(u^1, v^1) = (u^2, v^2)|_{\overline{Q_1}}, \quad \text{in } C^{2,1}(\overline{Q_1}).$$

Thus, we extend the vector function $(u^1, v^1) \in C^{2,1}(\overline{Q_1})$ to a vector function $(u^2, v^2) \in C^{2,1}(\overline{Q_2})$ and (u^2, v^2) satisfies (1.1) in $\overline{Q_2}$. For convenience, we still denote (u^i, v^i) by (u, v) .

Step by step, we obtain a vector function $(u, v) \in C^{2,1}(\Omega_T)$ satisfying (1.1) in Ω_T and a series of subsequences

$$\{n_{1,k}\} \supset \{n_{2,k}\} \supset \dots \supset \{n_{l,k}\} \supset \dots,$$

such that

$$\lim_{k \rightarrow \infty} (u_{n_{l,k}}, v_{n_{l,k}}) = (u, v), \quad \text{in } C^{2,1}(\overline{Q_l}).$$

By a diagonal procedure, we clearly have a subsequence $\{n_{k,k}\}$ and

$$(2.8) \quad \lim_{k \rightarrow \infty} (u_{n_{k,k}}, v_{n_{k,k}}) = (u, v), \quad \text{in } C^{2,1}(\Omega_T).$$

From (2.6), (2.8), we also have

$$(2.9) \quad u, v \geq k\phi(x)e^{-\rho t}, \quad \text{in } \Omega_T.$$

Define

$$(2.10) \quad \begin{aligned} u|_{\partial\Omega} &= 0, \quad v|_{\partial\Omega} = 0, \\ u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad \text{in } \Omega. \end{aligned}$$

If we can prove $(u, v) \in C(\overline{\Omega_T})$, combining with (2.8) and (2.9), then (u, v) is a positive solution of (1.1), (1.2).

First we establish the continuity of (u, v) up to $\Omega \times \{0\}$. Note a result of DiBenedetto (see [3, Theorem 1.2, p. 42] or [4]) that if the initial values are only Hölder continuous in an open domain $\Omega' \subset\subset \Omega$, then the solution of the Dirichlet problem would be Hölder continuous up to $\Omega' \times \{0\}$. Since $u_0, v_0 \in C^1(\overline{\Omega})$, (u, v) is continuous up to $t = 0$.

As for the continuity of (u, v) up to S_T , the proof is standard. We refer to [7] or [19].

Thus far it has been proved that a positive solution of (1.1), (1.2) exists in some interval $(0, T)$. To complete the local existence theory we state a continuation theorem for the positive solution (u, v) defined by (2.8), (2.10).

Theorem 2.4. *Assume (H1) holds and let T^* be the supremum over T for which (u, v) exists on $(0, T)$. Then there exists a positive solution of (1.1), (1.2) on $(0, T^*)$. If $T^* < \infty$, then (u, v) is unbounded in Ω_{T^*} .*

Proof. The proof is similar to [6, Theorem 2.5]. □

3. THE GLOBAL NONEXISTENCE

In this section we assume

$$(3.1) \quad \lambda_1 < \min\{a, b\}.$$

Let (u, v) be any positive solution of (1.1), (1.2). We will prove that, under assumption (3.1), (u, v) does not exist globally (but not means blowup in C -norm). First we give a lower bound of the positive solution. The lower bound is of the form $(k\phi(x), k\phi(x))$, where $\phi(x)$ is the first eigenfunction of (1.5), $k > 0$ is small such that

$$(3.2) \quad u_0(x) \geq k\phi(x), \quad v_0(x) \geq k\phi(x).$$

Denote by T^* the maximal existence time of (u, v) .

Lemma 3.1. *If*

$$\lambda_1 \leq \min\{a, b\},$$

then for all $0 < T < T^$,*

$$(3.3) \quad (u, v) \geq (k\phi(x), k\phi(x)), \quad \text{in } \Omega_T.$$

Proof. Let $w = u - k\phi, s = v - k\phi$, then

$$\begin{aligned} w_t &= v^p(\Delta w + aw) + kv^p(a - \lambda_1)\phi \\ &\geq v^p(\Delta w + aw), \\ s_t &= u^q(\Delta s + bs) + ku^q(b - \lambda_1)\phi \\ &\geq u^q(\Delta s + bs). \end{aligned}$$

Since (u, v) is bounded before its maximal existence time T^* , we have (3.3) according to (3.2) and Lemma 2.1. □

Theorem 3.2. *If $\lambda_1 < \min\{a, b\}$, then there are no nontrivial global positive solutions of (1.1), (1.2).*

Proof. From (3.1) there exists an open domain $\Omega' \subset\subset \Omega$ such that $\lambda_1 < \lambda'_1 < \min\{a, b\}$, where λ'_1 is the first eigenvalue on Ω' of (1.5). Lemma 3.1 implies that

$$u \geq k\phi(x) \geq 2\sigma, \quad v \geq k\phi(x) \geq 2\sigma, \quad \text{in } \Omega'_T$$

for some small $\sigma > 0$. Then $u_1 = u|_{\Omega'}, v_1 = v|_{\Omega'}$ satisfy (1.1) in Ω'_T with the initial-boundary condition

$$\begin{aligned} u_1|_{\partial\Omega'} &= u|_{\partial\Omega'} \geq 2\sigma, \quad v_1|_{\partial\Omega'} = v|_{\partial\Omega'} \geq 2\sigma, \\ u_1(x, 0) &= u_0(x) \geq 2\sigma, \quad v_1(x, 0) = v_0(x) \geq 2\sigma, \quad \text{in } \Omega'. \end{aligned}$$

Step 1. Consider the following system:

$$(3.4) \quad \begin{aligned} \underline{u}_t &= \underline{v}^p(\Delta \underline{u} + a\underline{u}), \\ \underline{v}_t &= \underline{u}^q(\Delta \underline{v} + b\underline{v}), \quad \text{in } \Omega'_T, \\ \underline{u}|_{\partial\Omega} &= f(t), \quad \underline{v}|_{\partial\Omega} = g(t), \\ \underline{u}(x, 0) &= \underline{v}(x, 0) = \sigma, \quad \text{in } \Omega', \end{aligned}$$

where $f(t), g(t)$ satisfy

$$\begin{aligned}
 & f(t), g(t) \in C^\infty([0, \infty)), \quad f'(t), g'(t) > 0, \\
 \text{(H3)} \quad & f(0) = g(0) = \sigma, \quad f(t), g(t) \leq 2\sigma, \\
 & f'(0) = a\sigma^{p+1}, \quad g'(0) = b\sigma^{q+1}.
 \end{aligned}$$

By the classical parabolic theory, under (H3), there exists a unique classical solution $(\underline{u}, \underline{v}) \in C^{2+\beta, 1+\beta/2}(\overline{\Omega'_T})$ to (3.4) in Ω'_T and

$$\text{(3.5)} \quad \underline{u}, \underline{v} \geq \sigma, \quad \text{in } \Omega'_T.$$

Similarly, by Lemma 2.1, we also have $\underline{u}_t, \underline{v}_t \geq 0$ in Ω'_T and

$$\text{(3.6)} \quad \Delta \underline{u} + a\underline{u} \geq 0, \quad \Delta \underline{v} + b\underline{v} \geq 0, \quad \text{in } \Omega'_T.$$

Thus from Proposition 2.2, we have

$$(\underline{u}, \underline{v}) \leq (u_1, v_1), \quad \text{in } \Omega'_T.$$

Therefore, it suffices to show that $(\underline{u}, \underline{v})$ blows up in finite time, because if so, its upper bound (u_1, v_1) does exist up to a finite time T^* .

By (3.5) and (3.6), we have

$$\begin{aligned}
 \text{(3.7)} \quad & \underline{u}_t \geq \sigma^{p-\alpha} \underline{v}^\alpha (\Delta \underline{u} + a\underline{u}), \\
 & \underline{v}_t \geq \sigma^{q-\alpha} \underline{u}^\alpha (\Delta \underline{v} + b\underline{v}), \quad \text{in } \Omega'_T
 \end{aligned}$$

with the corresponding initial and boundary condition and $0 < \alpha < 1$.

Step 2. Consider the problem

$$\begin{aligned}
 \text{(3.8)} \quad & z_t = Az^\alpha (\Delta z + Bz), \quad \text{in } \Omega'_T, \\
 & z|_{\partial\Omega} = \sigma, \\
 & z(x, 0) = \sigma, \quad \text{in } \Omega'
 \end{aligned}$$

with $\lambda'_1 < B \leq \min\{a, b\}$. According to [20, Lemma 3.1], $z(x, t)$ blows up in finite time. Since σ is a subsolution of (3.8), the usual subsolution argument (see [14, Lemma 4.1, p. 199]) yields that $z_t \geq 0$ and hence $\Delta z + Bz \geq 0$. In the following we construct a lower bound for $(\underline{u}, \underline{v})$ of the form (z, z) . Let $w = \underline{u} - z$, $s = \underline{v} - z$ and choose $A \leq \min\{\sigma^{p-\alpha}, \sigma^{q-\alpha}\}$. Then

$$\begin{aligned}
 w_t & \geq \sigma^{p-\alpha} \underline{v}^\alpha (\Delta w + aw) + \sigma^{p-\alpha} \underline{v}^\alpha (\Delta z + az) - Az^\alpha (\Delta z + Bz) \\
 & \geq \sigma^{p-\alpha} \underline{v}^\alpha (\Delta w + aw) + \alpha \sigma^{p-\alpha} \eta^{\alpha-1} (\Delta z + Bz)s, \\
 s_t & \geq \sigma^{q-\alpha} \underline{u}^\alpha (\Delta s + bs) + \sigma^{q-\alpha} \underline{u}^\alpha (\Delta z + bz) - Az^\alpha (\Delta z + Bz) \\
 & \geq \sigma^{q-\alpha} \underline{u}^\alpha (\Delta s + bs) + \alpha \sigma^{q-\alpha} \xi^{\alpha-1} (\Delta z + Bz)w,
 \end{aligned}$$

where η lies between \underline{v} and z , and ξ lies between \underline{u} and z . From (3.5), (3.8) and Lemma 2.1, we have

$$(\underline{u}, \underline{v}) \geq (z, z), \quad \text{in } \Omega'_T.$$

Hence $(\underline{u}, \underline{v})$ blows up in finite time. □

According to continuation Theorem 2.4, it follows that the positive solution (u, v) defined by (2.8), (2.10) blows up in finite time.

4. THE GLOBAL EXISTENCE

In this section, we assume

$$(4.1) \quad \lambda_1 \geq \min\{a, b\}.$$

We will prove that, under assumption (4.1), the positive solution (u, v) defined by (2.8), (2.10) exists globally. Without loss of generality, assume $a \leq b$. Let $K > 0$ be large such that

$$(4.2) \quad u_0(x), v_0(x) \leq K\phi(x),$$

where $\phi(x)$ is the first eigenfunction of (1.5). Denote by T^* its maximal existence time.

Theorem 4.1. *If $\lambda_1 \geq \min\{a, b\}$, then there exists a global positive solution of (1.1), (1.2).*

Proof. Suppose $T^* < +\infty$. Let $w = K\phi - u$, then from (4.1), we have

$$\begin{aligned} w_t &= v^p(\Delta w + aw) - K(a - \lambda_1)v^p\phi \\ &\geq v^p(\Delta w + aw). \end{aligned}$$

Hence (4.2) and Lemma 2.1 imply that $u(x, t) \leq K\phi(x)$ for all $0 < t < T^*$. Therefore, $K \exp(bK^qt)$ is an upper solution of the equation

$$\begin{aligned} v_t &= u^p(\Delta v + bv), \quad \text{in } \Omega_T, \\ v|_{\partial\Omega} &= 0, \\ v(x, 0) &= v_0(x), \quad \text{in } \Omega. \end{aligned}$$

It follows that $v(x, t) \leq K \exp(bK^qt)$ for all $0 < t < T^*$ by Lemma 2.1. This contradicts to continuation Theorem 2.4. \square

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