

OPERATOR WEAK AMENABILITY OF THE FOURIER ALGEBRA

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ABSTRACT. We show that for any locally compact group G , the Fourier algebra $A(G)$ is operator weakly amenable.

Let G be a locally compact group. It is shown by Johnson in [16] (and by Despić and Ghahramani in [6]) that the group algebra $L^1(G)$ is always weakly amenable. It is natural to ask whether the same holds for the Fourier algebra $A(G)$. In [17] it is shown for $G = SO(3)$ that $A(G)$ is not weakly amenable. We note that $A(G)$ is weakly amenable (and, in fact amenable) whenever G is Abelian, since $A(G) \cong L^1(\widehat{G})$ where \widehat{G} is the dual group. Also, $A(G)$ is weakly amenable whenever the connected component of the identity in G is Abelian [11]. It is conjectured that this characterizes the weak amenability of $A(G)$.

Since $A(G)$ is the predual of the von Neumann algebra $VN(G)$, it admits a natural structure as an operator space. Using this structure, Ruan [23] developed a completely bounded cohomology theory and proved that $A(G)$ is operator amenable exactly when G is an amenable group. This is analogous to Johnson's result [15] that $L^1(G)$ is amenable exactly when G is amenable. We note that the natural operator space structure on $L^1(G)$ as the predual of $L^\infty(G)$ is such that all bounded maps from $L^1(G)$ into any operator space are automatically completely bounded. Thus the notions of amenability and operator amenability coincide on $L^1(G)$, making Ruan's result truly a dual result of Johnson's, in the sense that $A(G)$ is the dual of $L^1(G)$ in a way which generalizes Pontryagin's Duality Theorem (see [9]).

The purpose of this note is to show that the natural operator space structure on $A(G)$ allows us to obtain another analogous result to one for $L^1(G)$: we show that $A(G)$ is always operator weakly amenable. We note that this result was obtained in [12], for the case that the connected component of the identity in G is amenable.

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1. PRELIMINARIES

If \mathcal{X} is a Banach space we always let \mathcal{X}^* denote its dual and $\mathcal{B}(\mathcal{X})$ denote the Banach algebra of bounded operators on \mathcal{X} . The symbol \mathcal{H} (possibly with a

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subscript) will always denote a Hilbert space and $\mathcal{U}(\mathcal{H})$ will denote the group of unitary operators on \mathcal{H} with the relativized weak operator topology.

Let G be a locally compact group. The *Fourier* and *Fourier-Stieltjes algebras*, $A(G)$ and $B(G)$, are defined in [10]. We recall that $B(G)$ is the space of matrix coefficients of all continuous unitary representations of G ; i.e. the space of functions of the form $s \mapsto \langle \pi(s)\xi|\eta \rangle$ where $\pi : G \rightarrow \mathcal{U}(\mathcal{H}_\pi)$ is a continuous homomorphism and $\xi, \eta \in \mathcal{H}_\pi$. $B(G)$ is the dual of the enveloping group C^* -algebra $C^*(G)$ via $\langle a, \langle \pi(\cdot)\xi|\eta \rangle \rangle = \langle \pi_*(a)\xi|\eta \rangle$, where $\pi_* : C^*(G) \rightarrow \mathcal{B}(\mathcal{H}_\pi)$ is the representation induced by π . It can be shown that $B(G)$ is a commutative Banach algebra (under pointwise operations) and $A(G)$ is the closed ideal in $B(G)$ generated by compactly supported matrix coefficients.

If π is a continuous unitary representation of G , let A_π be the norm closure of $\text{span}\{\langle \pi(\cdot)\xi|\eta \rangle : \xi, \eta \in \mathcal{H}_\pi\}$ in $B(G)$. Then, by [1, 2.2], $A_\pi^* \cong \text{VN}_\pi$, where VN_π is the von Neumann algebra generated by $\pi(G)$. If λ is the left regular representation of G on $L^2(G)$, then $A(G) = A_\lambda$ and we write $\text{VN}(G) = \text{VN}_\lambda$. Given two unitary representations π and σ of G , we let $\pi \oplus \sigma : G \rightarrow \mathcal{U}(\mathcal{H}_\pi \oplus \mathcal{H}_\sigma)$ denote their direct sum. If π and σ are disjoint (i.e. there are no subrepresentations π' of π and σ' of σ such that π' is spatially equivalent to σ'), then

$$(1.1) \quad \text{VN}_{\pi \oplus \sigma} = \text{VN}_\pi \oplus_\infty \text{VN}_\sigma \quad \text{and} \quad A_{\pi \oplus \sigma} = A_\pi \oplus_1 A_\sigma$$

by [22, 3.8.10] and [1, 3.13], respectively, where \oplus_p denotes the ℓ^p -direct sum for $p = 1, \infty$.

Our standard reference for operator spaces will be [8]. Given two operator spaces \mathcal{X} and \mathcal{Y} , we denote by $\mathcal{CB}(\mathcal{X}, \mathcal{Y})$ the space of completely bounded linear maps between \mathcal{X} and \mathcal{Y} and denote the norm on it by $\|\cdot\|_{cb}$. If $\mathcal{X} = \mathcal{Y}$, we will denote the Banach algebra $\mathcal{CB}(\mathcal{X}, \mathcal{X})$ by $\mathcal{CB}(\mathcal{X})$. Dual spaces will always be given the standard operator dual structure ([8, Sec. 3.2], [3]). Given two operator spaces \mathcal{X} and \mathcal{Y} , their direct product with the canonical product operator space structure will be denoted $\mathcal{X} \oplus_\infty \mathcal{Y}$. The direct sum $\mathcal{X} \oplus_1 \mathcal{Y}$ will be given the operator space structure it obtains from being imbedded into $(\mathcal{X}^* \oplus_\infty \mathcal{Y}^*)^*$. The product and sum are denoted by $\mathcal{X} \oplus_{CM} \mathcal{Y}$ and $\mathcal{X} \oplus_{CL} \mathcal{Y}$, respectively, in [7]. If \mathcal{M} is a von Neumann algebra, its predual \mathcal{M}_* will always be given the operator space structure it inherits from being imbedded in the dual \mathcal{M}^* . Moreover, \mathcal{M} is then the standard dual of \mathcal{M}_* ([8, 4.2.2], [3]). In particular, each space $A_\pi \cong (\text{VN}_\pi)_*$ will be endowed with this predual operator space structure.

The projective tensor product of Banach space theory admits an operator space analogue. Given two operator spaces \mathcal{X} and \mathcal{Y} , we denote their *operator space projective tensor product* by $\mathcal{X} \widehat{\otimes} \mathcal{Y}$. We will not need the explicit formula for the norm of this tensor product, but note that it is a completion of the algebraic tensor product $\mathcal{X} \otimes \mathcal{Y}$. We will use two important properties of this operator tensor product. First,

$$(1.2) \quad (\mathcal{X} \widehat{\otimes} \mathcal{Y})^* \cong \mathcal{CB}(\mathcal{X}, \mathcal{Y}^*) \quad \text{via} \quad \langle x \otimes y, T \rangle = \langle y, Tx \rangle.$$

See [8, 7.1.5] or [4]. This is analogous to the usual dual formula for the Banach space projective tensor product. Second, if \mathcal{M} and \mathcal{N} are von Neumann algebras, then

$$(1.3) \quad \mathcal{M}_* \widehat{\otimes} \mathcal{N}_* \cong (\mathcal{M} \widehat{\otimes} \mathcal{N})_*$$

where $\mathcal{M}\widehat{\otimes}\mathcal{N}$ is the von Neumann tensor product of \mathcal{M} and \mathcal{N} . See [8, 7.2.4]. In particular, since $\text{VN}(G\times G) \cong \text{VN}(G)\widehat{\otimes}\text{VN}(G)$ spatially, via the unitary which implements $L^2(G)\otimes L^2(G) \cong L^2(G\times G)$, we thus have that $\text{A}(G)\widehat{\otimes}\text{A}(G) \cong \text{A}(G\times G)$ completely isometrically. This identity holds isometrically for the Banach space projective tensor product only when G is Abelian. See [20].

If \mathcal{A} is an operator space which is also an algebra, it is called a *completely contractive Banach algebra* if the multiplication map $m_0 : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ extends to a complete contraction $m : \mathcal{A}\widehat{\otimes}\mathcal{A} \rightarrow \mathcal{A}$. The Fourier algebra $\text{A}(G)$ is a completely contractive Banach algebra since the multiplication map $m : \text{A}(G)\widehat{\otimes}\text{A}(G) \rightarrow \text{A}(G)$ corresponds to restriction to the diagonal subgroup, that is, the map $R : \text{A}(G\times G) \rightarrow \text{A}(G)$ given by $Ru(s) = u(s, s)$ ($s \in G$). Since the adjoint $R^* : \text{VN}(G) \rightarrow \text{VN}(G\times G)$ is a $*$ -homomorphism, it is a complete contraction and hence $R \cong m$ is a complete contraction.

If \mathcal{A} is a completely contractive Banach algebra and \mathcal{X} is an operator space which is also an \mathcal{A} -module for which the module multiplication maps $m_{l,0} : \mathcal{A} \otimes \mathcal{X} \rightarrow \mathcal{X}$ and $m_{r,0} : \mathcal{X} \otimes \mathcal{A} \rightarrow \mathcal{X}$ extend to complete contractions $m_l : \mathcal{A}\widehat{\otimes}\mathcal{X} \rightarrow \mathcal{X}$ and $m_r : \mathcal{X}\widehat{\otimes}\mathcal{A} \rightarrow \mathcal{X}$, then \mathcal{X} is called a *completely contractive \mathcal{A} -module*. A linear map $D : \mathcal{A} \rightarrow \mathcal{X}$ is called a derivation if $D(ab) = a\cdot D(b) + D(a)\cdot b$ for a, b in \mathcal{A} . If \mathcal{X} is a completely contractive \mathcal{A} -module, then \mathcal{X}^* is, too. \mathcal{A} is called *operator amenable* if every completely bounded derivation $D : \mathcal{A} \rightarrow \mathcal{X}^*$, where \mathcal{X} is a completely contractive \mathcal{A} -module, is inner (i.e. $D(a) = a\cdot f - f\cdot a$ for some f in \mathcal{X}^*). \mathcal{A} is called *operator weakly amenable* if every completely bounded derivation $D : \mathcal{A} \rightarrow \mathcal{A}^*$ is inner. If \mathcal{A} is commutative, this is equivalent to saying that the only completely bounded derivation $D : \mathcal{A} \rightarrow \mathcal{X}$, where \mathcal{X} is any symmetric completely contractive \mathcal{A} -module (i.e. $a\cdot x = x\cdot a$ for a in \mathcal{A} , x in \mathcal{X}), is 0. See [2] for this result in the Banach algebra case and [12] for the adaptation to the operator theoretic setting.

2. A THEOREM OF GROENBAEK

In this section we adapt a theorem of Groenbaek [13] to the completely contractive Banach algebra setting. Let \mathcal{A} be a completely contractive Banach algebra and \mathcal{X} an operator space which is a completely contractive \mathcal{A} -module. Let $\mathcal{A}_1 = \mathcal{A} \oplus_1 \mathbb{C}$ be the unitization of \mathcal{A} . Then \mathcal{X} is a completely contractive \mathcal{A}_1 -module by setting $1\cdot x = x$ and $x\cdot 1 = x$ for x in \mathcal{X} , where $1 = 0 \oplus 1$ in \mathcal{A}_1 . Indeed, $\mathcal{A}_1\widehat{\otimes}\mathcal{X} \cong (\mathcal{A}\widehat{\otimes}\mathcal{X}) \oplus_1 \mathcal{X}$ and the left multiplication map $m_1 : \mathcal{A}_1\widehat{\otimes}\mathcal{X} \rightarrow \mathcal{X}$ corresponds to the complete contraction $m \boxplus \text{id}_{\mathcal{X}} : (\mathcal{A}\widehat{\otimes}\mathcal{X}) \oplus_1 \mathcal{X} \rightarrow \mathcal{X}$ given by $m \boxplus \text{id}_{\mathcal{X}}((a \otimes x) \oplus y) = a\cdot x + y$. Similarly the right multiplication map can be extended. See [7].

The projection $\pi : \mathcal{A}_1 \rightarrow \mathcal{A}$ is a complete quotient map. Then if we let $i : \mathcal{A} \rightarrow \mathcal{A}_1$ be the natural injection, we get that the identity map $\text{id}_{\mathcal{A}\widehat{\otimes}\mathcal{X}}$ factors as

$$\mathcal{A}\widehat{\otimes}\mathcal{X} \xrightarrow{i\otimes\text{id}} \mathcal{A}_1\widehat{\otimes}\mathcal{X} \xrightarrow{\pi\otimes\text{id}} \mathcal{A}\widehat{\otimes}\mathcal{X}.$$

Hence we have for u in $M_n(\mathcal{A}\widehat{\otimes}\mathcal{X})$ ($n\times n$ -matrices over $\mathcal{A}\widehat{\otimes}\mathcal{X}$) that

$$\|u\| \leq \|\pi \otimes \text{id}\|_{cb} \|(i \otimes \text{id})_n u\| \leq \|u\|,$$

from which it follows that $\|(i \otimes \text{id})_n u\| = \|u\|$, so $\mathcal{A}\widehat{\otimes}\mathcal{X}$ is completely isometrically imbedded in $\mathcal{A}_1\widehat{\otimes}\mathcal{X}$.

Consider the sequence

$$0 \longrightarrow \mathcal{K} \xrightarrow{i} \mathcal{A}_1\widehat{\otimes}\mathcal{X} \xrightarrow{m} \mathcal{X} \longrightarrow 0$$

where m denotes the left module multiplication map (denoted m_1 above), $\mathcal{K} = \ker m$ and $i : \mathcal{K} \rightarrow \mathcal{A}_1 \widehat{\otimes} \mathcal{X}$ is the injection. Note that $\mathcal{A}_1 \widehat{\otimes} \mathcal{X}$ is a completely contractive \mathcal{A} -module via $a \cdot (b \otimes x) = (ab) \otimes x$ and $(b \otimes x) \cdot a = b \otimes (x \cdot a)$. Let

$$(2.1) \quad [\mathcal{K}; \mathcal{A}] = \overline{\text{span}}\{a \cdot u - u \cdot a : u \in \mathcal{K} \text{ and } a \in \mathcal{A}\}$$

and note that $[\mathcal{K}; \mathcal{A}] \subset \mathcal{K}$. The following proposition is [13, Prop. 3.1]. We rework it here to ensure that it holds in our context.

Proposition 2.1. *If an operator S in $\mathcal{CB}(\mathcal{A}_1, \mathcal{X}^*) \cong (\mathcal{A}_1 \widehat{\otimes} \mathcal{X})^*$ is a derivation, then it annihilates $[\mathcal{K}; \mathcal{A}]$. In particular, 0 is the only derivation in $\mathcal{CB}(\mathcal{A}_1, \mathcal{X}^*)$ if $[\mathcal{K}; \mathcal{A}] = \mathcal{K}$.*

Proof. First note that

$$\mathcal{K} = \{u - 1 \otimes m(u) : u \in \mathcal{A}_1 \widehat{\otimes} \mathcal{X}\} = \overline{\text{span}}\{b \otimes x - 1 \otimes b \cdot x : b \in \mathcal{A}_1 \text{ and } x \in \mathcal{X}\}.$$

Then for any a and b in \mathcal{A}_1 and x in \mathcal{X} , we have

$$\begin{aligned} \langle b \otimes x - 1 \otimes b \cdot x, S \cdot a - a \cdot S \rangle &= \langle x, S(ab) - a \cdot S(b) \rangle - \langle b \cdot x, S(a) - a \cdot S(1) \rangle \\ &= \langle x, S(ab) - a \cdot S(b) \rangle - \langle x, S(a) \cdot b \rangle \\ &= \langle x, S(ab) - (a \cdot S(b) + S(a) \cdot b) \rangle \end{aligned}$$

from which it follows that $S \cdot a - a \cdot S \in \mathcal{K}^\perp$ for all a . However, $S \cdot a - a \cdot S \in \mathcal{K}^\perp$ for all a , if and only if for all u in \mathcal{K} ,

$$0 = \langle u, S \cdot a - a \cdot S \rangle = \langle a \cdot u - u \cdot a, S \rangle.$$

Hence we obtain the first statement of the proposition.

Suppose that S in $\mathcal{CB}(\mathcal{A}_1, \mathcal{X}^*)$ is a derivation. For any u in $\mathcal{A}_1 \widehat{\otimes} \mathcal{X}$, we can write $u = (u - 1 \otimes m(u)) + 1 \otimes m(u)$, so $\mathcal{A}_1 \widehat{\otimes} \mathcal{X} = \mathcal{K} \oplus (1 \otimes \mathcal{X})$. (Note that $u \mapsto u - 1 \otimes m(u)$ is a completely bounded projection from $\mathcal{A}_1 \widehat{\otimes} \mathcal{X}$ onto \mathcal{K} , so the direct sum is one of operator spaces.) Observe that $\langle 1 \otimes x, S \rangle = \langle x, S1 \rangle = 0$ for any x in \mathcal{X} , so if $S \neq 0$, then there must be u in \mathcal{K} such that $\langle u, S \rangle \neq 0$. This is possible only if $[\mathcal{K}; \mathcal{A}] \neq \mathcal{K}$. \square

Now we will let $\mathcal{X} = \mathcal{A}$, and $m : \mathcal{A}_1 \widehat{\otimes} \mathcal{A}_1 \rightarrow \mathcal{A}_1$ be the multiplication map. Put

$$(2.2) \quad \mathcal{K}_1 = \ker m, \quad \mathcal{K} = \mathcal{K}_1 \cap (\mathcal{A}_1 \widehat{\otimes} \mathcal{A}) \quad \text{and} \quad \mathcal{K}_0 = \mathcal{K}_1 \cap (\mathcal{A} \widehat{\otimes} \mathcal{A}).$$

Since $\mathcal{A} \widehat{\otimes} \mathcal{A}$ imbeds into $\mathcal{A}_1 \widehat{\otimes} \mathcal{A}$, it is easily seen that \mathcal{K} is the same as in the notation above. If \mathcal{A} is commutative, then \mathcal{K}_1 and hence \mathcal{K} and \mathcal{K}_0 are closed ideals in $\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_1$.

With only trivial modifications to his proof, we get the following theorem of Groenbaek [13].

Theorem 2.2. *If \mathcal{A} is a commutative completely contractive Banach algebra, then the following are equivalent:*

- (i) \mathcal{A} is operator weakly amenable.
- (ii) $[\mathcal{K}; \mathcal{A}] = \mathcal{K}$.
- (iii) $\overline{\mathcal{K}_1^2} = \mathcal{K}_1$.
- (iv) $\overline{\mathcal{K}^2} = \mathcal{K}$.
- (v) $\overline{\mathcal{A}^2} = \mathcal{A}$ and $\overline{\mathcal{K}_0^2} = \overline{(\mathcal{A} \widehat{\otimes} \mathcal{A}) \cdot \mathcal{K}_1}$.

Furthermore, if \mathcal{A} has a bounded approximate identity, the above conditions are equivalent to

(iv) $\overline{\mathcal{K}_0^2} = \mathcal{K}_0$.

3. THE FOURIER ALGEBRA

We would now like to apply the above result to the Fourier algebra $A(G)$ of a locally compact group G . Unless it is specified otherwise, let G be a *non-compact* locally compact group.

If $\pi : G \rightarrow \mathcal{U}(\mathcal{H}_\pi)$ and $\sigma : G \rightarrow \mathcal{U}(\mathcal{H}_\sigma)$ are continuous unitary representations of G , let $\pi \times \sigma : G \times G \rightarrow \mathcal{U}(\mathcal{H}_\pi \otimes \mathcal{H}_\sigma)$ be the *Kronecker product* of π and σ , given by $\pi \times \sigma(s, t) = \pi(s) \otimes \sigma(t)$. We note, for future reference, that $VN_\pi \otimes VN_\sigma = VN_{\pi \times \sigma}$. Let $\lambda : G \rightarrow \mathcal{U}(L^2(G))$ be the left regular representation of G and $1 : G \rightarrow \mathbb{T} \cong \mathcal{U}(\mathbb{C})$ be the trivial representation.

Proposition 3.1. *The representations $1 \times 1, 1 \times \lambda, \lambda \times 1$ and $\lambda \times \lambda$ of $G \times G$ are all disjoint (i.e. no two of these representations have subrepresentations which are unitary equivalent).*

Proof. Since G is non-compact, λ has no fixed points, for a fixed point would give a constant function in $A(G)$. Hence none of $1 \times \lambda, \lambda \times 1$ or $\lambda \times \lambda$ have any fixed points for all of $G \times G$, so they are all disjoint from 1×1 . $1 \times \lambda$ fixes the entire subgroup $G \times \{e\}$ while neither $\lambda \times 1$ nor $\lambda \times \lambda$ have any fixed points for that subgroup, so $1 \times \lambda$ is disjoint from $\lambda \times 1$ and $\lambda \times \lambda$. Similarly, $\lambda \times 1$ and $\lambda \times \lambda$ are disjoint. □

Since λ and 1 are disjoint representations of G , we find that $VN_{\lambda \oplus 1} \cong VN_\lambda \oplus_\infty VN_1 = VN(G) \oplus_\infty \mathbb{C}$, by (1.1), and hence obtain the completely isometric identification $A(G)_1 = A(G) \oplus_1 \mathbb{C} \cong A_{\lambda \oplus 1}$. The implied map is clearly an algebra isomorphism.

If u and v are complex functions on G , let $u \times v$ denote the complex valued function on $G \times G$ given by $u \times v(s, t) = u(s)v(t)$. Also, let 1 denote the constant function on G (as well as the trivial representation).

Proposition 3.2. *We have the following completely isometric identifications:*

- (i) $A(G)_1 \widehat{\otimes} A(G) \cong A_{(\lambda \times \lambda) \oplus (1 \times \lambda)} = \text{span}\{u, 1 \times v : u \in A(G \times G) \text{ and } v \in A(G)\}$.
- (ii) $A(G)_1 \widehat{\otimes} A(G)_1 \cong A_{(\lambda \times \lambda) \oplus (1 \times \lambda) \oplus (\lambda \times 1) \oplus (1 \times 1)} = \text{span}\{u, 1 \times v, v \times 1, 1 \times 1 : u \in A(G \times G) \text{ and } v \in A(G)\}$.

These are implemented by algebra isomorphisms.

Proof. (i) We have $A(G)_1^* \cong VN_{\lambda \oplus 1}$ and $A(G)^* \cong VN(G) = VN_\lambda$. Thus, using (1.3), (1.1) and the lemma above, we obtain

$$\begin{aligned} (A(G)_1 \widehat{\otimes} A(G))^* &\cong VN_{\lambda \oplus 1} \widehat{\otimes} VN_\lambda = (VN_\lambda \oplus_\infty VN_1) \widehat{\otimes} VN_\lambda \\ &= (VN_\lambda \widehat{\otimes} VN_\lambda) \oplus_\infty (VN_1 \widehat{\otimes} VN_\lambda) \\ &= VN_{\lambda \times \lambda} \oplus_\infty VN_{1 \times \lambda} = VN_{(\lambda \times \lambda) \oplus (1 \times \lambda)}, \end{aligned}$$

where the last space is the dual of $A_{(\lambda \times \lambda) \oplus (1 \times \lambda)}$. The latter equality in the statement (i) is a straightforward identification of $A_{(\lambda \times \lambda) \oplus (1 \times \lambda)}$ in $B(G \times G)$. That the identifications are implemented by algebra isomorphisms is clear.

The proof of (ii) is similar. □

Theorem 3.3. *If G is a locally compact group, then $A(G)$ is operator weakly amenable.*

Proof. If G is compact, then $A(G)$ is operator amenable by [8, 16.2.3] (or Theorem 4.2, *infra*) and hence operator weakly amenable. Hence we are left to consider non-compact G . Let $\pi = (1 \times 1) \oplus (1 \times \lambda) \oplus (\lambda \times 1) \oplus (\lambda \times \lambda)$ so that A_π is the subalgebra of $B(G \times G)$ indicated in Proposition 3.2(ii) above. In the identification $A(G)_1 \widehat{\otimes} A(G)_1 \cong A_\pi$, the multiplication map $m : A(G)_1 \widehat{\otimes} A(G)_1 \rightarrow A(G)_1$ corresponds to the map $R : A_\pi \rightarrow A_{\lambda \oplus 1}$, which restricts functions to the diagonal subgroup $G_D = \{(s, s) : s \in G\} \cong G$, i.e. $Ru(s) = u(s, s)$ for s in G . Letting \mathcal{K}_1 and \mathcal{K}_0 be as in (2.2), we obtain identifications

$$\mathcal{K}_1 \cong \ker R = \text{span}\{u - 1 \times R(u), u - R(u) \times 1 : u \in A_\pi\}$$

and

$$\mathcal{K}_0 \cong \ker R \cap A(G \times G) = I(G_D).$$

Here, $I(G_D)$ denotes *the* ideal in $A(G \times G)$ with hull G_D : since G_D is a subgroup, it is a set of spectral synthesis by [14] (or see [17])[†]. From spectral synthesis we obtain that $\overline{I(G_D)^2} = I(G_D)$ so

$$(3.1) \quad \overline{\mathcal{K}_0^2} = \mathcal{K}_0.$$

Since $A(G \times G)$ is an ideal in A_π , we get

$$(3.2) \quad (A(G) \widehat{\otimes} A(G)) \cdot \mathcal{K}_1 \cong A(G \times G) \cdot \ker R \subset A(G \times G) \cap \ker R = I(G_D) \cong \mathcal{K}_0.$$

On the other hand, again using that G_D is a set of spectral synthesis for $A(G \times G)$,

$$(3.3) \quad \mathcal{K}_0 \cong I(G_D) = \overline{A(G \times G) \cdot I(G_D)} \subset \overline{A(G \times G) \cdot \ker R} \cong \overline{(A(G) \widehat{\otimes} A(G)) \cdot \mathcal{K}_1}.$$

We thus have, assembling (3.1) and the inclusions (3.2) and (3.3),

$$\overline{(A(G) \widehat{\otimes} A(G)) \cdot \mathcal{K}_1} = \overline{\mathcal{K}_0^2}.$$

Hence condition (v) of Theorem 2.2 is satisfied, since $\overline{A(G)^2} = A(G)$ by the Tauberian Theorem for $A(G)$. \square

If \mathcal{A} is a completely contractive Banach algebra and φ is a character of \mathcal{A} , then \mathbb{C} can be made into an \mathcal{A} -module via $a \cdot z = \varphi(a)z = z \cdot a$ for a in \mathcal{A} and z in \mathbb{C} . If φ is continuous, it is automatically completely bounded, and hence \mathbb{C} is a completely contractive \mathcal{A} -module. A *point derivation* is a derivation $D : \mathcal{A} \rightarrow \mathbb{C}$. If \mathcal{A} admits continuous non-zero point derivations, then it is not (operator) weakly amenable.

Corollary 3.4. *$A(G)$ has no continuous point derivations.*

In contrast to the case for $A(G)$, if G is Abelian and non-compact, then $B(G) \cong M(\widehat{G})$ (the measure algebra of the non-discrete Abelian group \widehat{G}) admits continuous point derivations by [5], and hence is not (operator) weakly amenable. If G is a locally compact group containing a closed non-compact Abelian subgroup H such that the restriction map $R_H : B(G) \rightarrow B(H)$ is surjective, then $B(G)$ is not (operator) weakly amenable. Note that if R_H is surjective, then $R_H^* : W^*(H) \rightarrow W^*(G)$ (enveloping von Neumann algebras) is a *-homomorphism, and hence a complete contraction. See [11] for further results on the weak amenability of $B(G)$.

[†] Note added in proof: The spectral synthesis result, in full generality, can be found in [24].

It has been recently announced by H. G. Dales, F. Ghahramani and A. Ya. Helemskii that the measure algebra $M(G)$ has continuous point derivations whenever G is not discrete. Hence we deduce that $M(G)$ is weakly amenable if and only if G is discrete. The reasonable dual conjecture to this is: $B(G)$ is operator weakly amenable if and only if G is compact.

4. OPERATOR AMENABILITY OF THE FOURIER ALGEBRA

In this section we give a quick proof that $A(G)$ is operator amenable when G is an amenable [SIN]-group. This proof uses elements of both [8, Sec. 16] (i.e. of [23, 3.5]) and [17, 5.3].

We say that G has the *small invariant neighbourhood* property, or that G is a [SIN]-group, if there is a neighbourhood base \mathcal{V} of the identity in G such that $sVs^{-1} = V$ for V in \mathcal{V} and s in G , i.e. each V in \mathcal{V} is invariant under inner automorphisms. Any compact, Abelian or discrete group is a [SIN]-group. [SIN]-groups are all unimodular. See [21] for more information.

We let $I(G_D)$ be as in the proof of Theorem 3.3. If we let $I_0(G_D)$ denote the set of functions in $A(G \times G)$ which are compactly supported with support disjoint from G_D , then $\overline{I_0(G_D)} = I(G_D)$, since G_D is a set of spectral synthesis for $A(G \times G)$.

Lemma 4.1. *If G is a [SIN]-group, then there is a bounded net $\{u_V\}_{V \in \mathcal{V}}$ in $B(G \times G)$ such that $u_V(s, s) = 1$ for s in G and $u_V v \rightarrow 0$ for v in $I(G_D)$.*

Proof. Let \mathcal{V} be a neighbourhood base of the identity in G consisting of relatively compact neighbourhoods, each of which is invariant under inner automorphisms. Let $\pi : G \times G \rightarrow \mathcal{U}(L^2(G))$ be given by $\pi(s, t)f = \lambda(s)\rho(t)f$ for f in $L^2(G)$, where λ and ρ are the left and right regular representations of G , respectively. Then π is a continuous unitary representation of $G \times G$. For v in \mathcal{V} , let $u_V = \frac{1}{\mu(V)} \langle \pi(\cdot)\chi_V | \chi_V \rangle$, where χ_V is the indicator function of V and μ is the Haar measure. Then for (s, t) in $G \times G$,

$$u_V(s, t) = \frac{1}{\mu(V)} \int_G \chi_V(s^{-1}rt)\chi_V(r)d\mu(r) = \frac{\mu(sVt^{-1} \cap V)}{\mu(V)}.$$

Clearly $u_V(s, s) = 1$ for s in G . Hence, since each u_V is positive definite, we have $\|u_V\| = u_V(e, e) = 1$. If $v \in I_0(G_D)$, then there is V in \mathcal{V} such that $u_V v = 0$. Hence $u_V v \rightarrow 0$ for v in $I(G_D)$, where V runs through decreasing elements of \mathcal{V} . \square

We remark that if G is discrete, we can use the positive definite function χ_{G_D} in place of the net above.

Theorem 4.2. *If G is an amenable [SIN]-group, then $A(G)$ is operator amenable.*

By [8, 16.1.4] (i.e. [23]), it suffices to show that $A(G)$ has an operator bounded approximate diagonal, i.e. a bounded net $\{w_\beta\}_{\beta \in B}$ in $A(G \times G) \cong A(G) \widehat{\otimes} A(G)$ such that

- (i) $\{Rw_\beta\}_{\beta \in B}$ is a bounded approximate identity for $A(G)$, and
- (ii) $\|u \times 1 w_\beta - w_\beta 1 \times u\| \rightarrow 0$ for all u in $A(G)$.

Recall that $R : A(G \times G) \rightarrow A(G)$ is the map $Ru(s) = u(s, s)$ for s in G . Let $\{u_\alpha\}_{\alpha \in A}$ be a bounded approximate identity for $A(G)$ (which we can obtain since G is amenable; see [19]), and $\{u_V\}_{V \in \mathcal{V}}$ be as in the lemma above. Let $B = A \times \mathcal{V}^A$ be the product directed set. For each $\beta = (\alpha, (V_{\alpha'})_{\alpha' \in A})$ in B let

$$w_\beta = u_{V_\alpha} u_\alpha \times u_\alpha.$$

Then $Rw_\beta = u_\alpha^2$, so (i) is satisfied. To see (ii), let $v \in A(G)$, so $u_\alpha \times u_\alpha(v \times 1 - 1 \times v) \in I(G_D)$ for each α , and hence

$$\begin{aligned}
 (\dagger) \quad \lim_{\beta} \|u \times 1 w_\beta - w_\beta 1 \times u\| &= \lim_{\beta=(\alpha, (V_{\alpha'})_{\alpha' \in A})} \|u_{V_\alpha} u_\alpha \times u_\alpha(v \times 1 - 1 \times v)\| \\
 &= \lim_{\alpha} \lim_V \|u_V u_\alpha \times u_\alpha(v \times 1 - 1 \times v)\| \\
 &= \lim_{\alpha} 0 = 0.
 \end{aligned}$$

The equality of limits at (\dagger) is from [18, p. 69]. \square

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