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DISTINCT GAPS BETWEEN FRACTIONAL PARTS OF SEQUENCES

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Abstract. Let α be a real number, N a positive integer and $\mathcal{N}$ a subset of $\{0, 1, 2, \ldots, N\}$. We give an upper bound for the number of distinct lengths of gaps between the fractional parts $\{n\alpha\}$, $n \in \mathcal{N}$.

1. Introduction

Questions on the distribution of fractional parts of sequences have a long history, and among the most intensively studied are those related to polynomial sequences. After the classical work of Weyl [11] on uniform distribution mod 1, other aspects of the distribution of fractional parts of polynomials, especially questions concerned with small fractional parts, have been investigated (see Schmidt [8] and Baker [1]). Recently, the distribution of gaps between fractional parts of sequences has attracted attention. Following the work of Rudnick and Sarnak [5] on the pair correlation of fractional parts of polynomials, other related questions have been studied in [2], [6] and [7]. We mention that the distribution of the local spacings between the fractional parts $\{n^d\alpha\}$, $n \in \mathbb{N}$, in the case $d = 1$ is completely different than in the case $d > 1$. If $d > 1$ one expects that for almost all α the distribution is Poissonian, and one knows for instance that the pair correlation is Poissonian indeed (see [5]). If $d = 1$ one knows for a fact that the distribution is not Poissonian, and this is a consequence of the following Three Gap Theorem of Steinhaus (see [4], [9] and [10]):

Let α be a real number and N a nonnegative integer. Then the fractional parts $\{n\alpha\}$, $0 \le n \le N$, partition the unit interval into $N + 1$ intervals which have at most 3 different lengths.

The correlation of fractional parts $\{n\alpha\}$, $n \in \mathbb{N}$, have been recently investigated by Marklof [3]. In this paper we take a real number α, a positive integer N, a subset $\mathcal{N}$ of $\{0, 1, 2, \ldots, N\}$ and look at the set of fractional parts

$$\mathcal{M} = \mathcal{M}(\alpha, \mathcal{N}) = \{\{n\alpha\} : n \in \mathcal{N}\},$$

with the intention of proving a result which is independent of $\mathcal{N}$. Clearly, as far as uniform distribution or small fractional parts are concerned, no such result is possible (for instance $\mathcal{N}$ might coincide with the set of those $1 \le n \le N$ for which $\{n\alpha\} \in [\frac{1}{3}, \frac{1}{2}]$). The same goes for the spacing distribution: if α is irrational, then the set $\{n\alpha\}$, $n \in \mathbb{N}$, is dense in $[0, 1]$, and one can choose for large N a sparse

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set $\mathcal{N}$ for which the distribution of $\mathcal{M}$ in $[0,1]$ approaches any given distribution. What we will do is to look at the gaps between the elements of $\mathcal{M}$ and see whether any kind of Steinhaus phenomenon still holds in this generality. Thus we arrange the elements of $\mathcal{M}$ in ascending order and consider the number $l(\alpha,\mathcal{N})$ of distinct lengths of gaps between consecutive elements of $\mathcal{M}(\alpha,\mathcal{N})$. Hence $l(\alpha,\mathcal{N}) \leq 3$ when $\mathcal{N} = \{0,1,2,\ldots,N\}$, by the Three Gap Theorem. For a general subset $\mathcal{N}$ of $\{0,1,2,\ldots,N\}$, $l(\alpha,\mathcal{N})$ can be much larger. For example, if N is a positive integer, $0 < \alpha < \frac{1}{N}$ and $\mathcal{N}$ consists of the squares $\{0,1,4,9,\ldots,[\sqrt{N}]^2\}$, then the numbers $n\alpha$, $n \in \mathcal{N}$, coincide with their fractional parts, and all the gaps between consecutive elements of $\mathcal{M}$ have distinct lengths. Thus $l(\alpha,\mathcal{N})$ can be as large as $\sqrt{N}$. The object of this paper is to prove the following theorem, which shows that $l(\alpha,\mathcal{N})$ cannot be much larger than $\sqrt{N}$.

Theorem 1. *For any real number α, any positive integer N and any subset $\mathcal{N}$ of $\{0,1,2,\ldots,N\}$ one has*

$$l(\alpha,\mathcal{N}) < (2+2\sqrt{2})\sqrt{N}\,.$$

2. Proof of Theorem 1

Fix a positive integer N, then choose a real number α and a subset $\mathcal{N}$ of $\{0,1,2,\ldots,N\}$ such that $l(\alpha,\mathcal{N})$ is largest. Note first that for fixed $\mathcal{N}$, the function $\alpha \mapsto l(\alpha,\mathcal{N})$ is periodic mod 1; thus we may assume in what follows that $0 \leq \alpha < 1$. In case $\alpha = 0$ all the fractional parts $\{n\alpha\}$ are zero, so the maximum value of $l(\alpha,\mathcal{N})$ is attained for some $\alpha \in (0,1)$. Next, notice that for $\mathcal{N}$ fixed, the lengths of the gaps between the elements of $\mathcal{M}(\alpha,\mathcal{N})$ are continuous functions of α. Thus there is an $\varepsilon = \varepsilon(\alpha,\mathcal{N}) > 0$ such that

$$l(\beta,\mathcal{N}) \geq l(\alpha,\mathcal{N}) \tag{1}$$

for any $\beta \in (\alpha-\varepsilon,\alpha+\varepsilon)$. If α and $\mathcal{N}$ are chosen as above such that $l(\alpha,\mathcal{N})$ is largest, then one has equality in (1). Replacing if necessary α by an irrational number $\beta \in (\alpha-\varepsilon,\alpha+\varepsilon)$ we may assume in the following that $0 < \alpha < 1$ is irrational. This last assumption is not essential in our proof, but it makes the presentation cleaner. For instance, in this case the fractional parts $\{n\alpha\}$, $n \in \mathcal{N}$, will be distinct, and we will discuss in detail the order of these fractional parts. To proceed, recall Dirichlet's theorem which asserts that for any positive integer M there are coprime integers a, q with $1 \leq q \leq M$ such that

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{qM}\,. \tag{2}$$

We use (2) with $M = 2N$, so let $a \in \mathbb{Z}$ and $1 \leq q \leq 2N$ such that $(a,q) = 1$ and

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{2qN}\,. \tag{3}$$

Since $0 < \alpha < 1$ we see that $0 \leq a \leq q$. From (3) it follows that for any $n \in \mathcal{N}$ one has

$$\left|n\alpha - \frac{na}{q}\right| < \frac{1}{2q}\,. \tag{4}$$

Let us consider the open intervals $J_k = \left(\frac{k}{q} - \frac{1}{2q},\ \frac{k}{q} + \frac{1}{2q}\right)$, $k = 0,1,\ldots,q-1$. For any $n \in \mathcal{N}$ let $k(n) \in \{0,1,\ldots,q-1\}$ be such that $an \equiv k(n) \pmod{q}$. Then the fractional part $\left\{\frac{an}{q}\right\}$ coincides with the center $\frac{k(n)}{q}$ of the interval $J_{k(n)}$, and from (4) it follows that $\{n\alpha\}$ belongs to $J_{k(n)}$. Therefore for any n, $n' \in \mathcal{N}$ for which $k(n) \neq k(n')$ the order of the elements $\{n\alpha\}$, $\{n'\alpha\} \in \mathcal{M}$ will simply be given by the order of the numbers $k(n)$ and $k(n')$. On the other hand, if $n, n' \in \mathcal{N}$ are such that $k(n) = k(n')$, then the order of $\{n\alpha\}$, $\{n'\alpha\}$ is determined by the sign of $\alpha - \frac{a}{q}$ and the order of the numbers n and n'. To be precise, let $\alpha - \frac{a}{q} = \eta$ and assume in what follows that $\eta > 0$. The case $\eta < 0$ is similar and will be left to the reader. Since $n\alpha = n\eta + \frac{na}{q}$, where as we know $|n\eta| < \frac{1}{2q}$, the relative "coordinate" of $\{n\alpha\}$ with respect to the center $\frac{k(n)}{q}$ of $J_{k(n)}$will equal $n\eta$. With our assumption on η, the order of $\{n\alpha\}$, $\{n'\alpha\}$ in case $k(n) = k(n')$ will be the same as the order of n, n'. Here the condition $k(n) = k(n')$ is equivalent to the condition $n \equiv n' \pmod q$. We now have a more clear picture of the distribution of the elements of $\mathcal{M}$. Write $\mathcal{N} = \bigcup_{r=0}^{q-1} \mathcal{N}_r$, where $\mathcal{N}_r = \{n \in \mathcal{N} : n \equiv r \pmod q\}$. Each $\mathcal{N}_r$ corresponds uniquely to a J_k, given by $k = k(r) \equiv ar \pmod q$, respectively $r = r(k) \equiv \overline{a}k \pmod q$, where $\overline{a}$ denotes the inverse of a mod q. For any r, the map $n \mapsto \{n\alpha\}$ sends $\mathcal{N}_r$ monotonically to a subset of $J_{k(r)}$. We now distinguish two kinds of gaps $(\{n\alpha\}, \{n'\alpha\})$ between consecutive elements $\{n\alpha\}$, $\{n'\alpha\}$ of $\mathcal{M}$, according as to whether $k(n) = k(n')$ or $k(n) \neq k(n')$, and count them separately. Denote by l_1, respectively l_2, the number of distinct lengths of gaps of the first kind, respectively of the second kind, between consecutive elements of $\mathcal{M}$. Some gaps of the first kind might have the same lengths as certain gaps of the second kind. Anyway one has

$$l(\alpha, \mathcal{N}) \leq l_1 + l_2. \tag{5}$$

In order to get an upper bound for l_1, we allow r to run over the set $\{0, 1, \dots, q-1\}$ and for each such value of r we look at the gaps formed by the image of $\mathcal{N}_r$ in $J_{n(r)}$. We already know that consecutive elements of $\mathcal{N}_r$ correspond to consecutive elements of $\mathcal{M}$. Moreover, if $n < n'$ are consecutive elements of $\mathcal{N}_r$, then the length of the gap between $\{n\alpha\}$ and $\{n'\alpha\}$ equals the difference between their coordinates in $J_{k(r)}$, which is $(n' - n)\eta$. Thus the lengths of these gaps in $\mathcal{M}$ are proportional to the lengths of the gaps $(n' - n)$ in $\mathcal{N}_r$, by a factor η which is independent of r. It follows that l_1 equals the cardinality of the set

$$A = \bigcup_{r=0}^{q-1} \{n' - n : n, n' \text{ consecutive in } \mathcal{N}_r\}.$$

Now the point is that since each element of A is a positive multiple of q, the sum of its l_1 (distinct) elements will be at least

$$q + 2q + \cdots + l_1 q = \frac{q l_1 (l_1 + 1)}{2}.$$

On the other hand, if we add all the elements of A counted with multiplicities, the sum will equal

$$\sum_{r=0}^{q-1} \sum_{n,n' \text{ consecutive in } \mathcal{N}_r} (n' - n) = \sum_{r=0}^{q-1} (\max \mathcal{N}_r - \min \mathcal{N}_r) < Nq\,.$$

It follows that $\frac{q l_1 (l_1+1)}{2} < Nq$, which implies

$$l_1 < \sqrt{2N}. \tag{6}$$

We now turn to l_2. Some of the above sets $\mathcal{N}_r$ might be empty, resulting in some intervals J_k having no points from $\mathcal{M}$. Let $0 \le k_1 < k_2 < \cdots < k_s \le q-1$ be those values of k for which $J_k \cap \mathcal{M}$ is nonempty. Then for each pair (k_j, k_{j+1}) we have exactly one gap of the second kind. Its left and right endpoints are the largest element of $\mathcal{M} \cap J_{k_j}$ and, respectively, the smallest element of $\mathcal{M} \cap J_{k_{j+1}}$. Thus the length of this gap, which we denote by δ_j, is given by

$$\delta_j = \{\underline{n}_{j+1}\alpha\} - \{\overline{n}_j\alpha\},$$

where for any j, $\underline{n}_j$ and $\overline{n}_j$ stand for the smallest, respectively the largest, element of $\mathcal{N}_{r(k_j)}$. The distance between the centers of J_{k_j} and $J_{k_{j+1}}$ equals $\frac{k_{j+1}-k_j}{q}$ and the coordinates of $\{\overline{n}_j\alpha\}$ and $\{\underline{n}_{j+1}\alpha\}$ with respect to these centers are $\overline{n}_j\eta$ and respectively $\underline{n}_{j+1}\eta$. Hence

$$\delta_j = \frac{k_{j+1}-k_j}{q} + \underline{n}_{j+1}\eta - \overline{n}_j\eta\,. \tag{7}$$

A trivial upper bound for l_2 is

$$l_2 \le s \le q\,. \tag{8}$$

For each positive integer b, let $n(b)$ be the number of distinct lengths δ_j of gaps of the second kind for which $k_{j+1} - k_j = b$. Thus l_2 can be written as

$$l_2 = \sum_{b \ge 1} n(b)\,. \tag{9}$$

Here we used the fact that if $k_{j+1} - k_j = b \ne b' = k_{j'+1} - k_{j'}$, then $\delta_j \ne \delta_{j'}$. This is a consequence of the inequalities $k_{j+1} - k_j - \frac{1}{2} < q\delta_j < k_{j+1} - k_j + \frac{1}{2}$, which in turn follow from (7) and the inequality $0 \le n\eta < \frac{1}{2q}$, valid for any $n \in \mathcal{N}$. Note that

$$\sum_{b \ge 1} n(b)\, b \le \sum_j (k_{j+1} - k_j) \le q\,. \tag{10}$$

We claim that for any b one has

$$n(b) \le \left[\frac{2N}{q}\right] + 1\,, \tag{11}$$

where $[\,\cdot\,]$ denotes the greatest integer part function. In order to prove the claim, let $j_1, \dots, j_{n(b)}$ be such that $\delta_{j_1}, \dots, \delta_{j_{n(b)}}$ are distinct and

$$k_{j_1+1} - k_{j_1} = \cdots = k_{j_{n(b)}+1} - k_{j_{n(b)}} = b\,.$$

By (7) we know that

$$\delta_j = \frac{b}{q} + \eta(\underline{n}_{j+1} - \overline{n}_j)$$

for any $j \in \{j_1, \dots, j_{n(b)}\}$. The numbers $\delta_{j_1}, \dots, \delta_{j_{n(b)}}$ being distinct, it follows that as j runs over the set $\{j_1, \dots, j_{n(b)}\}$, the numbers $\underline{n}_{j+1} - \overline{n}_j$ are distinct. Recall that $\overline{n}_j \in \mathcal{N}_{r(k_j)}$ and $\underline{n}_{j+1} \in \mathcal{N}_{r(k_{j+1})}$, so they satisfy the congruences

$$\overline{n}_j \equiv r(k_j) \equiv \overline{a}\, k_j \pmod{q}$$

and

$$\underline{n}_{j+1} \equiv r(k_{j+1}) \equiv \overline{a}\, k_{j+1} \pmod{q}\,.$$

Hence

$$\underline{n}_{j+1} - \overline{n}_j \equiv \overline{a}\,(k_{j+1} - k_j) \equiv \overline{a}\,b \pmod q$$

for any $j \in \{j_1, \ldots, j_{n(b)}\}$. Note also that for any j one has

$$-N \le \underline{n}_{j+1} - \overline{n}_j \le N\,.$$

There are at most $1 + \left[\frac{2N}{q}\right]$ integers in the interval $[-N, N]$ which are congruent to $\overline{a}\,b$ mod q, and this proves (11). Next, from (9) we know that the left-hand side of (10) is a sum of exactly l_2 terms, counting with multiplicities. By using (11) one sees that the left-hand side of (10) is at least as large as the sum

$$\Big(1 + \Big[\frac{2N}{q}\Big]\Big) \cdot 1 + \Big(1 + \Big[\frac{2N}{q}\Big]\Big) \cdot 2 + \cdots + \Big(1 + \Big[\frac{2N}{q}\Big]\Big)\, u + v\,(u+1)\,,$$

where u and v are given by

$$u = \Big[\frac{l_2}{1 + [2N/q]}\Big] \tag{12}$$

and

$$v = l_2 - \Big(1 + \Big[\frac{2N}{q}\Big]\Big)\, u\,.$$

We combine this with (10) to derive

$$\Big(1 + \Big[\frac{2N}{q}\Big]\Big)\, \frac{u\,(u+1)}{2} \le q\,,$$

which implies

$$u < \Big(\frac{2q}{1 + [2N/q]}\Big)^{\frac{1}{2}}\,. \tag{13}$$

Relations (12) and (13) give

$$\frac{l_2}{1 + [2N/q]} - 1 < \Big(\frac{2q}{1 + [2N/q]}\Big)^{\frac{1}{2}}\,,$$

from which we get the following upper bound for l_2 :

$$l_2 < 1 + \Big[\frac{2N}{q}\Big] + \Big(2q\Big(1 + \Big[\frac{2N}{q}\Big]\Big)\Big)^{\frac{1}{2}} < 1 + \frac{2N}{q} + (2q + 4N)^{\frac{1}{2}}\,. \tag{14}$$

Since $q \le 2N$, from (14) we obtain

$$l_2 < 1 + \frac{2N}{q} + 2\sqrt{2N}\,. \tag{15}$$

This inequality is sharp when q is at least of the size of $\sqrt{N}$. If q is smaller, then we use (8). From (8) and (15) we find that

$$l_2 < (2 + \sqrt{2})\,\sqrt{N} \tag{16}$$

regardless of the size of q. On combining (5), (6) and (16) we get

$$l(\alpha, \mathcal{N}) < (2 + 2\sqrt{2})\,\sqrt{N}\,,$$

which completes the proof of Theorem 1.

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