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# A REMARK ON REAL COBOUNDARY COCYCLES IN $L^\infty$-SPACE

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Abstract. Let $T$ be an ergodic automorphism of a probability measure space $(\Omega, \mathcal{A}, m)$ and let $f$ be a real-valued measurable function on $\Omega$. We deduce a necessary and sufficient condition for the existence of $L^\infty$-solutions of the cohomology equation $f = h \circ T - h$, by using the recent result of Alonso, Hong and Obaya.

Let $(\Omega, \mathcal{A}, m)$ be a probability measure space and let $T$ be an ergodic automorphism of $(\Omega, \mathcal{A}, m)$. Every real-valued measurable function $f$ on $\Omega$ defines an additive real $T$-cocycle

$$S_n f = \sum_{j=0}^{n-1} f \circ T^j, \qquad n > 0. \tag{1}$$

Recently Alonso, Hong and Obaya proved in [1] that a necessary and sufficient condition for the existence of $L^p$-solutions of the cohomology equation $f = h \circ T - h$, with $0 < p < \infty$, is that $f \in L^p(\Omega, m)$ and there exists a measurable subset $A \subset \Omega$ with $m(A) > 0$ and an increasing sequence $\{n_k\}$ of positive integers with

$$\sup_{k \geq 1} \frac{1}{n_k} \sum_{j=1}^{n_k} \int_A |S_j f|^p \, dm < \infty. \tag{2}$$

For related results we refer the reader to [2] and [3]. In this note we would like to consider the case $p = \infty$ and prove the following theorem.

**Theorem.** *Let $f \in L^p(\Omega, m)$ for some $p$ with $0 < p \leq \infty$. Then there exists a function $h$ in $L^\infty(\Omega, m)$ with $f = h \circ T - h$ if and only if there exists a measurable subset $A \subset \Omega$ with $m(A) > 0$ and an increasing sequence $\{n_k\}$ of positive integers with*

$$\sup_{k \geq 1} \frac{1}{n_k} \sum_{j=1}^{n_k} \|S_j f \cdot \chi_A\|_\infty < \infty. \tag{3}$$

*Proof.* Suppose $f = h \circ T - h$ for some $h \in L^\infty(\Omega, m)$. Then, since $S_j f = h \circ T^j - h$ for $j \geq 1$, we have $\|S_j f\|_\infty \leq 2\|h\|_\infty$, whence (3) holds for $A = \Omega$ and $n_k = k$.

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Conversely, suppose the second condition of the Theorem holds. Since $0 < r < p$ implies $L^p(\Omega, m) \subset L^r(\Omega, m)$, we may assume from the beginning that $0 < p \leq 1$. Then we have

$$\int_A |S_j f|^p \, dm \leq 1 + \|S_j f \cdot \chi_A\|_\infty \quad \text{for} \quad j \geq 1,$$

which implies

$$\sup_{k \geq 1} \frac{1}{n_k} \sum_{j=1}^{n_k} \int_A |S_j f|^p \, dm < \infty.$$

Thus, by Theorem 3.2 of [1], there exists a function $h$ in $L^p(\Omega, m)$ with $f = h \circ T - h$. It suffices to show that $h$ is a function in $L^\infty(\Omega, m)$.

To do so, we notice that there exists a positive number $\delta$ with

$$m(A \cap \{\omega : |h(\omega)| \leq \delta\}) > 0.$$

Therefore, we may assume from the beginning that

$$|h| \leq \delta \quad \text{on} \quad A. \tag{4}$$

Then, using the relations

$$\|h \circ T^j \cdot \chi_A\|_\infty = \|(S_j f + h) \cdot \chi_A\|_\infty \leq \|S_j f \cdot \chi_A\|_\infty + \delta,$$

we see from (3) that

$$M := \sup_{k \geq 1} \frac{1}{n_k} \sum_{j=1}^{n_k} \|h \circ T^j \cdot \chi_A\|_\infty < \infty. \tag{5}$$

By this inequality, we will prove below that $h \in L^\infty(\Omega, m)$.

For this purpose, let $\alpha$ be a positive number such that there exists a measurable subset $B \subset \Omega$ with

$$m(B) > 0 \quad \text{and} \quad |h| \geq \alpha \quad \text{on} \ B. \tag{6}$$

Since $T$ is an ergodic automorphism by hypothesis, the Birkhoff Ergodic Theorem implies that

$$\lim_{k \to \infty} \frac{1}{n_k} \sum_{j=1}^{n_k} \chi_A(T^{-j}\omega) = m(A) > 0$$

for almost all $\omega \in \Omega$. Hence, we can choose a measurable subset $B_1 \subset B$ with $m(B_1) > 0$ and a positive integer $K$ with

$$\frac{1}{n_K} \sum_{j=1}^{n_K} \chi_A(T^{-j}\omega) > 2^{-1} m(A) \quad \text{for all} \ \omega \in B_1. \tag{7}$$

Thus, taking into account (5) and (6), we get

$$\begin{aligned} \infty > M &\geq \frac{1}{n_K} \sum_{j=1}^{n_K} \|h \circ T^j \cdot \chi_A\|_\infty = \frac{1}{n_K} \sum_{j=1}^{n_K} \|h \cdot \chi_A \circ T^{-j}\|_\infty \\ &\geq \frac{1}{n_K} \sum_{j=1}^{n_K} \|\alpha \chi_B \cdot (\chi_A \circ T^{-j})\|_\infty \geq \alpha \cdot \left\| \chi_{B_1} \cdot \left( \frac{1}{n_K} \sum_{j=1}^{n_K} \chi_A \circ T^{-j} \right) \right\|_\infty \\ &\geq \alpha \cdot 2^{-1} m(A). \end{aligned}$$

This shows that $\alpha$ cannot be so large, and thus $h \in L^\infty(\Omega, m)$ must follow. This completes the proof. $\square$

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