PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 3, Pages 859–869
S 0002-9939(02)06321-9
Article electronically published on July 2, 2002

A VON NEUMANN TYPE INEQUALITY FOR CERTAIN DOMAINS IN $\mathbf{C}^n$

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(Communicated by Joseph A. Ball)

ABSTRACT. Strict contractions on a Hilbert space have a functional calculus with functions that are analytic in the unit disc of the complex plane; an estimate of the norm is then provided by von Neumann's inequality. We consider functions that satisfy related inequalities with respect to multioperators connected to certain domains in $\mathbf{C}^n$; a representation formula and a Nevanlinna–Pick type theorem are obtained.

1. Introduction

The von Neumann inequality plays a central role in the theory of contractions on a Hilbert space ([18]). Ando's theorem ([7]) implies an immediate generalization to the case of two commuting contractions, but to go beyond this case is not possible in all generality. Various classes of operators are considered instead, as, for instance, in [14], [16], or [5]; see also [3] and [9] for some interesting developments. In the present paper we follow an alternate approach by investigating functions that are supposed to satisfy an analogue of the von Neumann inequality with respect to multioperators connected to certain domains in $\mathbf{C}^n$. A description of these functions is obtained, as well as a Nevanlinna–Pick type interpolation theorem.

We start with a polynomial function

$$P : \mathbf{C}^n \to M_{p,q}(\mathbf{C})$$

and we define the basic domain $\Omega \subset \mathbf{C}^n$ by

$$\Omega := \{z \in \mathbf{C}^n \, ; \, \|P(z)\|_{M_{p,q}(\mathbf{C})} < 1\}. \tag{1}$$

Here, as below, the norm on $M_{p,q}(\mathbf{C})$ is the operator norm, obtained by identifying it with $\mathcal{L}(\mathbf{C}^q, \mathbf{C}^p)$; as usual, $\mathcal{L}(H)$ will denote the algebra of bounded linear operators on the Hilbert space H.

We will consider a certain class of analytic functions defined on Ω, namely, those that satisfy a von Neumann type inequality. The classical von Neumann inequality states that if T is a contraction on a Hilbert space H, and p is a polynomial such that $|p(z)| \leq 1$ whenever $|z| < 1$, then $\|p(T)\| \leq 1$. This can be extended to functions more general than polynomials, namely analytic functions f defined in

Received by the editors December 12, 2000 and, in revised form, February 19, 2001 and October 17, 2001.

2000 *Mathematics Subject Classification.* Primary 47A13, 47A57.

Key words and phrases. Von Neumann inequality, multioperators, Nevanlinna–Pick problem.

the unit disc $\mathbf{D} = \{z \in \mathbf{C}\ ;\ |z| < 1\}$ and bounded by 1 therein; to be able to define $f(T)$ in this case, one assumes that $\|T\| < 1$.

We are interested in obtaining a similar type of result for functions defined on the domain Ω. The idea is to consider the von Neumann inequality as a result ascertaining the domination of the identity function on a whole class of analytic functions: if the identity function applied to the operator yields a contraction, then the same is true with the whole class of analytic functions contractive in the unit disc. In the general case, the role of the identity function will be played by the matrix-valued polynomial P which defines Ω. The operator T is replaced by a multioperator $X = (X_1, \dots, X_n)$, and the relevant class is introduced using a domination relation between the operators defined by the analytic functional calculus.

There are different methods of defining the functional calculus for multioperators ([21]). They are equivalent as long as we consider analytic functions defined on the domain Ω, provided we apply them to multioperators whose Taylor spectrum is included in Ω. Actually, we will see below that in our case the last condition becomes redundant (see Lemma 1). Consequently, we define

$$\mathcal{S} = \{\phi : \Omega \to \mathbf{C} \text{ analytic}\ ;\ \|\phi(X)\| \le 1 \text{ whenever } \|P(X)\| < 1\}. \tag{2}$$

Thus, $\mathcal{S}$ is the class of analytic functions on Ω that satisfy a von Neumann type inequality. Our main goal in this paper is to obtain a more concrete description of $\mathcal{S}$.

2. A preliminary lemma

For the few basic facts concerning the spectrum of a multioperator and the functional calculus, we refer to [21], ch. III. Suppose $X = (X_1, \dots, X_n) \in \mathcal{L}(\mathcal{X})$ is a multioperator, $D \subset \mathbf{C}^n$ is an open set including the Taylor spectrum of X, while $f : D \to \mathbf{C}^m$ is analytic. If $f = (f_1, \dots, f_m)$ and $Y_j = f_j(X)$, then the Taylor spectrum of the multioperator $Y = (Y_1, \dots, Y_m)$ can be determined by the spectral mapping theorem

$$\sigma(Y) = f(\sigma(X)). \tag{3}$$

In our case the analytic functions f_j are actually polynomials, so they can be applied to any multioperator X. But a slight complication appears since $P(X)$ is a matrix whose entries are operators on the Hilbert space $\mathcal{X}$. We are actually interested in the action of this matrix as an operator with domain $\mathcal{X}^q$ and range $\mathcal{X}^p$, so one cannot directly apply the spectral mapping theorem. It is however possible to obtain a related result that is relevant in our situation, in which one replaces the exact formula (3) by an estimate involving the norm of the matrix. We will use for this purpose a result of Słodkowski and Żelazko that establishes a relation between the Taylor spectrum and the approximate point spectrum $\sigma_\pi(X)$ of a multioperator X, which is defined as the set of points $\lambda = (\lambda_1, \dots, \lambda_n)$ for which there exists a sequence of vectors $x_k \in \mathcal{X}$, with $\|x_k\| = 1$ and $(X_j - \lambda_j)x_k \to 0$ as $k \to \infty$ for all $j = 1, \dots, n$. It is then shown in [17] that $\sigma(X)$ is contained in the polynomially convex closure of $\sigma_\pi(X)$.

Lemma 1. *Let $X = (X_j)_{j=1}^n$, $X_j \in \mathcal{L}(\mathcal{X})$, be a commuting tuple on a Hilbert space $\mathcal{X}$. Then*

$$\sup_{z\in\sigma(X)} \|P(z)\|_{M_{p,q}(\mathbf{C})} \le \|P(X)\|_{\mathcal{L}(\mathcal{X}^q,\mathcal{X}^p)}.$$

Proof. Set $a := \|P(X)\|$. Let Y be the commuting pq–tuple of components $Y_{ij} := P_{ij}(X)$, $i = 1,\dots,p$, $j = 1,\dots,q$; thus, we ignore for the time being the fact that these operators are arranged in a matrix.

Take an arbitrary $\lambda = (\lambda_{ij})_{i,j}$ in the approximate point spectrum $\sigma_\pi(Y)$ of Y. Thus there exists a sequence of vectors $x_k \in \mathcal{X}$, with $\|x_k\| = 1$, such that for any i, j we have $(Y_{ij} - \lambda_{ij})x_k \to 0$ as $k \to \infty$.

Let $u = (u_1,\dots,u_q) \in \mathbf{C}^q$. If $y_k := (u_j x_k)_{j=1}^q \in \mathcal{X}^q$, then, since $\|P(X)y_k\| \le a\|y_k\|$, we have

$$\sum_{i=1}^p \|\sum_{j=1}^q Y_{ij}u_jx_k\|^2 \le a^2\sum_{j=1}^q \|u_jx_k\|^2 = a^2\sum_{j=1}^q |u_j|^2.$$

By taking $k \to \infty$, we can replace Y_{ij} by λ_{ij} in the left hand side, obtaining

$$\sum_{i=1}^p |\sum_{j=1}^q \lambda_{ij}u_j|^2 \le a^2\sum_{j=1}^q |u_j|^2.$$

It follows that the operator matrix $[\lambda_{ij}]_{i,j} \in M_{p,q}(\mathbf{C})$ satisfies the estimate $\|\lambda u\| \le a\|u\|$ for any $u \in \mathbf{C}^q$. Then λ is in the a-dilation $a\mathcal{B}$ of the (matrix) unit ball $\mathcal{B}$ in $M_{p,q}(\mathbf{C})$. It follows that $\sigma_\pi(Y)$ is a subset of $a\mathcal{B}$, which is convex (and hence also polynomially convex). But, by the result of [17] quoted above, $\sigma(Y)$, considered as a pq–tuple in $\mathbf{C}^{pq}$, is contained in the polynomially convex closure of $\sigma_\pi(Y)$. Hence $\sigma(Y) \subset a\mathcal{B}$. Now $P(\sigma(X)) = \sigma(P(X)) = \sigma(Y) \subset a\mathcal{B}$ and Lemma 1 is proved. □

As a consequence of the lemma, if $\|P(X)\| < 1$, then the Taylor spectrum $\sigma(X)$ is contained in Ω, and we may define, by the analytic functional calculus, $\phi(X)$ for any function ϕ analytic on Ω. Therefore, $\mathcal{S}$ is correctly defined by formula (2).

Let us denote, for further use, by $\mathbf{B}(\Omega)$ the class of analytic functions on Ω that take values in $\bar{\mathbf{D}}$. If we take as $\mathcal{X}$ a one-dimensional Hilbert space, and X the multioperator of multiplication with $\lambda \in \Omega$, then it follows immediately that $\mathcal{S} \subset \mathbf{B}(\Omega)$. Except for this simple remark, the definition of $\mathcal{S}$ is rather abstract. As stated in the Introduction, we intend to obtain a more concrete description, suggested by results in [1]. We will use a fractional type transform that we describe below.

3. The fractional transform

Suppose H_{ij}, $i = 1, 2$, are Hilbert spaces and $U = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : H_{11} \oplus H_{12} \to H_{21} \oplus H_{22}$ is a unitary operator. If $Y : H_{21} \to H_{11}$ is a strict contraction, then one can define

$$Y' = D + C(I_{H_{11}} - YA)^{-1}YB : H_{12} \to H_{22}.$$

It is well known that Y' is then a contraction (as a consequence of the formula

$$I - Y'^*Y' = B^*(I - Y^*A^*)^{-1}(I - Y^*Y)(I - AY)^{-1}B).$$

If K is an arbitrary Hilbert space, consider, for a fixed $z \in \Omega$ (or even $z \in \mathbf{C}^n$, for that matter), the matrix $P(z)$ as acting from the direct sum of q copies of K to the direct sum of p copies of K, by identifying $P(z)$ with the operator

$$P(z) \otimes I_K : \mathbf{C}^q \otimes K \to \mathbf{C}^p \otimes K.$$

When there is no danger of confusion, we will still denote by $P(z)$ the operator thus obtained; it is a strict contraction for any $z \in \Omega$. We are interested in fractional transforms associated to unitaries

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : (\mathbf{C}^p \otimes K) \oplus \mathbf{C} \to (\mathbf{C}^q \otimes K) \oplus \mathbf{C}. \tag{4}$$

Thus, we define

$$\Phi_U(z) = D + C(I_{\mathbf{C}^p \otimes K} - P(z)A)^{-1} P(z)B. \tag{5}$$

The result depends analytically on z; we have therefore obtained analytic functions on Ω and bounded by 1 therein: in other words, $\Phi_U \in \mathbf{B}(\Omega)$. However, the main interest in this class lies in their relation to the von Neumann inequality.

Proposition 2. *If Φ_U is defined by formula* (5)*, then $\Phi_U \in \mathcal{S}$.*

Proof. Note first that, since $\sigma(X)$ is a compact subset of Ω, Φ_U has the uniformly convergent development

$$\Phi_U(z) = D + \sum_{k=0}^{\infty} C(P(z)A)^k P(z)B$$

on a whole neighborhood of $\sigma(X)$. Therefore

$$\Phi_U(X) = DI_{\mathcal{X}} + \sum_{k=0}^{\infty} C(P(X)A)^k P(X)B.$$

But one can write one of the terms in the above series as

$$C(P(X)A)^k P(X)B = (I_{\mathcal{X}} \otimes C)[(P(X) \otimes I_K)(I_{\mathcal{X}} \otimes A)]^k (P(X) \otimes I_K)(I_{\mathcal{X}} \otimes B),$$

and therefore

$$\begin{aligned} \Phi_U(X) &= DI_{\mathcal{X}} + \sum_{k=0}^{\infty} (I_{\mathcal{X}} \otimes C)[(P(X) \otimes I_K)(I_{\mathcal{X}} \otimes A)]^k (P(X) \otimes I_K)(I_{\mathcal{X}} \otimes B) \\ &= (I_{\mathcal{X}} \otimes D) + (I_{\mathcal{X}} \otimes C)(I_{\mathcal{X} \otimes \mathbf{C}^p \otimes K} - (P(X) \otimes I_K)(I_{\mathcal{X}} \otimes A))^{-1} \\ &\quad \cdot (P(X) \otimes I_K)(I_{\mathcal{X}} \otimes B) \\ &= \Phi_{\tilde{U}}(X \otimes I_K) \end{aligned} \tag{6}$$

with

$$\tilde{U} = \begin{pmatrix} I_{\mathcal{X}} \otimes A & I_{\mathcal{X}} \otimes B \\ I_{\mathcal{X}} \otimes C & I_{\mathcal{X}} \otimes D \end{pmatrix}$$

unitary. From (6) it now follows that $\Phi_U(X)$ is contractive. □

In this paper we intend to show that one can prove a reciprocal to Proposition 2, and thus that formula (5) provides a representation formula for all functions in $\mathcal{S}$; that is, for all functions that satisfy a von Neumann type inequality.

4. Functions of positive type

Scalar valued functions of positive type are well known; the concept can be easily extended by allowing the values to be positive operators (see [4]), or, in our case, finite-dimensional matrices. Suppose Λ is a set and E is a Hilbert space that will usually be finite-dimensional; then we will call a function $\Gamma : \Lambda \times \Lambda \to \mathcal{L}(E)$ of positive type if

$$\sum_{i,j} \langle \Gamma(\lambda_i, \lambda_j)\xi_i, \xi_j \rangle \geq 0$$

for any finite number of $\lambda_i \in \Lambda$ and $\xi_i \in E$. As is the case with scalar valued functions of positive type, there exists a representation formula; namely, there is a Hilbert space (essentially unique) $\mathcal{E}$ and a map $F : \Lambda \to \mathcal{L}(E, \mathcal{E})$, such that

$$\Gamma(\lambda, \mu) = F(\mu)^* F(\lambda). \tag{7}$$

Conversely, if F is given, then formula (7) defines functions of positive type defined on Λ. In particular (this is a case that we will use below), take $\Lambda = \Omega$, $E = \mathbf{C}$, $\mathcal{E} = \mathbf{C}^q$, and $F(\lambda)^* = \begin{pmatrix} P_{11}(\lambda) & \dots & P_{1q}(\lambda) \end{pmatrix}$ (the first row of P). Denote

$$\tilde{P}(\lambda, \mu) = \sum_{k=1}^{q} P_{1k}(\lambda)\overline{P_{1k}(\mu)}$$

as the scalar function of positive type thus obtained.

Suppose now that Γ is an $\mathcal{L}(\mathbf{C}^m)$-valued function of positive type; then $\Gamma_0 = \operatorname{Tr}\Gamma$ is a scalar valued function of positive type. If $F : \Lambda \to \mathcal{L}(\mathbf{C}^m, \mathcal{E})$ is a representation (that is, if (7) is satisfied), then the same formula yields a representation of Γ_0, once we consider $\mathcal{L}(\mathbf{C}^m, \mathcal{E})$ as the Hilbert space of Hilbert-Schmidt operators. This can be rephrased in a more familiar way; let us denote, to this purpose, that H' is the dual of the Hilbert space H. There exists an identification J_m of $\mathcal{L}(\mathbf{C}^m, \mathcal{E})$ with $(\mathbf{C}^m)' \otimes \mathcal{E}$, which maps operators of rank one in elementary tensors. By composing it with F, we obtain a map $F_0 : \Lambda \to (\mathbf{C}^m)' \otimes \mathcal{E}$, such that $\Gamma_0(\lambda, \mu) = \langle F_0(\lambda), F_0(\mu) \rangle$. For further use, note also that, if $T \in \mathcal{L}(\mathbf{C}^l, \mathbf{C}^m)$, and $R \in \mathcal{L}(\mathbf{C}^m, \mathcal{E})$, then

$$J_l(RT) = (T' \otimes I_{\mathcal{E}}) J_m(R), \tag{8}$$

where T' denotes the dual of the operator T. Note that the identification J_m requires the use of dual space and dual operators, instead of the more familiar adjoint.

For scalar valued functions of positive type, Schur's theorem states that the (pointwise) product of two such functions is again of positive type. The same is then true of powers and of sums that are uniformly convergent. In the case above, since $\|P(\lambda)\| < 1$ for any $\lambda \in \Omega$, it follows that one may define a scalar function of positive type by the formula

$$\Delta(\lambda, \mu) = \sum_{i=0}^{\infty} \tilde{P}(\lambda, \mu)^i.$$

If $\gamma(\lambda, \mu)$ is an arbitrary scalar valued function of positive type, then $\Delta(\lambda, \mu)\gamma(\lambda, \mu)$ is also of positive type, and

$$\Delta(\lambda, \mu)\gamma(\lambda, \mu) - \tilde{P}(\lambda, \mu)\Delta(\lambda, \mu)\gamma(\lambda, \mu) = \gamma(\lambda, \mu). \tag{9}$$

5. The interpolation theorem

The main result of this paper is the following theorem, which allows the construction of a function in $\mathcal{S}$ with prescribed values on a given set $S \subset \Omega$.

Theorem 3. *Suppose $S \subset \Omega$ and $\phi : S \to \mathbf{C}$ is a given function. The following statements are then equivalent:*

(i) *there exists $\Phi \in \mathcal{S}$ such that $\Phi|S = \phi$;*

(ii) *there exists a function of positive type $\Gamma : S \times S \to \mathcal{L}(\mathbf{C}^p)$ such that*

$$1 - \phi(s)\overline{\phi(t)} = \mathrm{Tr}((I - P(s)P(t)^*)\Gamma(s,t)); \tag{10}$$

(iii) *there exist a Hilbert space K and a unitary operator $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : (\mathbf{C}^p \otimes K) \oplus \mathbf{C} \to (\mathbf{C}^q \otimes K) \oplus \mathbf{C}$ such that, if Φ_U is given by* (5), *then $\Phi_U|S = \phi$.*

Proof. (i) $\Rightarrow$ (ii). Suppose first that S is finite. Fix $\varepsilon > 0$ such that $\|s\| < 1/\varepsilon$ and $\|P(s)\| < 1/(1+\varepsilon)$ for any $s \in S$. Denote by $\mathcal{C}_\varepsilon$ the convex cone of all complex valued functions κ defined on $S \times S$ that have a representation

$$\kappa(s,t) = \mathrm{Tr}((I - (1+\varepsilon)^2 P(s)P(t)^*)\Gamma(s,t)) + \sum_{j=1}^{n} (1 - \varepsilon^2 s_j \bar{t}_j)\gamma_j(s,t), \tag{11}$$

where Γ is as in the right-hand side of formula (10), and γ_j are functions of positive type. Also, we will use the following notation in the proof: for $\psi, \psi' : S \to \mathbf{C}$, $\psi \otimes \psi' : S \times S \to \mathbf{C}$ is defined by $\psi \otimes \psi'(s,t) = \psi(s)\psi'(t)$.

Fix $s \in S$. If $\kappa \in \mathcal{C}_\varepsilon$, take a representation given by (11), and denote $G = \Gamma(s,s)$, $Y = (1+\varepsilon)P(s)$. Then $G \geq 0$, $\|Y\| < 1$, and

$$\kappa(s,s) \geq \mathrm{Tr}(I - YY^*)G = \mathrm{Tr}(I - YY^*)^{1/2} G (I - YY^*)^{1/2}.$$

Therefore

$$\begin{aligned} \|G\| &\leq \|G\|_1 = \|(I - YY^*)^{-1/2}(I - YY^*)^{1/2} G (I - YY^*)^{1/2}(I - YY^*)^{-1/2}\|_1 \\ &\leq \|(I - YY^*)^{-1/2}\|^2 \|(I - YY^*)^{1/2} G (I - YY^*)^{1/2}\|_1 \\ &\leq \frac{1}{1 - \|Y\|^2} \mathrm{Tr}(I - YY^*)^{1/2} G (I - YY^*)^{1/2} \qquad (12) \\ &\leq \frac{1}{1 - (1+\varepsilon)^2 \|P(s)\|^2} \kappa(s,s). \end{aligned}$$

We have thus obtained a bound for $\|\Gamma(s,s)\|$ that depends only on $\kappa(s,s)$ and $P(s)$. Since Γ is of positive type, one can obtain similar bounds for $\|\Gamma(s,t)\|$ depending only on the values of κ and P in s, t. A similar computation also yields bounds for $|\gamma_j(s,t)|$.

It then easily follows that $\mathcal{C}_\varepsilon$ is closed; indeed, if $\kappa_\iota \to \kappa$, then the corresponding Γ_ι's and $\gamma_{j;\iota}$'s belong to a bounded set, and hence have limit points Γ, resp. γ_j, which satisfy relation (11) with respect to κ.

It can also be seen that scalar functions of positive type on S always belong to $\mathcal{C}_\varepsilon$. Indeed, if $\kappa(s,t)$ is such a function, and Δ and $(1+\varepsilon)P$ are as stated at the end of section 4, set

$$g(s,t) = \begin{pmatrix} \Delta(s,t)\kappa(s,t) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}. \tag{13}$$

Then formula (9) implies that

$$\kappa(s,t) = \mathrm{Tr}(g(s,t) - (1+\varepsilon)^2 P(s)P(t)^* g(s,t)),$$

which is a particular case of (11).

Let us then assume that $1 - \phi \otimes \bar{\phi} \notin \mathcal{C}_\varepsilon$. It may then be separated from $\mathcal{C}_\varepsilon$ by a linear functional ℓ (on the space of functions defined on $S \times S$); more precisely, $\Re\ell|\mathcal{C}_\varepsilon \geq 0$, while $\Re\ell(1 - \phi \otimes \bar{\phi}) < 0$. Moreover, $1 - \phi \otimes \bar{\phi}$ as well as all functions in $\mathcal{C}_\varepsilon$ are selfadjoint (that is, they satisfy $\tilde{\kappa}(s,t) = \kappa(s,t)$, where $\tilde{\kappa}(s,t) = \overline{\kappa(t,s)}$), and therefore we may suppose that ℓ is real on selfadjoint functions (by replacing $\ell(\kappa)$ with $\frac{1}{2}(\ell(\kappa) + \overline{\ell(\tilde{\kappa})})$).

If ξ is any complex valued function on S, then $\kappa = \overline{\xi} \otimes \xi$ is a scalar positive definite function on $S \times S$, and consequently, as noted above, belongs to $\mathcal{C}_\varepsilon$. We can then define a prescalar product on the space of complex valued functions on S, by the formula $\langle \xi, \eta \rangle_\ell = \ell\big(\bar{\eta} \otimes \xi\big) = \ell\big(\overline{\eta(s)}\xi(t)\big)$. Denote by H the Hilbert space thus obtained, and consider, for $i = 1, \ldots, n$, the operators $X_i \in \mathcal{L}(H)$, defined by asking that their adjoints X_i^* act as $X_i^*(\xi)(s) = \bar{s}_i \xi(s)$. All X_i^* factor through the kernel of $\|\cdot\|_\ell$: indeed, $\overline{\xi(s)}\xi(t) - \varepsilon^2 s_i \overline{\xi(s)}\bar{t}_i \xi(t) \in \mathcal{C}_\varepsilon$ whence $\ell(\overline{\xi(s)}\xi(t) - \varepsilon^2 s_i \overline{\xi(s)}\bar{t}_i \xi(t)) \geq 0$, while $\|\xi\|^2 = \ell(\overline{\xi(s)}\xi(s))$ and $\|X_i^*(\xi)\|^2 = \ell(s_i \overline{\xi(s)}\bar{t}_i \xi(t))$; therefore $\varepsilon\|X_i^*\xi\| \leq \|\xi\|$.

Then $P(X)^*$ acts on H^p as left multiplication with $P(s)^*$. If $f \in H^p$, $f = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_p \end{pmatrix}$, then

$$\|f\|^2 = \ell\big(\,\mathrm{Tr}(f(t)f(s)^*)\big)$$

and

$$\|P(X)^* f\|^2 = \ell\big(\,\mathrm{Tr}(P(t)^* f(t) f(s)^* P(s))\big).$$

Taking $\Gamma(s,t) = f(t)f(s)^*$, it follows that

$$\mathrm{Tr}(f(t)f(s)^*) - \mathrm{Tr}((1+\varepsilon)^2 P(t)^* f(t) f(s)^* P(s)) \in \mathcal{C}_\varepsilon,$$

and therefore, since $\ell|\mathcal{C}_\varepsilon \geq 0$, $\|f\|^2 \geq \|(1+\varepsilon)P(X)^* f\|^2$, or $\|(1+\varepsilon)P(X)\| \leq 1$, $\|P(X)\| < 1$.

However, $\Phi(X)^*\xi = \overline{\phi}\xi$, and by taking ξ to be the constant function 1, $\ell(1 - \phi(s)\overline{\phi(t)}) < 0$ implies $\|\Phi(X)\| = \|\Phi(X)^*\| > 1$. We have thus contradicted the assumption $\Phi \in \mathcal{S}$. Hence ϕ has, for any $\varepsilon > 0$, a representation (11) with $\Gamma = \Gamma^\varepsilon$ and $\gamma_j = \gamma_j^\varepsilon$. By (12), there exist limit points Γ, resp. γ_j of Γ^ε, resp. γ_j^ε, such that

$$1 - \phi(s)\overline{\phi(t)} = \mathrm{Tr}((I - P(s)P(t)^*)\Gamma(s,t)) + \sum_{j=1}^n \gamma_j(s,t).$$

An argument similar to that used to show that positive definite scalar functions belong to $\mathcal{C}_\varepsilon$ will provide representations of type (10) for all γ_j, $j = 1, \ldots, n$. Hence ϕ has the representation (10).

The next step is to get rid of the finiteness condition imposed on S. This is easily done by considering, in the general case, the net of all finite subsets σ of S. For each of these subsets we obtain a representation of type (10), with a corresponding function of positive type Γ_σ. But it has been shown in the proof that the norms of $\Gamma_\sigma(s,t)$, for fixed $s, t \in \Omega$, are bounded independently of σ. A compactness argument then yields a (pointwise) limit point of all Γ_σ, which can be used for the desired representation.

(ii) $\Rightarrow$ (iii). Suppose that Γ has a representation given by (7). We can then write formula (10) as

$$1 - \phi(s)\overline{\phi(t)} = \mathrm{Tr}(F(t)^*F(s)) - \mathrm{Tr}(P(t)^*F(t)^*F(s)P(s))$$

or

$$\phi(s)\overline{\phi(t)} + \mathrm{Tr}(F(t)^*F(s)) = 1 + \mathrm{Tr}(P(t)^*F(t)^*F(s)P(s)). \tag{14}$$

Now, by using the results in section 4 (including formula (8)), it follows that there exists a map $F_0 : S \to (\mathbf{C}^p)' \otimes \mathcal{E}$, such that (14) can be rewritten as

$$\phi(s)\overline{\phi(t)} + \langle F_0(s), F_0(t)\rangle = 1 + \langle (P(s)' \otimes I_{\mathcal{E}})F_0(s), (P(t)' \otimes I_{\mathcal{E}})F_0(t)\rangle. \tag{15}$$

Then consider the map

$$\begin{pmatrix} (P(s)' \otimes I_{\mathcal{E}})F_0(s) \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} F_0(s) \\ \phi(s) \end{pmatrix}, \qquad s \in S,$$

defined on the corresponding subspaces of $((\mathbf{C}^q)' \otimes \mathcal{E}) \oplus \mathbf{C}$ and $((\mathbf{C}^p)' \otimes \mathcal{E}) \oplus \mathbf{C}$ respectively. It is an isometry by (15), and thus, by embedding $\mathcal{E}$ in a larger Hilbert space K, can be extended to a unitary operator

$$\begin{pmatrix} A_1 & C_1 \\ B_1 & D_1 \end{pmatrix} : ((\mathbf{C}^q)' \otimes \mathcal{E}) \oplus \mathbf{C} \to ((\mathbf{C}^p)' \otimes \mathcal{E}) \oplus \mathbf{C};$$

thus, the following relations are satisfied:

$$F_0(s) = A_1((P(s)' \otimes I_{\mathcal{E}})F_0(s)) + C_1 1, \tag{16}$$

$$\phi(s) = B_1((P(s)' \otimes I_{\mathcal{E}})F_0(s)) + D_1 1. \tag{17}$$

Since $\|(P(s)' \otimes I_{\mathcal{E}}\| < 1$, (16) implies that

$$F_0(s) = (I - A_1 P(s)' \otimes I_{\mathcal{E}})^{-1} C_1 1,$$

which can be plugged into (17) in order to obtain

$$\phi(s) = D_1 + B_1 P(s)' \otimes I_{\mathcal{E}} (I - A_1 P(s)' \otimes I_{\mathcal{E}})^{-1} C_1,$$

whence (5) follows by passing to dual spaces.

(iii) $\Rightarrow$ (i) is a consequence of Proposition 2. □

By taking $S = \Omega$, we obtain the reciprocal of Proposition 2.

Corollary 4. *If $\Phi \in \mathcal{S}$, then there exist a Hilbert space K and a unitary U such that $\Phi = \Phi_U$.*

On the other hand, by taking S finite one obtains a "usual" Nevanlinna-Pick type theorem for functions in $\mathcal{S}$.

Corollary 5. *Suppose $s_1, \dots, s_N \in \Omega$ and $w_1, \dots, w_N \in \mathbf{C}$. Then there exists $\Phi \in \mathcal{S}$ such that $\Phi(s_i) = w_i$ for all $i = 1, \dots, N$ if and only if there exists a block matrix $\Gamma = (\Gamma_{ij})_{i,j=1}^N \geq 0$, $\Gamma_{ij} \in M_p(\mathbf{C})$, such that*

$$1 - w_i \bar{w}_j = \mathrm{Tr}((I - P(s_i)P(s_j)^*)\Gamma_{ij}) \tag{18}$$

for all $i, j = 1, \dots, N$.

It may be useful to make this statement more explicit. It amounts to the existence of a nonnegative matrix with $p \times N$ rows and columns $\gamma = [\gamma_{i,s;j,t}]$, $i, j = 1, \ldots, p$, $s, t = 1, \ldots, N$, such that

$$1 - w_i \bar{w}_j = \sum_{i=1}^{p} \gamma_{i,s;i,t} - \sum_{i,j=1}^{p} \sum_{k=1}^{q} P_{ik}(s) \gamma_{i,s;j,t} \overline{P_{jk}(t)} \tag{19}$$

for all $s, t = 1, \ldots, N$.

6. Particular cases

Although the class $\mathcal{S}$ has not been studied for many domains Ω, there are a few important cases where it has already appeared.

1. The most basic case is that of the one-dimensional unit disc $\mathbf{D}$: $n = p = q = 1$, $P(z) = z$. A classical result of von Neumann then states that $\mathcal{S} = \mathbf{B}(\mathbf{D})$. Corollary 5 is then the classical Nevanlinna-Pick theorem. The representation given by formula (5) is known in system theory as the realization theorem for functions in H^∞; in the context of operator theory it is an immediate consequence of the theory of the characteristic function of a contraction ([18]).

2. We obtain the bidisc $\mathbf{D}^2 \subset \mathbf{C}^2$ by taking $n = p = q = 2$, $P(z_1, z_2) = \left(\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix}\right)$. It is a consequence of Ando's dilation theorem ([7]) that $\mathcal{S} = \mathbf{B}(\mathbf{D}^2)$. The representation formula (5) as well as the Nevanlinna-Pick theorem (Corollary 5) are proved by Agler in [1].

3. The situation is more complicated for the polydisc in more than two dimensions; that is, $\Omega = \mathbf{D}^n \subset \mathbf{C}^n$ $(n \geq 3)$ is defined by taking $p = q = n$ and

$$P(z_1, \ldots, z_n) = \begin{pmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & z_n \end{pmatrix}.$$

As a consequence of the failure of the von Neumann inequality (see [22]), $\mathcal{S}$ is a proper class of $\mathbf{B}(\mathbf{D}^n)$. It has also been introduced by Agler in [1], who has proved the corresponding results that have been a source of inspiration for the present paper ([1]; see also [2] and [11]). The results of Agler can be obtained as a consequence of Theorem 3 above. One should also note their applications in the theory of multiparametric linear scattering systems by Kalyuzhniy ([15]). Extensions to product domains are studied in [19].

4. Another interesting case is that of the n-dimensional unit ball $B_n = \{z \in \mathbf{C}^n \; ; \; |z_1|^2 + \cdots + |z_n|^2 < 1\}$. This can be obtained by taking $p = 1$, $q = n$, $P(z_1, \ldots, z_n) = \begin{pmatrix} z_1 & \ldots & z_n \end{pmatrix}$. It is a consequence of results in [12] or [6] that $\mathcal{S}$ is equal to the class of all multipliers of norm smaller than or equal to 1 of the space H^2 intensively studied by Arveson in [8]. Indeed, it is shown therein that the fractional representation (5) characterizes this class of multipliers. The above-quoted references also contain Nevanlinna-Pick analogues of Corollary 5.

5. The previous argument can also be extended to a more general setting. The main point is that a representation formula of type (5) is known to characterize contractive multipliers for a certain class of Hilbert spaces $\mathcal{H}$ of analytic functions, whose kernels k are of the so-called Nevanlinna-Pick type (see [13] or [12]). We may then obtain a similar description of $\mathcal{S}$ in case $1/k$ is a polynomial function, the domain of the functions in $\mathcal{H}$ is defined by the condition $k(z, z) > 0$, and the kernel

$1-1/k$ has bounded rank. However, the general setting of Nevanlinna-Pick kernels does not fit so nicely into our frame. In that case, a von Neumann type inequality for multipliers can be directly obtained from the lifting property (see [6] or [12]).

6. Examples 1–4 are particular cases from a larger class, namely that of classical Cartan domains of type I ([20]). These can be represented as operator unit balls of $M_{p,q}(\mathbf{C})$; thus, they correspond to the case $n = pq$ and $P(z) = z \in M_{p,q}(\mathbf{C})$ for $z = (z_{ij}) \in \mathbf{C}^n = \mathbf{C}^{pq}$. We do not know of an investigation of the class $\mathcal{S}$ in this case.

Corollary 5 yields in this case, as a condition for interpolation, the existence of a nonegative matrix with pN rows and columns $\gamma = [\gamma_{i,s;j,t}]$, $i, j = 1, \dots, p$, $s, t = 1, \dots, N$, such that

$$1 - w_i \bar{w}_j = \sum_{i=1}^{p} \gamma_{i,s;i,t} - \sum_{i,j=1}^{p} \sum_{k=1}^{q} s_{ik} \gamma_{i,s;j,t} \overline{t_{jk}}$$

for all $s, t = 1, \dots, N$.

7. Actually, other classical domains also admit representations of the type given by formula 1. (See [20] again for all references.) For Cartan domains of type II, which correspond to the unit ball in the space of symmetric matrices, we have to take $p = q$, $n = p(p+1)/2$ and, for $z \in \mathbf{C}^n$, $z = (z_{ij})$, $1 \le i \le j \le p$, $P(z) = \big(P_{ij}(z)\big) \in M_{p,q}(\mathbf{C})$, where $P_{ij}(z) = z_{ij}$ if $i \le j$ and $P_{ij}(z) = z_{ji}$ if $i > j$.

For Cartan domains of type III, which correspond to the unit ball of skew-symmetric matrices, we have to take $p = q$, $n = p(p-1)/2$ and, for $z \in \mathbf{C}^n$, $z = (z_{ij})$, $1 \le i < j \le p$, $P(z) = \big(P_{ij}(z)\big) \in M_{p,q}(\mathbf{C})$, where $P_{ij}(z) = z_{ij}$ if $i < j$, $P_{ij}(z) = -z_{ji}$ if $i > j$ and $P_{ii}(z) = 0$.

8. Domains of the type given by formula (1) are stable with respect to the simple operations of intersection and cartesian product. The following proposition, whose proof is a simple computation, makes this fact explicit.

Proposition 6. (i) *If Ω_i $(i = 1, 2)$ are defined by $P_i : \mathbf{C}^n \to M_{p_i,q_i}(\mathbf{C})$ respectively according to formula* (1)*, then $\Omega_1 \cap \Omega_2$ is defined, according to formula* (1)*, by* $P(z) = \left(\begin{smallmatrix} P_1(z) & 0 \\ 0 & P_2(z) \end{smallmatrix}\right)$.

(ii) *If Ω_i $(i = 1, 2)$ are defined by $P_i : \mathbf{C}^{n_i} \to M_{p_i,q_i}(\mathbf{C})$ respectively according to formula* (1)*, then $\Omega_1 \times \Omega_2$ is defined, according to formula* (1)*, by*

$$P(z_1, \dots, z_{n_1}, z_{n_1+1}, \dots, z_{n_1+n_2}) = \begin{pmatrix} P_1(z_1, \dots, z_{n_1}) & 0 \\ 0 & P_2(z_{n_1+1}, \dots, z_{n_1+n_2}) \end{pmatrix}.$$

We may then obtain theorems giving the characterization of the class $\mathcal{S}$ for domains that are either products or intersections of classical domains of types I, II or III.

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