

## VECTOR BUNDLES WITH INFINITELY MANY SOULS

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**ABSTRACT.** We construct the first examples of manifolds, the simplest one being  $S^3 \times S^4 \times \mathbb{R}^5$ , which admit infinitely many complete nonnegatively curved metrics with pairwise nonhomeomorphic souls.

According to the soul theorem of J. Cheeger and D. Gromoll [CG72], a complete open manifold of nonnegative sectional curvature is diffeomorphic to the total space of the normal bundle of a compact totally geodesic submanifold, called a soul. The soul is not unique but any two souls are mapped to each other by an ambient diffeomorphism inducing an isometry on the souls [Sha79]. In this note we show that the homeomorphism type of the soul generally depends on the metric; namely the following is true.

**Theorem 1.** *There exist infinitely many complete Riemannian metrics on  $S^3 \times S^4 \times \mathbb{R}^5$  with  $\sec \geq 0$  and pairwise nonhomeomorphic souls.*

The proof applies some classical techniques of geometric topology to recent examples of nonnegatively curved manifolds due to K. Grove and W. Ziller [GZ00]. As we explain below, it is much easier to produce a manifold with *finitely many* nonnegatively curved metrics having nonhomeomorphic souls, however the full power of [GZ00] is needed to get infinitely many such metrics.

Grove and Ziller [GZ00] showed that any principal  $S^3 \times S^3$ -bundle over  $S^4$  admits an  $S^3 \times S^3$ -invariant metric with  $\sec \geq 0$ . By O’Neill’s formula, all associated bundles admit metrics with  $\sec \geq 0$  which gives rise to a rich class of examples, including all sphere bundles over  $S^4$  with structure group  $SO(4)$ . Note that the souls in Theorem 1 are the total spaces of  $S^3$ -bundles over  $S^4$  with structure group  $SO(3)$ . Theorem 1 is a particular case of the following.

**Theorem 2.** *Let  $\xi$  be a rank  $n$  vector bundle over  $S^4$  with structure group  $SO(3)$ , let  $q: S \rightarrow S^4$  be a smooth  $S^{m-1}$ -bundle with structure group  $SO(3)$ , and let  $\eta$  be the  $q$ -pullback of  $\xi$ . If  $m = 4$ ,  $n > 4$ , or if  $m > 4$ ,  $n > m + 3$ , then the total space of  $\eta$  admits infinitely many complete Riemannian metrics with  $\sec \geq 0$  and pairwise nonhomeomorphic souls.*

The main topological tool used in this paper is a result of L. Siebenmann [Sie69] that generalizes the famous Masur’s theorem: any tangential homotopy equivalence of closed smooth  $n$ -manifolds is homotopic to a diffeomorphism after taking the

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product with the identity map of  $\mathbb{R}^{n+1}$ . Exotic 7-spheres are stably parallelizable, so they all become diffeomorphic after taking the product with  $\mathbb{R}^8$ , and in fact it suffices to take product with  $\mathbb{R}^5$ . A homotopy 7-sphere is called a *Milnor sphere* if it is diffeomorphic to the total space of an  $S^3$ -bundle over  $S^4$ ; it is known that 10 out of 14 homotopy 7-spheres are Milnor. According to [GZ00], any Milnor sphere carries a metric with  $\sec \geq 0$ , which leads to the following.

**Proposition 3.** *For every Milnor sphere  $\Sigma$ , the manifold  $S^7 \times \mathbb{R}^5$  has a complete Riemannian metric of  $\sec \geq 0$  with soul diffeomorphic to  $\Sigma$ .*

Gromoll and Tapp recently classified [GT01] all nonnegatively curved metrics on  $S^2 \times \mathbb{R}^2$ , and asked for a similar classification on  $S^n \times \mathbb{R}^k$ . The above proposition indicates that such a classification would be rather involved.

It would be interesting to classify the total spaces of  $S^3$ -bundles over  $S^4$  with nonzero Euler class up to tangential homotopy equivalence. Indeed, by Masur's theorem such a classification should lead to an analog of Proposition 3, with  $S^7$  replaced by any  $S^3$ -bundle  $S$  over  $S^4$  with nonzero Euler class. It is worth mentioning that nonvanishing of the Euler class implies that the homotopy type of  $S$  contains at most finitely many nondiffeomorphic  $S^3$ -bundles over  $S^4$  [Tam58] (cf. [CE00]).

It is easy to construct manifolds admitting two nonnegatively curved metrics with pairwise nonhomeomorphic souls. Here we describe two such situations where souls are lens spaces, or simply-connected homogeneous manifolds.

**Example 4.** It is well known [Coh73, pp. 96, 100] that the lens spaces  $L(7, 1)$  and  $L(7, 2)$  are homotopy equivalent, but not simply-homotopy equivalent (hence nonhomeomorphic). Orientable 3-manifolds are parallelizable, so any homotopy equivalence of  $L(7, 1)$ ,  $L(7, 2)$  is tangential, and therefore [Sie69, Theorem 2.3] we get a diffeomorphism of manifolds  $L(7, 1) \times \mathbb{R}^4$  and  $L(7, 2) \times \mathbb{R}^4$  admitting obvious product metrics of  $\sec \geq 0$  with souls  $L(7, 1) \times \{0\}$ ,  $L(7, 2) \times \{0\}$ . Since every homotopy type contains at most finitely many nonhomeomorphic lens spaces, this procedure yields only finitely many pairwise nonhomeomorphic souls on a given manifold.

**Proposition 5.** *If  $i$  is a positive integer divisible by 24, then there is a compact homogeneous space  $G/H$  homotopy equivalent but not homeomorphic to  $S^{4i-1} \times S^4$ , such that for  $n > 4i+3$ , the manifold  $G/H \times \mathbb{R}^n$  carries two complete nonnegatively curved metrics with souls diffeomorphic to  $G/H$  and  $S^{4i-1} \times S^4$ .*

The above-mentioned result of Siebenmann [Sie69, Theorem 2.2] states that if  $\xi$  is a rank  $> 2$  vector bundle over a compact manifold, and the total space  $E(\xi)$  of  $\xi$  contains a smooth compact submanifold  $S$  such that the inclusion  $S \rightarrow E(\xi)$  is a homotopy equivalence, then  $E(\xi)$  has a vector bundle structure with zero section  $S$ .

Then it easily follows [Sie69, Theorem 2.3] that if  $t$  is tangential homotopy equivalence of  $n$ -manifolds, each being the total space of a vector bundle over a compact smooth manifold of dimension less than the rank of the bundle, then  $t$  is homotopic to a diffeomorphism. The upshot is that the total space of vector bundle of sufficiently large rank often has many other vector bundle structures, perhaps with nonhomeomorphic base spaces, and this fact plays a crucial role in this note.

*Proof of Theorem 2.* In what follows we denote the total space of a vector bundle  $\zeta$  by  $E(\zeta)$ , and the associated sphere bundle by  $S(\zeta)$ . Principal  $SO(3)$ -bundles

over  $S^4$  are in one-to-one correspondence with  $\pi_4(BSO(3)) \cong \mathbb{Z}$ . Let  $P_k$  be the principal  $SO(3)$ -bundle over  $S^4$  corresponding to  $k \in \pi_4(BSO(3))$ . Let  $\xi_k^n$  be the rank  $n$  vector bundle over  $S^4$  associated with  $P_k$  via the standard inclusion  $SO(3) \rightarrow SO(n)$ .

Let  $P_{k,l}$  be the principal  $SO(3) \times SO(3)$ -bundle over  $S^4$  which is the pullback of  $P_k \times P_l$  via the diagonal map  $\Delta: S^4 \rightarrow S^4 \times S^4$ . According to [GZ00],  $P_{k,l}$  admits an  $SO(3) \times SO(3)$ -invariant metric with  $\sec \geq 0$ . Consider the  $S^{m-1} \times \mathbb{R}^n$ -bundle

$$\Delta^\#(S(\xi_k^m) \times \xi_l^n) = S^{m-1} \times_{SO(3) \times 1} P_{k,l} \times_{1 \times SO(3)} \mathbb{R}^n$$

associated with  $P_{k,l}$ . The total space of  $\Delta^\#(S(\xi_k^m) \times \xi_l^n)$  is also the total space of a rank  $n$  vector bundle over  $S(\xi_k^m)$  which we denote by  $\eta_{k,l}^{m,n}$ . Note that  $\eta_{k,l}^{m,n}$  is the pullback of  $\xi_l^n$  via the projection  $q_k^m: S(\xi_k^m) \rightarrow S^4$ . By O'Neill's formula for submersions,  $E(\eta_{k,l}^{m,n})$  carries a complete metric with  $\sec \geq 0$  with zero section being a soul.

First, note that  $S(\xi_k^m)$  is fiber homotopy equivalent to  $S(\xi_i^m)$  if  $k \equiv i \pmod{12}$ . Indeed,  $S^{m-1}$ -fibrations over  $S^4$  are classified, up to fiber homotopy equivalence, by  $\pi_3(SG_m)$  where  $SG_m$  is the space of orientation-preserving self-homotopy equivalences of  $S^{m-1}$ . The fibrations  $S(\xi_k^m)$  are classified by the image of

$$\phi: \pi_3(SO(3)) \rightarrow \pi_3(SO(m)) \rightarrow \pi_3(SG_m).$$

Since  $m \geq 4$ ,  $\phi$  factors as  $\pi_3(SO(3)) \rightarrow \pi_3(SF_3) \rightarrow \pi_3(SG_4) \rightarrow \pi_3(SG_m)$  where  $SF_3$  is the space of base-point-preserving elements of  $SG_4$ . It is well known that  $SF_3$  is the identity component of the loop space  $\Omega^3 S^3$  [MM79, Chapter 3], hence  $\pi_3(SF_3) \cong \pi_6(S^3)$ . Thus,  $\phi$  factors through the  $J$ -homomorphism  $\mathbb{Z} \cong \pi_3(SO(3)) \rightarrow \pi_6(S^3) \cong \mathbb{Z}_{12}$ , and the result follows.

Second, show that  $S(\xi_k^m)$  is homeomorphic to  $S(\xi_i^m)$  iff  $k = \pm l$  iff  $S(\xi_k^m)$  is diffeomorphic to  $S(\xi_l^m)$ . Indeed, the tangent bundle of  $S(\xi_k^m)$  is stably isomorphic to the  $q_k^m$ -pullback of  $\xi_k^m$ , because  $S(\xi_k^m)$  is a two-sided hypersurface in  $E(\xi_k^m)$ . Since  $\xi_k^m$  is an  $SO(3)$ -bundle, the first Pontrjagin class of  $\xi_k^m$  is the  $\pm 4k$ -multiple of a generator of  $H^4(S^4, \mathbb{Z})$  (cf. [Mil56]). Also  $S(\xi_k^m)$  has a section so that  $q_k^m$  induces an isomorphism on the 4th cohomology, hence  $p_1(TS(\xi_k^m))$  is the  $\pm 4k$ -multiple of a generator. By the topological invariance of rational Pontrjagin classes,  $S(\xi_k^m)$  is not homeomorphic to  $S(\xi_i^m)$  unless  $k = \pm i$ . Finally,  $P_k$  is the pullback of  $P_{-k}$  via an orientation-reversing self-diffeomorphism of  $S^4$ , so  $S(\xi_k^m)$  and  $S(\xi_{-k}^m)$  are diffeomorphic.

Third, note that  $E(\eta_{k,i}^{m,n})$  and  $E(\eta_{i,j}^{m,n})$  are tangentially homotopy equivalent provided  $k \equiv i \pmod{12}$ , and  $k+l = i+j$ . Indeed, fix  $k, l, i, j$  with these properties. The tangent bundle to  $E(\eta_{k,i}^{m,n})$  is determined by its restriction to the zero section. This restriction is the  $q_k^m$ -pullback of  $\xi_k^m \oplus \xi_l^n$  for any  $k, l$ . The images of  $\xi_k^m, \xi_l^n$  under the homomorphism

$$\pi_4(BSO(3)) \rightarrow \pi_4(BSO)$$

add up to  $k+l$ , and the addition in  $\pi_4(BSO)$  is given by the Whitney sum  $\oplus$ . Thus  $k+l = i+j$  implies that  $\xi_k^m \oplus \xi_l^n$  and  $\xi_i^m \oplus \xi_j^n$  are stably isomorphic. Since  $k \equiv i \pmod{12}$  there is a fiber homotopy equivalence  $g: S(\xi_k^m) \rightarrow S(\xi_i^m)$ , so that  $g \circ q_i^m$  is homotopic to  $q_k^m$ . Therefore,  $g$  induces a tangential homotopy equivalence  $t: E(\eta_{k,i}^{m,n}) \rightarrow E(\eta_{i,j}^{m,n})$ , which is the composition of the projection of  $\eta_{k,i}^{m,n}$ , followed by  $g$ , and then by the zero section of  $\eta_{i,j}^{m,n}$ .

Next we show that  $E(\eta_{k,l}^{m,n})$  and  $E(\eta_{i,j}^{m,n})$  are diffeomorphic if  $k \equiv i \pmod{12}$ , and  $k+l = i+j$ . By Haefliger's embedding theorem [Hae61], the restriction of  $t$  to the zero section of  $\eta_{k,l}^{m,n}$  is homotopic to a smooth embedding  $f$  because  $2n \geq m+6$ , which means we are in metastable range. Since  $n \geq 3$ , [Sie69, Theorem 2.2] implies that  $E(\eta_{i,j}^{m,n})$  has a vector bundle structure with zero section  $f$ . Since  $t$  is tangential,  $\eta_{k,l}^{m,n}$  is stably isomorphic to  $\nu_f$  which is the normal bundle to  $f$ .

In fact, our assumptions on  $n, m$  imply that  $\eta_{k,l}^{m,n}$  and  $\nu_f$  are isomorphic. Indeed, if  $m > 4$ ,  $n > m+3 = \dim(S(\xi_k^m))$ , then we are in stable range, hence  $\eta_{k,l}^{m,n} \cong \nu_f$ . If  $m = 4$ ,  $n > 4$ , we apply obstruction theory comparing  $\nu_f$ ,  $\eta_{k,l}^{m,n}$ , which are thought of as classifying maps from  $S(\xi_k^m)$  to  $BSO(n)$ . Since  $S(\xi_k^m)$  has a section, it can be obtained by attaching a 7-cell to  $S^3 \vee S^4$ . The bundles  $\nu_f$ ,  $\eta_{k,l}^{m,n}$  are isomorphic on the 6-skeleton  $S^3 \vee S^4$ , because they are stably isomorphic and  $n > 4 = \dim(S^3 \vee S^4)$ . Comparing  $\nu_f$ ,  $\eta_{k,l}^{m,n}$  on the 7-cell, we get a map  $S^7 \rightarrow BSO(n)$  which is nullhomotopic since  $\pi_7(BSO(n)) = 0$  if  $n > 4$  [Mim95, page 970]). Thus,  $\nu_f$ ,  $\eta_{k,l}^{m,n}$  are isomorphic, and hence  $E(\eta_{k,l}^{m,n})$  and  $E(\eta_{i,j}^{m,n})$  are diffeomorphic.

To summarize, each manifold  $E(\eta_{k,l}^{m,n})$  has infinitely many vector bundle structures  $\eta_{i,j}^{m,n}$  with base manifolds  $S(\xi_i^m)$  for any  $i, j$  satisfying  $k \equiv i \pmod{12}$ , and  $k+l = i+j$ , and the proof is complete.  $\square$

*Proof of Proposition 3.* Let  $\Sigma$  be a homotopy 7-sphere of  $\sec \geq 0$  [GM74, GZ00]. The product metric on  $\Sigma \times \mathbb{R}^n$  has  $\sec \geq 0$  with a soul  $\Sigma \times \{0\}$ . By [Hae61] any homotopy equivalence  $S^7 \rightarrow \Sigma \times \mathbb{R}^n$  is homotopic to a smooth embedding if  $n \geq 5$ , so  $\Sigma \times \mathbb{R}^n$  gets a structure of a vector bundle over  $S^7$  which is necessarily trivial since  $\pi_7(BSO(n)) = 0$  for  $n \geq 5$ . Thus,  $\Sigma \times \mathbb{R}^5$  is diffeomorphic to  $S^7 \times \mathbb{R}^5$ , as promised.  $\square$

*Proof of Proposition 5.* Kamerich [Kam77, page 116] (cf. [Oni94, page 275]) showed that if  $i$  is a positive integer divisible by 24, then there is a compact homogeneous space  $G/H$  homotopy equivalent but not homeomorphic to  $S^{4i-1} \times S^4$ . Here  $G = Sp(i) \times Sp(2)$  and  $H = Sp(i-1) \times Sp(1) \times Sp(1)$ , where  $Sp(i-1)$  is embedded in  $Sp(i)$  in the standard way, the first  $Sp(1)$  is embedded into  $Sp(i) \times Sp(2)$  diagonally so that a quaternion goes to the last diagonal entry of the matrix in  $Sp(i)$ , and also to the first diagonal entry in  $Sp(2)$ , while the second  $Sp(1)$  goes into the last diagonal entry of  $Sp(2)$ .

Since  $n > 4i+3$ , the homotopy equivalence  $S^{4i-1} \times S^4 \rightarrow G/H \times \mathbb{R}^n$  is homotopic to a smooth embedding  $f$ . By [Sie69, Theorem 2.2],  $G/H \times \mathbb{R}^n$  admits an  $\mathbb{R}^n$ -bundle structure with zero section  $f$ .

Note that any  $\mathbb{R}^n$ -bundle  $\xi$  over  $S^{4i-1} \times S^4$  is the pullback of a  $\mathbb{R}^n$ -bundle over  $S^4$  via the projection  $S^{4i-1} \times S^4 \rightarrow S^4$ . This is proved by obstruction theory for maps  $S^{4i-1} \times S^4 \rightarrow BSO$  by comparing  $\xi$  with the bundle  $\xi_4$  obtained by pullback  $\xi$  to  $S^4$  via an inclusion  $S^4 \rightarrow S^{4i-1} \times S^4$ , and then pullback it back to  $S^{4i-1} \times S^4$  via the projection  $S^{4i-1} \times S^4 \rightarrow S^4$ . Since  $\pi_{4i-1}(BSO) = 0$ ,  $\xi$  and  $\xi_4$  agree on  $S^{4i-1} \vee S^4$ , and they are homotopic on the top  $4i+3$ -cell as  $\pi_{4i+3}(BSO) = 0$ .

Since each vector bundle over  $S^4$  carries  $\sec \geq 0$  with zero section being a soul [GZ00], so does the product of the bundle and  $S^{4i-1}$ . Thus  $G/H \times \mathbb{R}^n$  gets a metric with  $\sec \geq 0$  and soul  $S^{4i-1} \times S^4$ . On the other hand,  $G/H \times \mathbb{R}^n$  has the product metric with soul  $G/H$ .  $\square$

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## REFERENCES

- [CE00] D. Crowley and C. M. Escher, *A classification of  $S^3$ -bundles over  $S^4$* , to appear in *Differential Geom. Appl.*
- [CG72] J. Cheeger and D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. **96** (1972), 413–443. MR **46**:8121
- [Coh73] M. M. Cohen, *A course in simple-homotopy theory*, Springer-Verlag, 1973. MR **50**:14762
- [GM74] D. Gromoll and W. Meyer, *An exotic sphere with nonnegative sectional curvature*, Ann. of Math. **100** (1974), 401–406. MR **51**:11347
- [GT01] D. Gromoll and K. Tapp, *Nonnegatively curved metrics on  $S^2 \times \mathbb{R}^2$* , to appear in *Geom. Dedicata*.
- [GZ00] K. Grove and W. Ziller, *Curvature and symmetry of Milnor spheres*, Ann. Math. **151** (2000), 1–36. MR **2000i**:53047
- [Hae61] A. Haefliger, *Plongements différentiables de variétés dans variétés*, Comment. Math. Helv. **36** (1961), 47–82.
- [Kam77] B. N. P. Kamerich, *Transitive transformation groups of products of two spheres*, Ph.D. thesis, Catholic University of Nijmegen, 1977.
- [Mil56] J. Milnor, *On manifolds homeomorphic to the 7-sphere*, Ann. of Math. **64** (1956), 399–405. MR **18**:498d
- [Mim95] M. Mimura, *Homotopy theory of Lie groups*, Handbook of algebraic topology, North-Holland, 1995, pp. 951–991. MR **97c**:57038
- [MM79] I. Madsen and R. J. Milgram, *The classifying spaces for surgery and cobordism of manifolds*, Princeton University Press, 1979. MR **81b**:57014
- [Oni94] A. L. Onishchik, *Topology of transitive transformation groups*, Johann Ambrosius Barth Verlag GmbH, 1994. MR **95e**:57058
- [Sha79] V. A. Sharafutdinov, *Convex sets in a manifold of nonnegative curvature*, Math. Notes **26** (1979), no. 1–2, 556–560. MR **81d**:53039
- [Sie69] L. Siebenmann, *On detecting open collars*, Trans. Amer. Math. Soc. **142** (1969), 201–227. MR **39**:7605
- [Tam58] I. Tamura, *Homeomorphy classification of total spaces of sphere bundles over spheres*, J. Math. Soc. Japan **10** (1958), 29–43. MR **20**:2717

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