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ON STABLE EQUIVALENCES OF MORITA TYPE FOR FINITE DIMENSIONAL ALGEBRAS

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Abstract. In this paper, we assume that algebras are finite dimensional algebras with 1 over a fixed field k and modules over an algebra are finitely generated left unitary modules. Let A and B be two algebras (where k is a splitting field for A and B) with no semisimple summands. If two bimodules ${}_AM_B$ and ${}_BN_A$ induce a stable equivalence of Morita type between A and B, and if $N\otimes_A -$ maps any simple A-module to a simple B-module, then $N\otimes_A -$ is a Morita equivalence. This conclusion generalizes Linckelmann's result for selfinjective algebras. Our proof here is based on the construction of almost split sequences.

1. Introduction

Given two finite dimensional k-algebras A and B, suppose that M is an A-B-bimodule and N is a B-A-bimodule. Following [2] we say that M and N induce a stable equivalence of Morita type between A and B if M and N are projective both as left and right modules, and if

$$M\otimes_B N\cong A\oplus P$$

as A-A-bimodules, where P is a projective A-A-bimodule, and

$$N\otimes_A M\cong B\oplus Q$$

as B-B-bimodules, where Q is a projective B-B-bimodule.

Since the projective A-A-bimodule P tensoring any A-module is a projective A-module and since the projective B-B-bimodule Q tensoring any B-module is a projective B-module (see Lemma 2.1), the functor $N\otimes_A -$ (defined by the above bimodule N) induces a stable equivalence between the stable categories $\underline{\mathrm{mod}}A$ and $\underline{\mathrm{mod}}B$, and $M\otimes_B -$ induces its quasi-inverse. Note that a stable equivalence $\alpha:\underline{mod}A\longrightarrow\underline{mod}B$ gives a one-to-one correspondence between the isomorphism classes of indecomposable non-projective modules in mod$A$ and mod$B$ (see [1, Proposition 1.1, p. 336]).

Important examples of stable equivalences of Morita type are selfinjective algebras which are derived equivalent [7, Corollary 5.5]. Linckelmann in [5] proved that, for two selfinjective algebras A and B having no simple projective modules, if there is an exact functor F which induces a stable equivalence between A and

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B (in fact, F induces a stable equivalence of Morita type by Richard [8, Theorem 3.2]), and if F maps any simple A-module to a simple B-module, then F is a Morita equivalence. Using this result, Linckelmann classified stable equivalences of Morita type between p-groups over a complete discrete valuation ring and proved that a stable equivalence of Morita type between the group algebras of two p-groups forces these p-groups to be isomorphic (see [5], [6]). Recently, Xi studied the representation dimension in [9] and proved in [10] that two finite dimensional algebras which are stably equivalent of Morita type have the same representation dimension. This implies that stable equivalences of Morita type between finite dimensional algebras preserve some nice properties. In this paper, we shall generalize Linckelmann's result ([5, Proposition 2.5]) to arbitrary finite dimensional algebras with no semisimple summands. Namely, we obtain the following theorem.

Theorem 1.1. *Let A and B be finite dimensional k-algebras (where k is a splitting field for A and B) with no semisimple summands. If two bimodules ${}_AM_B$ and ${}_BN_A$ induce a stable equivalence of Morita type between A and B, and if $N \otimes_A -$ maps any simple A-module to a simple B-module, then $N \otimes_A -$ is a Morita equivalence.*

2. Preliminaries

For an algebra A, we denote by mod$A$ and by $\underline{\text{mod}}A$ the category of finitely generated left A-modules and its stable category, respectively. Note that mod$A$ is a Krull-Schmidt category. For X in mod$A$, we define the top of X by $\text{top}(X) = X/\text{rad}(X)$, where $\text{rad}(X)$ is the radical of X. Recall that an A-module X is called a generator if the regular module ${}_AA$ is a direct summand of a finite direct sum of X.

Suppose that two algebras A and B are stably equivalent of Morita type. We can define functors $T_M : modB \longrightarrow modA$ by $X \longmapsto M \otimes_B X$ and $T_N : modA \longrightarrow modB$ by $Y \longmapsto N \otimes_A Y$. Similarly, we have the functors T_P and T_Q. Related to these functors, we have the following two lemmas.

Lemma 2.1 (see [10, Theorem 4.1])**.** (1) *T_M, T_N, T_P and T_Q are exact functors.*

(2) *$T_M \circ T_N \longrightarrow id_{modA} \oplus T_P$ and $T_N \circ T_M \longrightarrow id_{modB} \oplus T_Q$ are natural isomorphisms.*

(3) *The images of T_P and T_Q consist of projective modules.* □

Lemma 2.2. *The bimodules ${}_AM_B$ and ${}_BN_A$ are projective generators both as left and right modules. Therefore T_M and T_N are faithful functors.*

Proof. We prove that ${}_AM$ is a generator. Since ${}_BN$ is projective, there is a natural number n such that ${}_BN$ is a direct summand of ${}_BB^n$. It follows that ${}_AM \otimes_B N \cong_A (A \oplus P)$ is a direct summand of ${}_AM \otimes_B B^n \cong_A M^n$. Therefore ${}_AA$ is a direct summand of ${}_AM^n$; this implies that ${}_AM$ is a generator in mod$A$. □

For an A-module X, we have a unique (up to isomorphism) decomposition $X = X_1 \oplus X'$, where X_1 has no nonzero projective summands and X' is projective. We call X_1 the non-projective part of X. The following lemma is obvious.

Lemma 2.3. *If $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ is exact in modA, then there is an exact sequence*

$$0 \longrightarrow X \xrightarrow{f_1} Y_0 \xrightarrow{g_1} Z_1 \longrightarrow 0,$$

where $Z \cong Z_1 \oplus Z'$, $Y \cong Y_0 \oplus Z'$, $f = \left[\begin{smallmatrix} f_1 \\ 0 \end{smallmatrix}\right]$, $g = \left[\begin{smallmatrix} g_1 & g_2 \\ 0 & g_3 \end{smallmatrix}\right]$, and g_3 is an isomorphism. Moreover, g is a split epimorphism if and only if g_1 is a split epimorphism. $\square$

Lemma 2.4. *Let $0 \longrightarrow X \xrightarrow{i} Y \longrightarrow Z \longrightarrow 0$ be an exact sequence in modA. If X is a simple module and Z has simple top, then either Y has simple top or $Y \cong X \oplus Z$.*

Proof. If $i(X)$ lies in $\mathrm{rad}(Y)$, then $\mathrm{top}(Y) = \mathrm{top}(Z)$; therefore Y has simple top.

If $i(X)$ is not in $\mathrm{rad}(Y)$, then i induces an isomorphism $X \longrightarrow S$ which we also denote by i, where S is a direct summand of $\mathrm{top}(Y)$. We define $j : Y \longrightarrow X$ to be the composition $Y \xrightarrow{\rho} S \xrightarrow{i^{-1}} X$ where ρ is the canonical projection. Then we have $j \circ i = id_X$. This implies that i is split, and therefore $Y \cong X \oplus Z$. $\square$

Let A be a finite dimensional algebra. Suppose that $\{S_1, \ldots, S_n\}$ is a complete set of non-isomorphic simple A-modules and $\{P_1, \ldots, P_n\}$ is the corresponding projective covers. Recall that the Cartan matrix $C(A)$ of the algebra A is an $n \times n$-matrix, with its i-j-entry given by the number of composition factors of P_j which are isomorphic to S_i.

The following lemma is well-known (see, for example, [3, §54.16]).

Lemma 2.5. *Let A be a finite dimensional k-algebra where k is a splitting field for A, and let X be an A-module. Then for any $1 \leq i \leq n$, the number of composition factors of X which are isomorphic to S_i is $dim_k Hom(P_i, X)$.* $\square$

Remark. k is a splitting field for k-algebra A if and only if every A-endomorphism ring of simple A-module is isomorphic to the base field k. In particular, any algebraically closed field k is a splitting field for algebras over k (see [3, §29]).

3. Proof of Theorem 1.1

In this section, we shall give a proof of the generalization of Linckelmann's theorem. Our proof here is based on the construction of almost split sequences. For the basic theory of almost split sequences, we refer to [1].

Proposition 3.1. *Let A and B be finite dimensional k-algebras with no semisimple summands, and let $\{S_1, \ldots, S_n\}$ be a complete set of non-isomorphic simple A-modules and $\{P_1, \ldots, P_n\}$ be the corresponding projective covers. If two bimodules ${}_AM_B$ and ${}_BN_A$ induce a stable equivalence of Morita type between A and B, and if $N \otimes_A -$ maps any simple A-module to a simple B-module, then we have the following:*

(1) $\{T_N(S_i) | i = 1, \ldots, n\}$ is a complete set of non-isomorphic simple B-modules;

(2) $\{T_N(P_i) | i = 1, \ldots, n\}$ is a complete set of non-isomorphic indecomposable projective B-modules.

Proof. (1) First, we show that every projective B-module E lies in

$$add(\bigoplus_{i=1}^{n} T_N(P_i)).$$

Since $T_M(E)$ is a projective A-module, we have $T_M(E) \in add(\bigoplus_{i=1}^n P_i)$. It follows from $T_N \circ T_M(E) \cong E \oplus T_Q(E) \in add(\bigoplus_{i=1}^n T_N(P_i))$ that E lies in $add(\bigoplus_{i=1}^n T_N(P_i))$. Since all composition factors of $T_N(P_i) (1 \leq i \leq n)$ occur in $\{T_N(S_i) | i = 1, \ldots, n\}$, we know that $\{T_N(S_i)|i=1,\dots,n\}$ contains all isomorphism classes of simple B-modules. To finish the proof we need to show that $T_N(S_i)\ncong T_N(S_j)$ for all $i\neq j$. It suffices to prove this when S_i and S_j are projective modules. Note that S_i and S_j are non-injective modules since A has no semisimple summands. By [1, Proposition 2.6, p.151], we have an almost split sequence $0\longrightarrow S_i\xrightarrow{f}T_i\xrightarrow{g}TrDS_i\longrightarrow 0$, where T_i is projective. Applying T_N we get an exact sequence $0\longrightarrow T_N(S_i)\xrightarrow{T_N(f)}T_N(T_i)\xrightarrow{T_N(g)}T_N(TrDS_i)\longrightarrow 0$ in mod$B$. By Lemma 2.3, we get an exact sequence

$$(*)\qquad 0\longrightarrow T_N(S_i)\xrightarrow{T_N(f)_1}T_N(T_i)_0\xrightarrow{T_N(g)_1}T_N(TrDS_i)_1\longrightarrow 0,$$

where $T_N(TrDS_i)_1$ is the non-projective part of $T_N(TrDS_i)$, $T_N(TrDS_i)=T_N(TrDS_i)_1\oplus T_N(TrDS_i)'$, $T_N(T_i)\cong T_N(T_i)_0\oplus T_N(TrDS_i)'$, $T_N(f)=\begin{bmatrix}T_N(f)_1\\0\end{bmatrix}$, and $T_N(g)=\begin{bmatrix}T_N(g)_1 & T_N(g)_2\\0 & T_N(g)_3\end{bmatrix}$. Since $T_N(S_i)$ is simple, $T_N(g)_1$ is a projective cover. We want to show that $(*)$ is an almost split sequence.

Clearly, there is an almost split sequence

$$(**)\qquad 0\longrightarrow X\longrightarrow E\xrightarrow{h}T_N(TrDS_i)_1\longrightarrow 0.$$

We claim that E is projective. Assume that E is not projective. Write $E=E_1\oplus E'$, where $E_1\neq 0$ is the non-projective part of E. Since $T_N(TrDS_i)_1$ is indecomposable non-projective, and since $E=E_1\oplus E'\longrightarrow T_N(TrDS_i)_1$ is a minimal right almost split morphism, there exists a morphism $F\longrightarrow T_M(T_N(TrDS_i)_1)_1$ with F projective in mod$A$ such that $T_M(E_1)_1\oplus F\longrightarrow TrDS_i$ is a minimal right almost split morphism ([1, Proposition 1.3, p.337]), where $T_M(E_1)_1\neq 0$ is the non-projective part of $T_M(E_1)$, and $T_M(T_N(TrDS_i)_1)_1\cong TrDS_i$ is the non-projective part of $T_M(T_N(TrDS_i)_1)$. But $T_i\longrightarrow TrDS_i$ is a minimal right almost split morphism with T_i projective; this contradicts the uniqueness of minimal right almost split morphism! Since E must be projective, it follows from [1, Theorem 3.3, p.154] that h is a projective cover. Hence the exact sequences $(*)$ and $(**)$ are isomorphic. This implies that $(*)$ is also an almost split sequence.

Similarly, we have an almost split sequence

$$0\longrightarrow T_N(S_j)\longrightarrow T_N(T_j)_0\longrightarrow T_N(TrDS_j)_1\longrightarrow 0,$$

where $T_N(TrDS_j)_1$ is the non-projective part of $T_N(TrDS_j)$. Suppose that $T_N(S_i)\cong T_N(S_j)$. Then $T_N(TrDS_i)_1\cong T_N(TrDS_j)_1$ by the basic properties of almost split sequences. But $TrDS_i$ and $TrDS_j$ are non-isomorphic, indecomposable non-projective modules. This contradicts the fact that T_N induces a stable equivalence, and therefore (1) follows.

(2) By the previous proof, we only need to show that $T_N(P_i)$ is indecomposable for all $1\leq i\leq n$. In fact, we shall prove the following more general result: if an A-module X has simple top, then the B-module $T_N(X)$ also has simple top. We prove this by induction on the length of X. For $l(X)=1$, the module X is simple, and $T_N(X)$ is simple by assumption. For $l(X)=m>1$, take an exact sequence $0\longrightarrow S\longrightarrow X\longrightarrow X/S\longrightarrow 0$, where S is a simple submodule of X and therefore X/S has simple top. Applying T_N we get an exact sequence $0\longrightarrow T_N(S)\longrightarrow T_N(X)\longrightarrow T_N(X/S)\longrightarrow 0$, where $T_N(S)$ is simple, and $T_N(X/S)$ has simple top by induction. By Lemma 2.4, either $T_N(X)$ has simple top or $T_N(X)\cong T_N(S)\oplus T_N(X/S)$. Suppose that $T_N(X)\cong T_N(S)\oplus T_N(X/S)$. Applying

T_M we have $X \cong S \oplus X/S$ since X/S is an indecomposable non-projective module. This contradicts the indecomposability of X, and completes our proof. $\square$

Proof of Theorem 1.1. By Lemma 2.2, T_N is a faithful functor and induces a monomorphism between algebras: $End_A(A) \longrightarrow End_B(T_N(A))$. By Proposition 3.1, T_N induces a bijection between the sets of isomorphism classes of indecomposable projective modules over A and B. On the other hand, T_N is an exact functor which gives a one-to-one correspondence between the sets of isomorphism classes of simple modules over A and B. It follows that T_N preserves the Cartan matrix. By Lemma 2.5, Cartan matrix is given by k-dimensions of homomorphism spaces between indecomposable projective modules. Therefore we have

$$dim_k Hom_A(P_i, P_j) = dim_k Hom_B(T_N(P_i), T_N(P_j))$$

for all $1 \leq i, j \leq n$. Assume that $A \cong \bigoplus_{i=1}^n P_i^{m_i}$. Then we have

$$dim_k End_A(A) = \sum_{i,j=1}^n dim_k Hom_A(P_i, P_j) m_i m_j$$

and

$$dim_k End_B(T_N(A)) = \sum_{i,j=1}^n dim_k Hom_B(T_N(P_i), T_N(P_j)) m_i m_j \ .$$

Thus $End_A(A) \longrightarrow End_B(T_N(A))$ is an isomorphism. Since $A \cong End_A(A)^{op}$ and $T_N(A) \cong N$ is a projective generator as B-module, we know that $A \cong End_B(N)^{op}$. Hence T_N is a Morita equivalence between A and B by the Morita theorem (see, for example, [4, Theorem 8.4.5]). $\square$

Remark. If k is a perfect field, then by [3, §54.19] we know that the condition k is a splitting field for A and B can be weakened as follows: $dim_k End_A(S) = dim_k End_B(T_N(S))$ for any simple A-module S.

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References

[1] M. Auslander, I. Reiten and S. O. Smalø, Representation theory of Artin algebras, Cambridge University Press, 1995. MR **96c:**16015

[2] M. Broué, Equivalences of blocks of group algebras, Finite dimensional algebras and related topics, V. Dlab and L. L. Scott (eds.), Kluwer, 1994, 1-26. MR **97c:**20004

[3] C. W. Curtis and I. Reiner, Representation theory of finite groups and associative algebras, Wiley-Interscience, New York, 1962. MR **26:**2519; reprint MR **90g:**16001

[4] Y. A. Drozd and V. V. Kirichenko, Finite dimensional algebras, Springer-Verlag, Berlin, 1994. MR **95i:**16001

[5] M. Linckelmann, Stable equivalences of Morita type for selfinjective algebras and p-groups, Math. Zeit. 223(1996), 87-100.

[6] M. Linckelmann, On stable equivalences of Morita type, LNM 1685, Derived equivalences for group rings, 1998, 221-232.

[7] J. Rickard, Derived equivalences as derived functors, J. London Math. Soc. 43(1991), 37-48. MR **92b:**16043

[8] J. Rickard, Some recent advances in modular representation theory, CMS Conf. Proc. 23(1998), 157-178. MR **99h:**20011

[9] C. C. Xi, On the representation dimension of finite dimensional algebras, J. of Algebra 226, no. 1(2000), 332-346. MR **2001d:**16027

[10] C. C. Xi, Representation dimension and quasi-hereditary algebras, Adv. in Math. 168(2002), 193-212. CMP 2002:15

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