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K-THEORY OF SG-PSEUDO-DIFFERENTIAL ALGEBRAS

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ABSTRACT. We are concerned with the so-called SG-pseudo-differential calculus. We describe the spectrum of the unital and commutative C^*-algebra given by the norm closure of the space of 0-order pseudo-differential operators modulo compact operators; other related algebras are also considered. Finally, their K-theory is computed.

1. Introduction

In this note we consider $*$-algebras of pseudo-differential operators in $\mathbb{R}^n$ whose symbols satisfy estimates of product type, namely the so-called SG-operators. In Hörmander's terminology, they correspond to the symbol classes $S(m,g)$ with weight function $m(x,\xi) = \langle x\rangle^{\mu}\langle\xi\rangle^{m}$ and slowly varying metric $g_{x,\xi} = \langle x\rangle^{-2}|dx|^2 + \langle\xi\rangle^{-2}|d\xi|^2$; see [8], Chapter XVIII. These operator algebras were studied by Shubin [19], Parenti [16], Feygin [6], Grushin [7], Cordes [3], [4], Schrohe [17], Schulze [18], and Coriasco [5]. A different approach, due to Melrose [12], [13] (see also Melrose and Zworski [14]), highlights strict links between these symbol classes and the so-called radial compactification of $\mathbb{R}^n$. The starting point, as we shall recall in Section 2, is a useful geometric interpretation of classical symbols, namely those symbols which admit an asymptotic expansion in homogeneous terms. We shall adopt this point of view to investigate the spectrum and the K-theory of SG-classical operator algebras (cf. Schulze [18]), essentially following the approach developed in [15] for Melrose's b-calculus; see also the paper by Lauter, Mothubert and Nistor [9].

Precisely, we know that the $*$-algebra $L^{0,0}_{\rm cl}(\mathbb{R}^n)$ of SG-classical 0-order operators is contained in the C^*-algebras $\mathbb{B}$ of bounded operators in $L^2(\mathbb{R}^n)$. Its norm closure $\mathbb{A}^{0,0} = \overline{L^{0,0}_{\rm cl}(\mathbb{R}^n)}$ is then a C^*-algebra. Among other algebras, we shall mostly be concerned with the quotients $\mathbb{A}^{0,0}/\mathbb{K}$, where $\mathbb{K}$ denotes the ideal of compact operators in $L^2(\mathbb{R}^n)$. The calculus says that this unital algebra is commutative, so that by Gelfand-Naimark's Theorem it is isomorphic to the C^*-algebra of continuous functions on a compact space X, *the spectrum* of the algebra (that is, the space of all multiplicative linear continuous functionals $\mathbb{A}^{0,0}/\mathbb{K} \to \mathbb{C}$ with the w^*-topology). Our main result states the homeomorphism $X \simeq \mathbb{S}^{2n-1}$ (see Theorem 3.5 below). Notice that the existence of such a homeomorphism would follow immediately for classical operators in Shubin's classes $G^0_{\rm cl}$ (see [20]) because in that case the symbol

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map extends to a continuous map $\overline{G^0_{\rm cl}} \to \mathcal{C}(\mathbb{S}^{2n-1})$ with kernel $\mathbb{K}$. In our case, we take advantage of the exactness of the sequence induced from the joint symbol map $\sigma^0_\psi \oplus \sigma^0_{\rm e}$ (here σ^0_ψ is the 0-order interior symbol and $\sigma^0_{\rm e}$ is the 0-order exit symbol), and the above result is then shown to follow from some gluing arguments.

Finally, we compute the K-theory groups of the algebra $\mathbb{A}^{0,0}$ (see Theorem 3.6) and of some quotient algebras, as listed in Table 1.

After completing the present paper, we were informed of the recent contribution of Lauter and Moroianu [10], which obtain results corresponding to ours in the frame of the double-edge pseudo-differential calculus on manifolds with fibered boundaries. For further references to the existing literature concerning this subject we refer to the bibliography of the forthcoming paper by Melo, Nest and Schrohe [11].

2. The pseudo-differential operator algebras

Let $\mathbb{S}^n_+ = \mathbb{S}^n \cap \{x = (x', x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \geq 0\}$ and consider the so-called radial compactification map

$$\mathrm{RC} : \mathbb{R}^n \to \mathbb{S}^n_+, \quad z \mapsto (z/\langle z\rangle, 1/\langle z\rangle). \tag{2.1}$$

We set $\mathrm{RC}^2 = \mathrm{RC} \times \mathrm{RC} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{S}^n_+ \times \mathbb{S}^n_+$. Furthermore, we fix a function $\rho \in \mathcal{C}^\infty(\mathbb{S}^n_+)$ positive away from $\partial\mathbb{S}^n_+$, and such that $(\mathrm{RC}^*\rho)(z) = |z|$ and for large $|z|$. Then $\rho_\psi := 1 \otimes \rho$ and $\rho_{\rm e} := \rho \otimes 1$ are boundary defining functions for the boundary hypersurfaces $\mathbb{S}^n_+ \times \mathbb{S}^{n-1}$ and $\mathbb{S}^{n-1} \times \mathbb{S}^n_+$ respectively.

Now we can define the symbol classes we shall consider and related operators.

Definition 2.1. Let m, μ be real numbers. We define the class of SG-classical (or poli-homogeneous) symbols of order (m, μ) by

$$S^{m,\mu}_{{\rm cl}(\xi,x)}(\mathbb{R}^n \times \mathbb{R}^n) := (\mathrm{RC}^2)^*(\rho_\psi^{-m}\rho_{\rm e}^{-\mu}C^\infty(\mathbb{S}^n_+ \times \mathbb{S}^n_+)).$$

By $L^{m,\mu}_{\rm cl}(\mathbb{R}^n)$ we denote the class of the corresponding pseudo-differential operators: if $a(x,\xi) \in S^{m,\mu}_{{\rm cl}(\xi,x)}$ one sets as standard

$$Au(x) = \mathrm{Op}(a)u(x) = (2\pi)^{-n} \int e^{ix\xi} a(x,\xi)\hat{u}(\xi)\, d\xi.$$

We see that there are well defined symbol maps

$$\sigma^{m-k}_\psi : L^{m,\mu}_{\rm cl}(\mathbb{R}^n) \to \rho_{\rm e}^{-\mu}\mathcal{C}^\infty(\mathbb{S}^n_+ \times \mathbb{S}^{n-1}), \tag{2.2}$$

$$\sigma^{\mu-j}_{\rm e} : L^{m,\mu}_{\rm cl}(\mathbb{R}^n) \to \rho_\psi^{-m}\mathcal{C}^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^n_+), \tag{2.3}$$

$$\sigma^{m-k,\mu-h}_{\psi,{\rm e}} : L^{m,\mu}_{\rm cl}(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}), \tag{2.4}$$

for $k, j, h \in \mathbb{N}$. Precisely, given $A = \mathrm{Op}(a)$ with $a = (\mathrm{RC}^2)^*(\rho_\psi^{-m}\rho_{\rm e}^{-\mu}a')$, $a' \in \mathcal{C}^\infty(\mathbb{S}^n_+ \times \mathbb{S}^n_+)$, we consider the Taylor expansion near $\mathbb{S}^n_+ \times \mathbb{S}^{n-1}$ of $a' \sim \sum_{i=0}^\infty \rho_\psi^i a_i$, $a_i \in \mathcal{C}^\infty(\mathbb{S}^n_+ \times \mathbb{S}^{n-1})$; then we define

$$\sigma^{m-k}_\psi(A) = \rho_{\rm e}^{-\mu}a_k, \quad k \in \mathbb{N}.$$

Similarly, we expand $a' \sim \sum_{i=0}^\infty \rho_{\rm e}^i b_i$ near $\mathbb{S}^{n-1} \times \mathbb{S}^n_+$, with $b_i \in \mathcal{C}^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^n_+)$, and we set

$$\sigma^{\mu-j}_{\rm e}(A) = \rho_\psi^{-m}b_j, \quad j \in \mathbb{N}.$$

Finally, near $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$ we can write $a'\sim\sum_{i,j=0}^{\infty}\rho_\psi^j\rho_{\mathrm{e}}^i c_{i,j}$, for suitable $c_{i,j}\in\mathcal{C}^\infty(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$. This leads us to define

$$\sigma_{\psi,\mathrm{e}}^{m-k,\mu-h}(A)=c_{h,k},\quad h,k\in\mathbb{N}.$$

Proposition 2.2. *We have the following exact sequences of involutive algebras:*

$$0\to L_{\mathrm{cl}}^{-1,-1}(\mathbb{R}^n)\hookrightarrow L_{\mathrm{cl}}^{0,0}(\mathbb{R}^n)\xrightarrow{\sigma_\psi^0\oplus\sigma_{\mathrm{e}}^0}\Sigma^{0,0}\to 0, \tag{2.5}$$

$$0\to L_{\mathrm{cl}}^{-1,i}(\mathbb{R}^n)\hookrightarrow L_{\mathrm{cl}}^{0,i}(\mathbb{R}^n)\xrightarrow{\sigma_\psi^0}\rho_{\mathrm{e}}^{-i}\mathcal{C}^\infty(\mathbb{S}_+^n\times\mathbb{S}^{n-1})\to 0, \tag{2.6}$$

$$0\to L_{\mathrm{cl}}^{i,-1}(\mathbb{R}^n)\hookrightarrow L_{\mathrm{cl}}^{i,0}(\mathbb{R}^n)\xrightarrow{\sigma_{\mathrm{e}}^0}\rho_\psi^{-i}\mathcal{C}^\infty(\mathbb{S}^{n-1}\times\mathbb{S}_+^n)\to 0, \tag{2.7}$$

$$0\to L_{\mathrm{cl}}^{-1,0}(\mathbb{R}^n)+L_{\mathrm{cl}}^{0,-1}(\mathbb{R}^n)\hookrightarrow L_{\mathrm{cl}}^{0,0}(\mathbb{R}^n)\xrightarrow{\sigma_{\psi,\mathrm{e}}^{0,0}}\mathcal{C}^\infty(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})\to 0, \tag{2.8}$$

$$0\to L_{\mathrm{cl}}^{-1,-1}(\mathbb{R}^n)\hookrightarrow L_{\mathrm{cl}}^{-1,0}(\mathbb{R}^n)+L_{\mathrm{cl}}^{0,-1}(\mathbb{R}^n)\xrightarrow{\sigma_\psi^0\oplus\sigma_{\mathrm{e}}^0}\Sigma_{\psi,\mathrm{e}}^{0,0}\to 0, \tag{2.9}$$

for $i=-1,0$, *where* $\Sigma^{0,0}=\{(f,g)\in\mathcal{C}^\infty(\mathbb{S}_+^n\times\mathbb{S}^{n-1})\oplus\mathcal{C}^\infty(\mathbb{S}^{n-1}\times\mathbb{S}_+^n)\colon f|_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}}=g|_{\mathbb{S}}^{n-1}\times\mathbb{S}^{n-1}\}$ *and* $\Sigma_{\psi,\mathrm{e}}^{0,0}=\rho_{\mathrm{e}}\mathcal{C}^\infty(\mathbb{S}_+^n\times\mathbb{S}^{n-1})\oplus\rho_\psi\mathcal{C}^\infty(\mathbb{S}^{n-1}\times\mathbb{S}_+^n)$.

Proof. The proof that (2.5) is an exact sequence is given in Schulze [18] (see also Melrose [12]). The other cases can be handled in the same way. □

The statement of Proposition 2.2 is purely algebraic. The following proposition establishes the continuity of all symbol maps.

Proposition 2.3. *Let us regard* $L_{\mathrm{cl}}^{0,0}(\mathbb{R}^n)$ *as a ($*$-closed) sub-algebra of the* C^*-*algebra* $\mathbb{B}$ *of bounded operators on* $L^2(\mathbb{R}^n)$. *Then, if* $a\in S_{\mathrm{cl}(\xi,x)}^{0,0}(\mathbb{R}^n\times\mathbb{R}^n)$ *and* $A=\mathrm{Op}(a)$, *we have*

$$\|\sigma_\psi^0(A)\|_{\mathcal{C}(\mathbb{S}_+^n\times\mathbb{S}^{n-1})}\le\|A\|_{\mathbb{B}}, \tag{2.10}$$

$$\|\sigma_{\mathrm{e}}^0(A)\|_{\mathcal{C}(\mathbb{S}^{n-1}\times\mathbb{S}_+^n)}\le\|A\|_{\mathbb{B}}, \tag{2.11}$$

$$\|\sigma_{\psi,\mathrm{e}}^{0,0}(A)\|_{\mathcal{C}(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})}\le\|A\|_{\mathbb{B}}. \tag{2.12}$$

Proof. From Theorem 1.1.85 and 1.4.39 by Schulze [18] it follows that for every $\epsilon>0$ and $\omega\in\mathbb{S}_+^n\times\mathbb{S}^{n-1}$, $\omega'\in\mathbb{S}^{n-1}\times\mathbb{S}_+^n$, there are unitary operators S,W in $L^2(\mathbb{R}^n)$ such that

$$\|S^{-1}ASu-\sigma_\psi^0(A)(\omega)u\|<\epsilon,$$
$$\|W^{-1}AWu-\sigma_{\mathrm{e}}^0(A)(\omega')u\|<\epsilon,$$

for every $u\in L^2(\mathbb{R}^n)$. Since ϵ is arbitrary, from the triangle inequality we deduce at once (2.10) and (2.11).

Finally, (2.12) follows from (2.11) (or (2.10)), because

$$\sigma_{\psi,\mathrm{e}}^{0,0}(A)=\sigma_{\mathrm{e}}^0(A)|_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}}$$

by definition.

The proof is complete. □

Now we establish a simple property we shall need in the next section.

Proposition 2.4. *For every* $a \in S^{0,0}_{\mathrm{cl}(\xi,x)}(\mathbb{R}^n \times \mathbb{R}^n)$ *there exists* $A \in L^{0,0}_{\mathrm{cl}}(\mathbb{R}^n)$ *with* $A \equiv \mathrm{Op}(a) \mod L^{-1,-1}_{\mathrm{cl}}(\mathbb{R}^n)$ *and*

$$\|A\|_{\mathbb{B}} \leq 4 \sup_{(x,\xi)\in\mathbb{R}^{2n}} |a(x,\xi)|.$$

Proof. Of course, it suffices to prove that, for every $a \in S^{0,0}_{\mathrm{cl}(\xi,x)}(\mathbb{R}^n \times \mathbb{R}^n)$ *real valued*, there exists $A \in L^{0,0}_{\mathrm{cl}}(\mathbb{R}^n)$ with $A \equiv \mathrm{Op}(a) \mod L^{-1,-1}_{\mathrm{cl}}(\mathbb{R}^n)$ and

$$\|A\|_{\mathbb{B}} \leq 2 \sup_{(x,\xi)\in\mathbb{R}^{2n}} |a(x,\xi)|. \tag{2.13}$$

Let $M = \sup_{(x,\xi)\in\mathbb{R}^{2n}} |a(x,\xi)|$ (we can suppose $M > 0$) and $a = (\mathrm{RC}^2)^* a'$ with $a' \in \mathcal{C}^\infty(\mathbb{S}^n_+ \times \mathbb{S}^n_+)$. Since the symbol $2M - a(x,\xi)$ is elliptic of order $(0,0)$, i.e. it is equivalent, as a weight function, to the constant function 1, the function $2M - a'$ is strictly positive up to the boundary; so $(2M - a')^{1/2} \in \mathcal{C}^\infty(\mathbb{S}^n_+ \times \mathbb{S}^n_+)$ and therefore $(2M - a)^{1/2} = (\mathrm{RC}^2)^*(2M - a')^{1/2}$ is a classical symbol of order $(0,0)$. Now let $B = \mathrm{Op}((2M - a)^{1/2})$. The calculus gives

$$2M - \mathrm{Op}(a) = B^*B + R \tag{2.14}$$

with $R \in L^{-1,-1}_{\mathrm{cl}}(\mathbb{R}^n)$. From (2.14) we deduce that $A := \mathrm{Op}(a) + R$ satisfies (2.13), as one sees using the fact that A is self-adjoint. □

3. The completed algebras

We would like to replace in (2.5)-(2.9) all algebras by their completions.

For $i, j = -1, 0$, we set $\mathbb{A}^{i,j} = \overline{L^{i,j}_{\mathrm{cl}}(\mathbb{R}^n)}$, $\mathbb{A}^0_{-1} = \overline{L^{0,-1}_{\mathrm{cl}}(\mathbb{R}^n) + L^{-1,0}_{\mathrm{cl}}(\mathbb{R}^n)}$, with respect to the norm in $\mathbb{B}$.

The following proposition can be seen as a particular case of the results of Lauter, Monthubert and Nistor [9]; to be definite, we shall give here an independent proof.

Proposition 3.1. *It turns out that* $\mathbb{A}^{-1,-1} = \mathbb{K}$*, the ideal of compact operators on* $L^2(\mathbb{R}^n)$.

Proof. $L^{-1,-1}_{\mathrm{cl}}(\mathbb{R}^n)$ contains the subspace $L^{-\infty}(\mathbb{R}^n)$ of smoothing operators, namely having kernel in $\mathcal{S}(\mathbb{R}^{2n})$. Now, $\mathcal{S}(\mathbb{R}^{2n})$ is dense in $L^2(\mathbb{R}^{2n})$, hence it follows that $L^{-\infty}(\mathbb{R}^n)$ is dense in the C^*-algebra $\mathbb{B}_2$ of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$. $\mathbb{B}_2$ in turn is dense in $\mathbb{K}$, because it contains all finite-rank operators. □

Proposition 3.2. *We have the following exact sequences of* C^**-algebras:*

$$0 \to \mathbb{K} \hookrightarrow \mathbb{A}^{0,0} \xrightarrow{\sigma^0_\psi \oplus \sigma^0_{\mathrm{e}}} \overline{\Sigma^{0,0}} \to 0, \tag{3.1}$$

$$0 \to \mathbb{A}^{-1,i} \hookrightarrow \mathbb{A}^{0,i} \xrightarrow{\sigma^0_\psi} \overline{\rho_{\mathrm{e}}^{-i}\mathcal{C}^\infty(\mathbb{S}^n_+ \times \mathbb{S}^{n-1})} \to 0, \tag{3.2}$$

$$0 \to \mathbb{A}^{i,-1} \hookrightarrow \mathbb{A}^{i,0} \xrightarrow{\sigma^0_{\mathrm{e}}} \overline{\rho_\psi^{-i}\mathcal{C}^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^n_+)} \to 0, \tag{3.3}$$

$$0 \to \mathbb{A}^0_{-1} \hookrightarrow \mathbb{A}^{0,0} \xrightarrow{\sigma^{0,0}_{\psi,\mathrm{e}}} \overline{\mathcal{C}^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})} \to 0, \tag{3.4}$$

$$0 \to \mathbb{K} \hookrightarrow \mathbb{A}^0_{-1} \xrightarrow{\sigma^0_\psi \oplus \sigma^0_{\mathrm{e}}} \overline{\Sigma^{0,0}_{\psi,\mathrm{e}}} \to 0, \tag{3.5}$$

for $i = -1, 0$*, where* $\sigma^0_\psi, \sigma^0_{\mathrm{e}}, \sigma^{0,0}_{\psi,\mathrm{e}}$ *now denote the extensions by continuity of the corresponding symbol maps and the closures are done with respect to the sup-norm.*

Proof. We verify that (3.1) is an exact sequence.

First, $\sigma^0_\psi \oplus \sigma^0_{\mathrm{e}}$ is onto. In fact, the map $\sigma^0_\psi \oplus \sigma^0_{\mathrm{e}}$ preserves the involution, i.e. $\sigma^0_\psi(A^*) = \overline{\sigma^0_\psi(A)}$ and $\sigma^0_{\mathrm{e}}(A^*) = \overline{\sigma^0_{\mathrm{e}}(A)}$, so that its range is closed. On the other hand, its range contains $\Sigma^{0,0}$.

To prove that (3.1) is exact in $\mathbb{A}^{0,0}$, we have only to verify that $\ker \sigma^0_\psi \oplus \sigma^0_{\mathrm{e}} \subset \mathbb{K}$, because $\sigma^0_\psi \oplus \sigma^0_{\mathrm{e}}$ is continuous and (2.5) is exact (in $L^{0,0}_{\mathrm{cl}}(\mathbb{R}^n)$). Consider $A \in \mathbb{A}^{0,0}$ with $\sigma^0_\psi(A) = \sigma^0_{\mathrm{e}}(A) = 0$. If $A_j \in L^{0,0}_{\mathrm{cl}}(\mathbb{R}^n)$ is a sequence with $A_j \to A$ in norm, then $\sigma^0_\psi(A_j) \to 0$ and $\sigma^0_{\mathrm{e}}(A_j) \to 0$ uniformly. Let us set $p_j := (\mathrm{RC} \times \mathrm{id})^* \sigma^0_\psi(A_j)$, $q_j := (\mathrm{id} \times \mathrm{RC})^* \sigma^0_{\mathrm{e}}(A_j)$, $r_j := \sigma^{0,0}_{\psi,\mathrm{e}}(A_j)$, and define $a_j(x,\xi) = \chi(\xi) p_j(x, \xi/|\xi|) + \chi(x) q_j(x/|x|, \xi) - \chi(x)\chi(\xi) r_j(x/|x|, \xi/|\xi|)$, where χ is any excision function in $\mathbb{R}^n$. We have $a_j(x,\xi) \to 0$ uniformly in $\mathbb{R}^{2n}$ and, by Proposition 2.4, we can take $B_j \in L^{0,0}_{\mathrm{cl}}(\mathbb{R}^n)$ with $B_j \equiv \mathrm{Op}(a_j) \mod L^{-1,-1}_{\mathrm{cl}}(\mathbb{R}^n)$ and $\|B_j\|_{\mathbb{B}} \leq 4 \sup |a_j(x,\xi)| \to 0$. Since A_j and B_j have the same image by $\sigma^0_\psi \oplus \sigma^0_{\mathrm{e}}$, we deduce that $A_j - B_j \in \ker \sigma^0_\psi \oplus \sigma^0_{\mathrm{e}} \cap L^{0,0}_{\mathrm{cl}}(\mathbb{R}^n) = L^{-1,-1}_{\mathrm{cl}}(\mathbb{R}^n)$, where the equality follows from the exactness of (2.5). On the other hand $A_j - B_j \to A$ in norm. This proves that $A \in \mathbb{A}^{-1,-1}$. Proposition 3.1 then gives the conclusion.

Similar arguments prove the exactness of (3.2)-(3.5). $\square$

Lemma 3.3. *Let $X, Y \subset \mathbb{R}^N$ be compact smooth manifolds with boundary, with $X \cap Y = \partial X = \partial Y$, and consider the $*$-algebra $\mathcal{A} = \{(f,g) \in \mathcal{C}^\infty(X) \oplus \mathcal{C}^\infty(Y) : f|_{\partial X} = g|_{\partial Y}\}$. Then the closure of $\mathcal{A}$ as a subspace of $C(X) \oplus C(Y)$ is given by $\overline{\mathcal{A}} = \{(f,g) \in \mathcal{C}(X) \oplus \mathcal{C}(Y) : f|_{\partial X} = g|_{\partial Y}\}$.*

Proof. Of course we have $\overline{\mathcal{A}} \subset \mathcal{B} := \{(f,g) \in \mathcal{C}(X) \oplus \mathcal{C}(Y) : f|_{\partial X} = g|_{\partial Y}\}$. In the other direction, given $(f,g) \in \mathcal{B}$, we construct a sequence $(f_n, g_n)_{n \geq 1}$ in $\mathcal{A}$ with $\|f_n - f\|_{\mathcal{C}(X)} \leq 1/n$, $\|g_n - g\|_{\mathcal{C}(Y)} \leq 1/n$. Since $\overline{C^\infty(Z)} = C(Z)$ for any compact smooth manifold Z, for every $n \in \mathbb{N}, n \geq 1$, we can choose a function $\phi_n \in \mathcal{C}^\infty(\partial X)$ such that $\|\phi_n - f|_{\partial X}\|_{\mathcal{C}(\partial X)} \leq 1/4n$. Then we extend ϕ_n to a $\mathcal{C}^\infty$ function on a collar neighborhood $U \subset X$ of ∂X; we can suppose, by continuity, that $|\phi_n(x) - f(x)| \leq 1/2n$ on U. On the other hand, we can choose $\psi_n \in C^\infty(X)$ with $\|\psi_n - f\|_{\mathcal{C}(X)} \leq 1/2n$. If $\chi \in \mathcal{C}^\infty_c(U)$ is a real valued function with $0 \leq \chi(x) \leq 1$, $\chi(x) = 1$ when $x \in \partial X$, we see that $f_n := \chi\phi_n + (1-\chi)\psi_n$ satisfies the required estimates. In the same way, starting from the same ϕ_n, we construct g_n; then $(f_n, g_n) \in \mathcal{A}$ by construction. $\square$

Proposition 3.4. *We have the following isomorphisms of C^*-algebras:*

$$\overline{\Sigma^{0,0}} \simeq \mathcal{C}(\mathbb{S}^{2n-1}), \tag{3.6}$$

$$\overline{\rho_{\mathrm{e}} \mathcal{C}^\infty(\mathbb{S}^n_+ \times \mathbb{S}^{n-1})} \simeq \mathcal{C}_0(\mathbb{R}^n \times \mathbb{S}^{n-1}), \tag{3.7}$$

$$\overline{\rho_\psi \mathcal{C}^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^n_+)} \simeq \mathcal{C}_0(\mathbb{S}^{n-1} \times \mathbb{R}^n), \tag{3.8}$$

$$\overline{\Sigma^{0,0}_{\psi,\mathrm{e}}} \simeq \mathcal{C}_0(\mathbb{R}^n \times \mathbb{S}^{n-1}) \oplus \mathcal{C}_0(\mathbb{S}^{n-1} \times \mathbb{R}^n). \tag{3.9}$$

Furthermore, we have

$$\overline{\mathcal{C}^\infty(\mathbb{S}^n_+ \times \mathbb{S}^{n-1})} = \mathcal{C}(\mathbb{S}^n_+ \times \mathbb{S}^{n-1}), \tag{3.10}$$

$$\overline{\mathcal{C}^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^n_+)} = \mathcal{C}(\mathbb{S}^{n-1} \times \mathbb{S}^n_+), \tag{3.11}$$

$$\overline{\mathcal{C}^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})} = \mathcal{C}(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}). \tag{3.12}$$

Proof. The second part of the statement is obvious, since $\overline{\mathcal{C}^\infty(X)} = \mathcal{C}(X)$, for any compact smooth manifold X.

Now consider $\Sigma^{0,0}$. By Lemma 3.3, the completion of this space with respect to the norm $\|(f,g)\| = \sup\{\|f\|, \|g\|\}$ is given by the subspace of $\mathcal{C}(\mathbb{S}^n_+ \times \mathbb{S}^{n-1}) \oplus \mathcal{C}(\mathbb{S}^{n-1} \times \mathbb{S}^n_+)$ of all couples (f,g) satisfying the compatibility condition

$$f|_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} = g|_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}}. \tag{3.13}$$

This C^*-algebra can be regarded as the algebra of continuous functions on the compact space X that we obtain by gluing $\mathbb{S}^n_+ \times \mathbb{S}^{n-1}$ and $\mathbb{S}^{n-1} \times \mathbb{S}^n_+$ along their common boundary $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$. We are going to verify that this topological quotient space is homeomorphic to $\mathbb{S}^{2n-1}$. In fact we have

$$X \simeq Y \simeq Z \simeq W \simeq \mathbb{S}^{2n-1}, \tag{3.14}$$

where

$Y = \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times [0,1]/{\sim_1}$, with $(x,y,1) \sim_1 (x,y',1)$ and $(x,y,0) \sim_1 (x',y,0)$;
$Z = [0,1]^{n-1} \times [0,1]^{n-1} \times [0,1]/\sim_2$, with $(x,y,1) \sim_2 (x,y',1)$, $(x,y,0) \sim_2 (x',y,0)$; $(x,y,a) \sim_2 (x,y',a)$ when $y,y' \in \partial[0,1]^{n-1}$ and $(x,y,b) \sim_2 (x',y,b)$ when $x,x' \in \partial[0,1]^{n-1}$;
$W = \mathbb{S}^{2n-2} \times [0,1]/\sim_3$, with $(x,1) \sim_3 (y,1)$, $(x,0) \sim_3 (y,0)$.

The third homeomorphism in (3.14) follows by observing that all points $(x,y,0)$ are $\sim_2$-equivalent, as well as all points of the type $(x,y,1)$. This proves (3.6).

As (3.7) is concerned, we observe that the map $(\mathrm{RC}\times\mathrm{id})^* : \mathcal{C}(\mathbb{S}^n_+ \times \mathbb{S}^{n-1}) \to \mathcal{C}(\mathbb{R}^n \times \mathbb{S}^{n-1})$ is (a $*$-homomorphism and) an isometry. The image of the subspace $\rho_{\mathrm{e}}\mathcal{C}^\infty(\mathbb{S}^n_+ \times \mathbb{S}^{n-1})$ of course contains $\mathcal{C}^\infty_c(\mathbb{R}^n \times \mathbb{S}^{n-1})$ and in turn is contained in $\mathcal{C}_0(\mathbb{R}^n \times \mathbb{S}^{n-1})$. The equality $\overline{\mathcal{C}^\infty_c(X)} = \mathcal{C}_0(X)$, valid for any smooth manifold X, then gives (3.7). Similarly one proves (3.8) and therefore (3.9). □

Theorem 3.5. *There are the following isomorphisms of C^*-algebras:*

$$\mathbb{A}^{0,0}/\mathbb{K} \simeq \mathcal{C}(\mathbb{S}^{2n-1}), \tag{3.15}$$

$$\mathbb{A}^{0,0}/\mathbb{A}^{-1,0} \simeq \mathcal{C}(\mathbb{S}^n_+ \times \mathbb{S}^{n-1}), \quad \mathbb{A}^{0,0}/\mathbb{A}^{0,-1} \simeq \mathcal{C}(\mathbb{S}^{n-1} \times \mathbb{S}^n_+), \tag{3.16}$$

$$\mathbb{A}^{0,-1}/\mathbb{K} \simeq \mathcal{C}_0(\mathbb{R}^n \times \mathbb{S}^{n-1}), \quad \mathbb{A}^{-1,0}/\mathbb{K} \simeq \mathcal{C}_0(\mathbb{S}^{n-1} \times \mathbb{R}^n), \tag{3.17}$$

$$\mathbb{A}^{0,0}/\mathbb{A}^0_{-1} \simeq \mathcal{C}(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}), \tag{3.18}$$

$$\mathbb{A}^0_{-1}/\mathbb{K} \simeq \mathcal{C}_0(\mathbb{R}^n \times \mathbb{S}^{n-1}) \oplus \mathcal{C}_0(\mathbb{S}^{n-1} \times \mathbb{R}^n). \tag{3.19}$$

Proof. The isomorphisms (3.15)-(3.19) follow at once from Proposition 3.2 and Proposition 3.4. □

The K-theory of the above C^*-algebras is listed in Table 1. Indeed, the K-theory of $\mathbb{A}^{0,0}/\mathbb{K}$ follows from (3.15), since $K_i(\mathcal{C}(\mathbb{S}^{2n-1})) = \mathbb{Z}$ for $i = 0,1$; see for instance [21], 9.D.

The second row of the table can be deduced from (3.16) if we think of $\mathcal{C}(\mathbb{S}^n_+ \times \mathbb{S}^{n-1})$ as $\mathcal{C}(\mathbb{S}^n_+; \mathcal{C}(\mathbb{S}^{n-1}))$; since $\mathbb{S}^n_+$ is compact and contractible we have $K_i(\mathcal{C}(\mathbb{S}^n_+ \times \mathbb{S}^{n-1})) = K_i(\mathcal{C}(\mathbb{S}^{n-1}))$.

To verify the third row we use (3.17) and note that $\mathcal{C}_0(\mathbb{R}^n \times \mathbb{S}^{n-1}) = \mathcal{C}_0(\mathbb{R}^n) \otimes \mathcal{C}(\mathbb{S}^{n-1}) =: S^n\mathcal{C}(\mathbb{S}^{n-1})$, where S denotes the suspension of C^*-algebras. Then the result follows from the Bott Periodicity Theorem.

TABLE 1. K-theory

	K_0		K_1	
	n even	*n odd*	*n even*	*n odd*
$\mathbb{A}^{0,0}/\mathbb{K}$	$\mathbb{Z}$		$\mathbb{Z}$	
$\mathbb{A}^{0,0}/\mathbb{A}^{-1,0}, \mathbb{A}^{0,0}/\mathbb{A}^{0,-1}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$	0
$\mathbb{A}^{0,-1}/\mathbb{K}, \mathbb{A}^{-1,0}/\mathbb{K}$	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}^2$
$\mathbb{A}^{0,0}/\mathbb{A}^{0}_{-1}$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	0
$\mathbb{A}^{0}_{-1}/\mathbb{K}$	$\mathbb{Z}^2$	0	$\mathbb{Z}^2$	$\mathbb{Z}^4$

As $\mathbb{A}^{0,0}/\mathbb{A}^{0}_{-1}$ is concerned, we look at (3.18), and after writing $\mathcal{C}(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$ as $\mathcal{C}(\mathbb{S}^{2n-1})\otimes\mathcal{C}(\mathbb{S}^{2n-1})$, we apply the Kunneth formula ([21], 9.3.3) for the K-theory of tensor products. As an alternative: we know that for a compact space X it turns out that $K_i(\mathcal{C}(X)) = K^i(X)$, the topological K theory of X; then we apply the formula for the topological K-theory of products (notice that $\mathbb{S}^{n-1}\wedge\mathbb{S}^{n-1}\simeq\mathbb{S}^{2n-2}$); see for example Atiyah [1], Corollary 2.4.7.

Finally, (3.19) and the additivity of K_i give the last row.

We observe that other algebras, like $\mathbb{A}^{0}_{-1}/\mathbb{A}^{0,-1}$, $\mathbb{A}^{0}_{-1}/\mathbb{A}^{-1,0}$, etc., could be studied in the same way. To avoid an overweight of the paper we have not considered these cases.

Theorem 3.6. *We have* $K_0(\mathbb{A}^{0,0}) = \mathbb{Z},\ K_1(\mathbb{A}^{0,0}) = 0.$

Proof. The result follows from the six-term exact sequence applied to the short exact sequence $0\to\mathbb{K}\to\mathbb{A}^{0,0}\to\mathbb{A}^{0,0}/\mathbb{K}\to 0$, taking into account the K-theory groups in Table 1 and the fact that the connecting morphism $\partial = \mathrm{ind} : K_1(\mathbb{A}^{0,0}/\mathbb{K})\simeq\mathbb{Z}\to K_0(\mathbb{K})\simeq\mathbb{Z}$ actually is an isomorphism because it is onto. We can see this, for example, as a consequence of Fedosov's index formula. □

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