

GRAPH-DISTANCE CONVERGENCE AND UNIFORM LOCAL BOUNDEDNESS OF MONOTONE MAPPINGS

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(Communicated by Jonathan M. Borwein)

ABSTRACT. In this article we study graph-distance convergence of monotone operators. First, we prove a property that has been an open problem up to now: the limit of a sequence of graph-distance convergent maximal monotone operators in a Hilbert space is a maximal monotone operator. Next, we show that a sequence of maximal monotone operators converging in the same sense in a reflexive Banach space is uniformly locally bounded around any point from the interior of the domain of the limit mapping. The result is an extension of a similar one from finite dimensions. As an application we give a simplified condition for the stability (under graph-distance convergence) of the sum of maximal monotone mappings in Hilbert spaces.

1. INTRODUCTION

This paper is concerned with properties of convergent sequences of monotone mappings. One of them is uniform local boundedness of a sequence of mappings, which has been recognized as a crucial property when studying stability of sums of monotone mappings as in Attouch, Moudafi and Riahi [3], and in Pennanen, Rockafellar and Théra [10]. It was shown in [10] that, in finite dimensions, a *graphically convergent sequence* of monotone mappings (that means convergence of the graphs in the Painlevé-Kuratowski sense) is uniformly locally bounded at every interior point of the domain of the limit mapping. Also, a counterexample was given to show that the result does not directly extend to infinite-dimensional spaces. This raised the question of whether the result would hold if Painlevé-Kuratowski convergence of the graphs was replaced by the stronger graph-distance convergence, which is equivalent to the former in finite dimensions (see below the precise definitions). One of the main purposes of this note is to show that, in reflexive Banach spaces, it does.

Received by the editors May 14, 2002.

2000 *Mathematics Subject Classification*. Primary 47H05; Secondary 54B20.

The first author's research was partially supported by LACO (Laboratoire d'Arithmétique, Calcul Formel et Optimisation), UMR-CNRS 6090 of the University of Limoges, as well as by the Région Limousin under a Research Grant.

The second author's research was partially supported by the LACO (Laboratoire d'Arithmétique, Calcul Formel et Optimisation), UMR-CNRS 6090 of the University of Limoges, by Bulgarian NFSR under grant No. MM-1105/01 and by NATO-CLG 978488.

The third author's research was partially supported by the French Chilean Scientific Cooperation Programme ECOS under grant C00E05 and by NATO-CLG 978488.

We apply this result to the study of the stability of sums of monotone mappings under graph-distance convergence in Hilbert spaces. In particular, our uniform local boundedness result yields a significant simplification of the sum-rule for graph-distance convergence given in [3, Proposition 3.4].

Despite the fact that the graphical convergence of monotone operators has been investigated since the beginning of the eighties (cf. the monograph of Attouch [1]) and despite the intensive study of set convergence during the eighties and nineties, to our best knowledge, it has not been known so far whether in infinite dimensions the limit of a graphically (or graph-distance) convergent sequence of maximal monotone operators is again a maximal monotone operator. One of our main results (Theorem 2) gives a positive answer to this question for the graph-distance convergence in Hilbert spaces.

2. GRAPH-DISTANCE CONVERGENCE

In the sequel, unless otherwise specified, X will stand for a real Banach space and X^* for its dual. As usual, $\langle \cdot, \cdot \rangle$ will denote the pairing between X and X^* . For a set-valued mapping $T : X \rightrightarrows X^*$, its *graph* is the set

$$\text{gph } T = \{(x, x^*) \in X \times X^* \mid x^* \in T(x)\},$$

and the projection of $\text{gph } T$ into X is its *domain*

$$\text{dom } T = \{x \in X \mid T(x) \neq \emptyset\}.$$

T is said to be *monotone* if

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq 0, \quad \text{for every } (x_1, x_1^*), (x_2, x_2^*) \in \text{gph } T.$$

When the inequality is always strict, the operator T is called *strictly monotone*. If the graph of a monotone mapping T is not contained properly in the graph of another monotone mapping, then T is called *maximal monotone*. Such mappings play a key role in variational analysis and optimization. For instance, if $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper extended real-valued closed convex function, its *subdifferential* ∂f , which is defined by

$$\partial f(x) := \{x^* \in X^* \mid f(y) \geq f(x) + \langle x^*, y - x \rangle \quad \forall y \in X\}, \quad x \in X,$$

is maximal monotone; see Rockafellar [13]. For the use of maximal monotone operators in another branch—the domain of differential equations—the reader can consult for example the book of Brezis [7].

Let us also recall a classical result of Minty, originally proved for Hilbert spaces (see Rockafellar [14] for the reflexive case): Let X be reflexive with suitable (always existing) renorming of X and X^* making the duality map J between X and X^* single-valued and strictly monotone. Then a monotone operator $T : X \rightrightarrows X^*$ is maximal if and only if the mapping $T + J$ is surjective.

We will denote the closed unit ball in X (resp. in X^*) by \mathbb{B}_X (resp. by \mathbb{B}_{X^*}). For the following notions and facts related to set-convergence the reader is referred to [4, 5]. In what follows, we will be interested in the following convergence in the family of the non-empty closed subsets of X : a sequence $\{C_n\}_{n \in \mathbb{N}}$ of non-empty closed subsets of X is declared *bounded Hausdorff convergent* to a closed set C , and we write $C_n \xrightarrow{bH} C$, if for each $\rho > 0$ and each $\epsilon > 0$ there is an N , such that

$$C \cap \rho \mathbb{B}_X \subset C_n + \epsilon \mathbb{B}_X \quad \text{and} \quad C_n \cap \rho \mathbb{B}_X \subset C + \epsilon \mathbb{B}_X \quad \forall n \geq N.$$

Equivalently, for $D, D' \subset X$, let $e(D, D') := \sup\{d(x, D') : x \in D\}$ (where $d(\cdot, D')$ is the distance function generated by D') designate as usual the *excess* of D to D' (with the convention $e(\emptyset, D') = 0, D' \neq \emptyset$). Given $\rho > 0$, the ρ -Hausdorff distance between D and D' is defined as follows:

$$\text{haus}_\rho(D, D') := \max\{e(D \cap \rho\mathbb{B}_X, D'), e(D' \cap \rho\mathbb{B}_X, D)\}.$$

Then the sequence of closed sets $\{C_n\}_{n \in \mathbb{N}}$ bounded Hausdorff converges to the closed set C , if for any $\rho > 0$ we have $\lim_n \text{haus}_\rho(C_n, C) = 0$. In fact, this convergence in the family of the non-empty closed subsets of the Banach space X is completely metrizable by the following metric:

$$(1) \quad d_{bH}(C, D) := \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(C, D)}{1 + d_k(C, D)},$$

where

$$d_k(C, D) := \sup_{\|x\| \leq k} |d(x, C) - d(x, D)|.$$

Bounded Hausdorff convergence (also known as *Attouch-Wets convergence*) has turned out to be very useful in quantitative analysis of different variational problems such as optimization problems, variational inequalities, monotone inclusions, etc. (see e.g. [2, 4, 5, 6] and the references therein).

It is well known that bounded Hausdorff convergence implies Painlevé-Kuratowski convergence, and that in finite dimensions they are equivalent. When applied to graphs of operators, this concept gives rise to the following strengthened version of graph-convergence: A sequence of graph-closed operators $\{T_n : X \rightrightarrows X^*\}_{n \in \mathbb{N}}$ is said to *graph-distance converge* to an operator $T : X \rightrightarrows X^*$ (necessarily with closed graph, as well), and we write $T_n \xrightarrow{gd} T$, if $\text{gph } T_n \xrightarrow{bH} \text{gph } T$. Here, and in what follows, we will use the box-norm in Cartesian products of Banach spaces. Recall that the graphs of maximal monotone operators between X and its dual X^* are non-empty closed sets in the Cartesian product $X \times X^*$, equipped with the product topology.

In a Hilbert space X , given a maximal monotone operator T , its *resolvent* of index $\lambda > 0$ is the (single-valued) operator $J_\lambda^T = (I + \lambda T)^{-1}$ (I is the identity in X). This operator is an everywhere defined contraction of X into itself. The *Yosida regularization* of T of index $\lambda > 0$ is $T_\lambda = (I - J_\lambda^T)/\lambda = (T^{-1} + \lambda I)^{-1}$, an everywhere defined Lipschitz continuous maximal (single-valued) monotone mapping. Let us mention that one always has $T_\lambda(x) \in T(J_\lambda^T(x))$ for any $x \in X$.

Given two maximal monotone operators S and T in a Hilbert space X and $\lambda, \rho > 0$, let us denote

$$d_{\lambda, \rho}(S, T) := \sup_{\|x\| \leq \rho} \|J_\lambda^S(x) - J_\lambda^T(x)\| = \lambda \sup_{\|x\| \leq \rho} \|S_\lambda(x) - T_\lambda(x)\|.$$

The following estimate from [3] is very useful.

Proposition 1 (Attouch-Moudafi-Riahi [3]). *Let S, T be maximal monotone operators in a Hilbert space X and $\rho > 0$. Then*

$$d_{1, \rho}(S, T) \leq 3 \text{haus}_{\rho + \|J_1^S(0)\|}(S, T).$$

Now, we are ready to prove our first main result (for the particular case when $X = \mathbb{R}^n$, see also [[15], Theorem 12.32]):

Theorem 2. *Let X be a Hilbert space. The class of maximal monotone operators in X is closed for the graph-distance convergence.*

Proof. Let $\{T_n\}_{n \in \mathbb{N}}$ be a sequence of maximal monotone operators graph-distance converging to an operator T (necessarily with non-empty closed graph). It is clear that T is a monotone operator. In order to prove the maximal monotonicity of T , it suffices by Minty's criterion to show that $I + T$ is surjective, or equivalently, that the equation $f \in x + Tx$ has a solution for every $f \in X$.

So fix an $f \in X$. By Minty's criterion again, there exists for every $n \in \mathbb{N}$ an x_n such that

$$f \in x_n + T_n x_n.$$

We will show that the sequence $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence. Let $\rho > \|f\|$ be arbitrary. Then (by the definition of the resolvent) for every p, q we have

$$(2) \quad \|x_p - x_q\| = \|J_1^{T_p}(f) - J_1^{T_q}(f)\| \leq d_{1,\rho}(T_p, T_q).$$

By virtue of Proposition 1, we have:

$$(3) \quad d_{1,\rho}(T_p, T_q) \leq 3 \operatorname{haus}_{\rho + \|J_1^{T_p}(0)\|}(T_p, T_q).$$

We claim that the sequence $\{J_1^{T_n}(0)\}_{n \in \mathbb{N}}$ is bounded. To this end, fix y_0 and v_0 such that $v_0 \in T(y_0)$. Since $\{T_n\}_{n \in \mathbb{N}}$ graph-distance converges to T , we can select sequences $\{y_n\}_{n \in \mathbb{N}}$ and $\{v_n\}_{n \in \mathbb{N}}$, with limits y_0 and v_0 respectively, such that $v_n \in T_n(y_n)$ for every $n \in \mathbb{N}$.

Set $u_n := J_1^{T_n}(0)$. This means $-u_n \in T_n(u_n)$ and hence, using the monotonicity of T_n , we have

$$\begin{aligned} \|u_n - y_n\|^2 &= \langle u_n - y_n, u_n - y_n \rangle \\ &= \langle u_n - y_n, u_n + v_n \rangle + \langle u_n - y_n, -v_n - y_n \rangle \leq \langle u_n - y_n, -v_n - y_n \rangle \end{aligned}$$

and therefore

$$\|u_n - y_n\| \leq \|y_n + v_n\| \leq \|y_n\| + \|v_n\|.$$

Thus, we have $\|u_n\| \leq M$ for every n and some positive constant M . Combining this with (2) and (3) we derive

$$(4) \quad \|x_p - x_q\| \leq 3 \operatorname{haus}_{\rho+M}(T_p, T_q).$$

To complete the proof, we use the following generalized triangle inequality for the ρ -Hausdorff distance haus_ρ due to Penot [11]: if A, B, C are non-empty closed subsets of a Banach space and $\mu_0 = \max\{d(0, A), d(0, B), d(0, C)\}$, then for every $\mu > \mu_0$ one has

$$\operatorname{haus}_\mu(A, B) \leq \operatorname{haus}_{2\mu+\mu_0}(A, C) + \operatorname{haus}_{2\mu+\mu_0}(B, C).$$

Using the above convergent sequence $(y_n, v_n) \in \operatorname{gph} T_n$ we see that there is $\mu_0 > 0$ so that $d(0, \operatorname{gph} T) \leq \mu_0$ and $d(0, \operatorname{gph} T_n) \leq \mu_0$ for every n . Then we conclude that for $\rho_0 > 0$ large enough we have

$$\|x_p - x_q\| \leq 3 \operatorname{haus}_{\rho_0}(T_p, T) + 3 \operatorname{haus}_{\rho_0}(T, T_q)$$

for every p, q . This together with the graph-distance convergence of T_n to T shows that the sequence $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence and hence convergent to some $x_0 \in X$.

For every n we have $f = x_n + w_n$ for some $w_n \in T_n(x_n)$. Obviously w_n also converges to some w_0 and once again using the graph-distance convergence of T_n

to T we obtain $w_0 \in T(x_0)$. Thus, since $f = x_0 + w_0$, $I + T$ is surjective and hence T is maximal monotone. \square

We mentioned above that the family of non-empty closed sets in a Banach space X equipped with the bounded Hausdorff convergence is completely metrizable by the metric d_{bH} given in (1). Hence, if we denote the corresponding metric in the family of the non-empty closed sets of $X \times X$ again by d_{bH} , the following is an immediate corollary from this fact and Theorem 2.

Corollary 3. *Let \mathcal{M} be the family of all maximal monotone mappings in a Hilbert space X , equipped with the graph-distance convergence. Then this convergence is compatible with the metric d_{bH} and the space (\mathcal{M}, d_{bH}) is a complete metric space.*

Let us mention that a similar result for the graphical convergence (i.e. the Painlevé-Kuratowski convergence on the graphs) of maximal monotone operators in a reflexive separable Banach space was proved by Attouch [1, Theorem 3.62]. But neither the result nor the construction from [1] implies that the family of maximal monotone operators is closed (wrt to the graphical convergence) in the family of all closed sets in the corresponding Cartesian product.

We conclude this section by recalling the following two facts about the graph-distance convergence of maximal monotone operators in Hilbert spaces.

Theorem 4 (Attouch-Moudafi-Riahi [3]). *Let T be a maximal monotone mapping in a Hilbert space. Then the family $\{T_\lambda\}_{\lambda>0}$ graph-distance converges to T as $\lambda \searrow 0$.*

Theorem 5 (Attouch-Moudafi-Riahi [3]). *Let $\{S_n, S \mid n \in \mathbb{N}\}$ and $\{T_n, T \mid n \in \mathbb{N}\}$ be families of maximal monotone mappings in a Hilbert space such that $S_n \xrightarrow{gd} S$ and $T_n \xrightarrow{gd} T$. Assume that the following qualification condition holds:*

(Q) *there exist some $x_0 \in \text{dom } S \cap \text{int dom } T$ and some positive constants $\rho_0 > 0, M > 0$ such that for all $x \in X$ satisfying $x \in x_0 + \rho_0 \mathbb{B}$ and all $n \in \mathbb{N}$ we have that the element of minimal norm in $T_n(x)$ is bounded from above by M .*

Then, $S_n + T_n \xrightarrow{gd} S + T$.

As we will see in Theorem 12 below, condition (Q) can be significantly simplified to the assumption that $\text{dom } S \cap \text{int dom } T \neq \emptyset$.

3. UNIFORM LOCAL BOUNDEDNESS

A set-valued mapping $T : X \rightrightarrows X^*$ is *locally bounded* at a point $x \in \text{dom } T$ if there is a neighborhood V of x and a (norm-)bounded set $B \subset X^*$ such that $T(V) \subset B$.

Theorem 6 (Rockafellar [12]). *Let X be reflexive. A maximal monotone mapping $T : X \rightrightarrows X^*$ is locally bounded at a point $x \in \text{dom } T$ if and only if $x \in \text{int dom } T$.*

In the above theorem, reflexivity is not needed for local boundedness at interior points, but only for the reverse implication.

When studying collections of mappings, e.g., approximations of a single mapping, it is often useful to have a bound that applies to all the mappings [3, 15, 10]. In this paper we are interested in the following concept.

Definition 7. A collection $\{T_i\}_{i \in \mathcal{I}}$ of set-valued mappings from X into X^* is *uniformly locally bounded* at a point $x \in \bigcap_{i \in \mathcal{I}} \text{dom } T_i$ if there is a neighborhood V of x and a bounded set $B \subset X^*$ such that

$$T_i(V) \subset B \quad \text{for all } i \in \mathcal{I}.$$

The following lemma, which follows the lines of [10, Lemma 2.1] is the key to our main result on uniform local boundedness. As usual, $\text{co}C$ denotes the convex hull of a set $C \subset X$ and $\mathbb{B}(x, r)$ is the open ball centered at $x \in X$ and with radius $r > 0$.

Lemma 8. *Let $\{T_n\}_{n \in \mathbb{N}}$ be a sequence of monotone mappings from X into X^* , $C \subset X$, and $\epsilon_1, \rho_1, \rho_2 > 0$ be such that $C \subset \rho_1 \mathbb{B}_X$, and*

$$(5) \quad T_n(\mathbb{B}(x, \epsilon_1)) \cap \rho_2 \mathbb{B}_{X^*} \neq \emptyset \quad \forall x \in C, \forall n \in \mathbb{N}.$$

Then for any $V \subset X$ and ϵ_2 such that $\epsilon_2 > \epsilon_1$ and $V + \epsilon_2 \mathbb{B}_X \subset \text{co}C$, one has

$$T_n(V) \subset \rho \mathbb{B}_{X^*} \quad \forall n \in \mathbb{N},$$

where

$$\rho = \frac{\rho_2(2\rho_1 + \epsilon_1 - \epsilon_2)}{\epsilon_2 - \epsilon_1}.$$

Proof. Let $n \in \mathbb{N}$, $x \in V$, $v \in T_n(x)$, and let $x' \in C$ be arbitrary. By (5), we can find an $(x'', v'') \in \text{gph} T_n$, such that $\|x'' - x'\| \leq \epsilon_1$ and $\|v''\| \leq \rho_2$. By monotonicity we obtain

$$\begin{aligned} \langle v, x' - x \rangle &\leq \langle v, x'' - x \rangle + \epsilon_1 \|v\| \leq \langle v'', x'' - x \rangle + \epsilon_1 \|v\| \\ &\leq \|v''\|(\|x'\| + \|x'' - x'\| + \|x\|) + \epsilon_1 \|v\|, \end{aligned}$$

and thus,

$$\langle v, x' - x \rangle - \epsilon_1 \|v\| \leq \rho_2(\rho_1 + \epsilon_1 + (\rho_1 - \epsilon_2)) = \rho_2(2\rho_1 + \epsilon_1 - \epsilon_2).$$

Since $x' \in C$ was arbitrary, this yields

$$\langle v, y \rangle - \epsilon_1 \|v\| \leq \rho_2(2\rho_1 + \epsilon_1 - \epsilon_2) \quad \forall y \in \text{co}C - x.$$

Then $\epsilon_2 \mathbb{B}_X \subset \text{co}C - x$ implies

$$\epsilon_2 \|v\| - \epsilon_1 \|v\| \leq \rho_2(2\rho_1 + \epsilon_1 - \epsilon_2),$$

or $\|v\| \leq \rho$. Since $n \in \mathbb{N}$, $x \in V$, and $v \in T_n(x)$ were arbitrary, the claim follows. \square

Note that the above proof does not use the completeness of the space.

Lemma 9. *Let X be a Banach space, $T : X \rightrightarrows X^*$ maximal monotone and $C \subset X$ a convex set such that T is bounded on C (i.e. $T(C) \subset X^*$ is bounded). If $C \cap \text{int dom} T \neq \emptyset$, then $C \subset \text{int dom} T$.*

Proof. Let us fix some $x_0 \in C \cap \text{int dom} T$. Below we will use the following well-known result of Rockafellar [12], Theorem 1: if $\text{int dom} T \neq \emptyset$, then $\text{int dom} T$ is convex and $\overline{\text{dom} T} = \overline{\text{int dom} T}$ (for more results of this kind see also Phelps [8], Theorem 1.9, and Simons [16], Theorems 18.3 and 18.4). Here \overline{A} has the usual meaning of the (norm-)closure of a set $A \subset X$.

Suppose C is not contained in $\text{int dom} T$, i.e. there is some $z \in C \setminus \text{int dom} T$. Then, because of the above cited theorem, on the line segment $[z, x_0]$ there is a point $\bar{x} \in C$ such that $\bar{x} \in \overline{\text{dom} T} \setminus \text{int dom} T$. We show first that $\bar{x} \in \text{dom} T$. Indeed, take a sequence $\{x_n\}_{n \in \mathbb{N}}$ on the (right-open) segment $[x_0, \bar{x}[$ which strongly converges to \bar{x} . Since $[x_0, \bar{x}[\subset \text{dom} T$, for every $n \in \mathbb{N}$ there is some $x_n^* \in T(x_n)$. The sequence $\{x_n^*\}_{n \in \mathbb{N}}$ is bounded because $x_n \in C$ for every n and $T(C)$ is bounded. Hence, there is a subnet $\{x_{n_\alpha}^*\}_{\alpha \in \Lambda}$ which converges to some $\bar{x}^* \in X^*$ for the weak-star

topology in X^* . Take an arbitrary $(y, y^*) \in \text{gph } T$. Then, using the monotonicity of T , for any $\alpha \in \Lambda$ we have

$$\begin{aligned} \langle \bar{x}^* - y^*, \bar{x} - y \rangle &= \langle \bar{x}^* - x_{n_\alpha}^*, \bar{x} - y \rangle + \langle x_{n_\alpha}^* - y^*, x_{n_\alpha} - y \rangle \\ &\quad + \langle x_{n_\alpha}^* - y^*, \bar{x} - x_{n_\alpha} \rangle \\ &\geq \langle \bar{x}^* - x_{n_\alpha}^*, \bar{x} - y \rangle + \langle x_{n_\alpha}^* - y^*, \bar{x} - x_{n_\alpha} \rangle. \end{aligned}$$

After passing to the limit, we obtain $\langle \bar{x}^* - y^*, \bar{x} - y \rangle \geq 0$, and since $(y, y^*) \in \text{gph } T$ is arbitrary and T is maximal we get $\bar{x}^* \in T(\bar{x})$ and, thus, $\bar{x} \in \text{dom } T$.

Furthermore, observe that the operator $T'(\cdot) = T(\cdot) - \bar{x}^*$ is maximal monotone and has the same domain as T , and, moreover, $T'(C) = T(C) - \bar{x}^*$. Thus, we may suppose that $0_{X^*} \in T(\bar{x})$. Since $\bar{x} \notin \text{int dom } T$, using again the convexity of $\text{int dom } T$, there exists a non-zero element $\tilde{x}^* \in X^*$ so that

$$\langle \tilde{x}^*, \bar{x} \rangle \geq \langle \tilde{x}^*, y \rangle \quad \forall y \in \text{dom } T.$$

Let us now fix some $\lambda > 0$ and take again an arbitrary $(y, y^*) \in \text{gph } T$. Using the last inequality, the monotonicity of T and $0_{X^*} \in T(\bar{x})$, we have

$$\langle \lambda \tilde{x}^* - y^*, \bar{x} - y \rangle = \langle \lambda \tilde{x}^*, \bar{x} - y \rangle + \langle 0_{X^*} - y^*, \bar{x} - y \rangle \geq 0.$$

Thus, $\lambda \tilde{x}^* \in T(\bar{x})$ because of the maximal monotonicity of T . Since $\lambda > 0$ was arbitrary, this shows that $T(\bar{x})$ is not bounded—a contradiction because $\bar{x} \in C$ and $T(C)$ was bounded. Therefore, $C \subset \text{int dom } T$ and the proof of the lemma is completed. \square

Lemma 9 together with Theorem 6 implies, in particular, the following.

Corollary 10. *Let X be a reflexive Banach space. If a collection $\{T_i\}_{i \in \mathcal{I}}$ of maximal monotone mappings from X to X^* is uniformly locally bounded at a point $x \in \bigcap_{i \in \mathcal{I}} \text{dom } T_i$, then there is a non-empty open convex set $U \subset \bigcap_{i \in \mathcal{I}} \text{dom } T_i$.*

Proof. By local convexity of the space, we can assume that the set V in Definition 7 is convex. The result then follows by combining Lemma 9 and Theorem 6. \square

Our second main result is the following extension of [10, Theorem 2.2].

Theorem 11. *Let X be reflexive and let $\{T_n, T \mid n \in \mathbb{N}\}$ be maximal monotone mappings from X into X^* such that $T_n \xrightarrow{gd} T$. If $x_0 \in \text{int dom } T$, then for N large enough, $\{T_n\}_{n \geq N}$ is uniformly locally bounded at x_0 .*

Proof. By Theorem 6, we can find $\epsilon_0, \rho_0 > 0$, such that $\mathbb{B}(x_0, \epsilon_0) \subset \text{dom } T$ and $T(\mathbb{B}(x_0, \epsilon_0)) \subset \rho_0 \mathbb{B}_{X^*}$. Let $\epsilon \in (0, \epsilon_0/3)$ be arbitrary. By the graph-distance convergence, we can then find an N such that

$$(6) \quad T_n(\mathbb{B}(x, \epsilon)) \cap (\rho_0 + \epsilon) \mathbb{B}_{X^*} \neq \emptyset \quad \forall x \in \mathbb{B}(x_0, \epsilon_0), \quad \forall n \geq N.$$

Defining $V = \mathbb{B}(x_0, \epsilon)$, and $\epsilon_2 = 2\epsilon$, we get $V + \epsilon_2 \mathbb{B}_X \subset \mathbb{B}(x_0, \epsilon_0)$; so by Lemma 8, there is a $\rho > 0$, such that

$$(7) \quad T_n(\mathbb{B}(x_0, \epsilon)) \subset \rho \mathbb{B}_{X^*} \quad \text{for all } n \geq N.$$

To finish the proof, it suffices to show that $\mathbb{B}(x_0, \epsilon) \subset \text{dom } T_n$ for all $n \geq N$. Let $n \geq N$. Choosing $x = x_0$ in (6), we see that there is a point $x' \in \mathbb{B}(x_0, \epsilon)$, such that $T_n(x') \neq \emptyset$. By Theorem 6, $x' \in \text{int dom } T_n$, and then (7) allows us to apply Lemma 9 in order to conclude that $\mathbb{B}(x_0, \epsilon) \subset \text{dom } T_n$. \square

Theorem 11 shows that the boundedness condition in (Q) of Theorem 5 is already implied by $x_0 \in \text{dom } S \cap \text{int dom } T$. We can thus state Theorem 5 in the following much simpler form.

Theorem 12. *Let $\{S_n, S \mid n \in \mathbb{N}\}$ and $\{T_n, T \mid n \in \mathbb{N}\}$ be families of maximal monotone mappings in a Hilbert space such that $S_n \xrightarrow{gd} S$ and $T_n \xrightarrow{gd} T$. If $\text{dom } S \cap \text{int dom } T \neq \emptyset$, then*

$$S_n + T_n \xrightarrow{gd} S + T.$$

Much as in [3], Corollary 3.5, this yields the sum-rule for the bounded Hausdorff convergence of convex functions obtained by Beer and Lucchetti [6] in a more general setting.

Remark. Having in mind that our local uniform boundedness result (Theorem 11) is proved in a reflexive Banach space, a question may be raised whether the sum-rule from Theorem 12 holds true outside the setting of Hilbert spaces. This question, as well as the similar one for our Theorem 2, remain open for the moment, since it is not seen how the existing proof of Theorem 5 (as well as our proof of Theorem 2) can be extended to the reflexive case. The reason is that these proofs substantially use estimations (as in Proposition 1) which rely on the fact that the underlying space is Hilbertian.

ACKNOWLEDGMENTS

The authors are very grateful to Hedy Attouch for his valuable idea of using Proposition 1 in the proof of Theorem 2. We wish to express our gratitude also to an anonymous referee for several remarks that helped to improve and clarify the presentation of the results.

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