THERE IS NO SEPARABLE UNIVERSAL II$_1$-FACTOR

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(Communicated by David R. Larson)

ABSTRACT. Gromov constructed uncountably many pairwise nonisomorphic discrete groups with Kazhdan's property (T). We will show that no separable II$_1$-factor can contain all these groups in its unitary group. In particular, no separable II$_1$-factor can contain all separable II$_1$-factors in it. We also show that the full group C^*-algebras of some of these groups fail the lifting property.

1. RESULTS

We recall that a discrete group Γ is said to have Kazhdan's property (T) if the trivial representation is isolated in the dual $\hat{\Gamma}$ of Γ, equipped with the Fell topology. This is equivalent to saying that there exist a finite subset E of generators in Γ and a decreasing function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$ such that the following is true: if π is a unitary representation of Γ on a Hilbert space $\mathcal{H}$ and $\xi \in \mathcal{H}$ is a unit vector with $\varepsilon = \max_{s \in E} \|\pi(s)\xi - \xi\|$, then there is a vector $\eta \in \mathcal{H}$ with $\|\xi - \eta\| < f(\varepsilon)$ (in particular, $\eta \neq 0$ when ε is small enough) such that $\pi(s)\eta = \eta$ for all $s \in \Gamma$. We refer the reader to [HV] and [V] for the information of Kazhdan's property (T). We recall that a discrete group Γ is said to be quasifinite if all of its proper subgroups are finite, and it is said to be of infinite conjugacy classes (abbreviated to ICC) if all nontrivial conjugacy classes in Γ are infinite. We note that a discrete group Γ is ICC if and only if its group von Neumann algebra $L\Gamma$ is a factor. We also observe that a group which is quasifinite and ICC has to be simple.

Gromov (Corollary 5.5.E in [G]) claimed that any torsion-free noncyclic hyperbolic group has a quotient group all of whose proper subgroups are cyclic of prescribed orders (cf. Theorem 3.4 in [V]). This claim was partly confirmed by Olshanskii (Corollary 4 in [O]). Actually, what Olshanskii proved there is that any torsion-free noncyclic hyperbolic group has a nontrivial quasifinite quotient group. We observe that Olshanskii's argument gives us the following.

Theorem 1 (Gromov-Olshanskii). *Any torsion-free noncyclic hyperbolic group has uncountably many pairwise nonisomorphic quotient groups all of which are quasifinite and ICC. In particular, there is a discrete group Γ with Kazhdan's property* (T) *which has uncountably many pairwise nonisomorphic quotient groups $\{\Gamma_\alpha\}_{\alpha \in I}$ all of which are simple and ICC.*

Connes conjectured that a discrete group Δ with Kazhdan's property (T) and the ICC property is uniquely determined by its group von Neumann algebra $L\Delta$. The following theorem and its corollary, which was suggested by S. Popa, confirm Connes' conjecture for $\{\Gamma_\alpha\}_{\alpha \in I}$ "modulo countable sets" and solve Problem 4.4.29 in [S], Conjecture 4.5.5 in [P1] and Problem III.45 in [H]. See also Theorem 1 in [P2] and its remarks.

Theorem 2. *Let Γ and $\{\Gamma_\alpha\}_{\alpha \in I}$ be as in Theorem 1 and let M be a separable II_1-factor. Then, the set*

$$\{\alpha \in I : \text{the unitary group } \mathcal{U}(M) \text{ of } M \text{ contains a subgroup isomorphic to } \Gamma_\alpha\}$$

is at most countable.

Recall that two II_1-factors M and N are said to be stably equivalent if there are $n \in \mathbb{N}$ and a projection $p \in \mathbb{M}_n(M)$ such that $p\mathbb{M}_n(M)p$ is isomorphic to N.

Corollary 3. *Let Γ and $\{\Gamma_\alpha\}_{\alpha \in I}$ be as in Theorem 1 and let M be a separable II_1-factor. Then, the set*

$$\{\alpha \in I : M \text{ contains a subfactor which is stably equivalent to } L\Gamma_\alpha\}$$

is at most countable.

In connection with Connes' embedding problem [C], it would be interesting to know whether all (or at least one of) the Γ_α's are embeddable into the unitary group $\mathcal{U}(R^\omega)$ of the ultrapower R^ω of hyperfinite II_1-factors. Since each Γ_α arises as a limit of hyperbolic groups, we observe that if all hyperbolic groups are embeddable into $\mathcal{U}(R^\omega)$, then so is Γ_α. We remark that whether all hyperbolic groups are residually finite (and thus embeddable into $\mathcal{U}(R^\omega)$) is one of the major open problems in geometric group theory.

Let us consider the category of unital C^*-algebras and unital completely positive maps. A C^*-algebra A is said to be complementary universal for a class $\mathcal{C}$ of C^*-algebras if for every member B of $\mathcal{C}$, there are unital completely positive maps $\psi \colon B \to A$ and $\varphi \colon A \to B$ such that $\varphi\psi = \mathrm{id}_B$. It follows from Kirchberg's theorem [K2] that any separable C^*-algebra not of type I is complementary universal for the class of separable nuclear C^*-algebras. The full group C^*-algebra $C^*\mathbb{F}_\infty$ of the free group $\mathbb{F}_\infty$ on countably many generators is complementary universal for the class of separable C^*-algebras with the lifting property (abbreviated to LP). See [K1] for information on the LP. It is not known whether there exists a separable complementary universal C^*-algebra for the class of separable exact C^*-algebras.

Theorem 4. *Let Γ and $\{\Gamma_\alpha\}_{\alpha \in I}$ be as in Theorem 1 and let $\mathcal{C} = \{C^*\Gamma_\alpha : \alpha \in I\}$ or $\mathcal{C} = \{C^*_{\mathrm{red}}\Gamma_\alpha : \alpha \in I\}$. Then, there is no separable complementary universal C^*-algebra for $\mathcal{C}$.*

From the above discussion, we immediately obtain the following corollary.

Corollary 5. *The full group C^*-algebra $C^*\Gamma_\alpha$ of Γ_α fails the LP for some $\alpha \in I$.*

2. Proofs

Proof of Theorem 1. Since there exists a torsion-free noncyclic hyperbolic group with Kazhdan's property (T) (e.g., a co-compact lattice in $Sp(n, 1)$ or in $F_{4(-20)}$), the second part is a straight consequence of the first. We just indicate how to modify the proof of Corollary 3 in [O] to obtain the first part of our Theorem

1. So, we stick by the notation used in [O]. Let $G = \{g_1, g_2, \ldots\}$ be a torsion-free noncyclic hyperbolic group. It follows that G is ICC since the set $E(G)$ of all $x \in G$ whose conjugacy class is finite is a finite subgroup in G (cf. Proposition 1 in [O]). Recall that the quasifinite quotient group G' was the inductive limit of a sequence of epimorphisms $G = G_0 \to G_1 \to G_2 \to \cdots$. By the construction, every $E(G_i)$ is trivial, or equivalently every G_i is ICC. Hence, manipulating the construction, for every i and $j \leq i$, we can carry at least i mutually distinct elements from the conjugacy class of g_j in G_i injectively into G' unless $g_j = 1$ in G_i. This ensures the ICC property of G'. In the construction, there has to be infinitely many i's such that g_i is torsion-free in G_{2i-2}. For such i, we may choose an arbitrarily large number for the order of g_i in G_{2i-1} which will be equal to that in G'. Combined with a diagonal argument, this implies that there are uncountably many normal subgroups in G all of whose corresponding quotient groups are quasifinite and ICC. Theorem 1 now follows from this result and Lemma III.42 in [H]. $\square$

Proof of Theorem 2. To prove the theorem by reductio ad absurdum, suppose that

$$I_0 = \{\alpha \in I : \mathcal{U}(M) \text{ contains a subgroup isomorphic to } \Gamma_\alpha\}$$

is uncountable. For each $\alpha \in I_0$, let $u_\alpha \colon \Gamma \to \mathcal{U}(M)$ be a nontrivial homomorphism which factors through Γ_α. We fix a standard representation of M on $\mathcal{H}$ with a unit cyclic separating trace vector ξ in $\mathcal{H}$. It follows that there are $\delta > 0$ and an uncountable subset I_1 of I_0 such that $\max_{s \in \Gamma} \|u_\alpha(s)\xi - \xi\| > \delta$ for all $\alpha \in I_1$.

Let a finite subset E of generators in Γ and a function f be as in the above definition of Kazhdan's property (T). Take $\varepsilon > 0$ small enough so that $2f(\varepsilon) < \delta$. Since $\mathcal{H}$ is separable and I_1 is uncountable, there are distinct α and β in I_1 such that $\max_{s \in E} \|u_\alpha(s)\xi - u_\beta(s)\xi\| < \varepsilon$. We consider the unitary representation $\pi \colon \Gamma \ni s \mapsto u_\alpha(s)Ju_\beta(s)J \in \mathbb{B}(\mathcal{H})$, where J is the canonical conjugation on $\mathcal{H}$ associated with M and ξ. Then, we have $\max_{s \in E} \|\pi(s)\xi - \xi\| < \varepsilon$. It follows from Kazhdan's property (T) of Γ that there is a vector $\eta \in \mathcal{H}$ with $\|\xi - \eta\| < f(\varepsilon)$ such that $\pi(s)\eta = \eta$ for all $s \in \Gamma$. Let $\Delta = \{s \in \Gamma : u_\alpha(s)\eta = \eta\}$. It is easy to see that Δ is a subgroup of Γ and that Δ contains the normal subgroups $\ker u_\alpha$ and $\ker u_\beta$. Since Γ_α and Γ_β are simple and $\ker u_\alpha$ and $\ker u_\beta$ are distinct, we actually have $\Delta = \Gamma$. It follows that $\max_{s \in \Gamma} \|u_\alpha(s)\xi - \xi\| < 2f(\varepsilon) < \delta$, which is absurd. $\square$

Proof of Corollary 3. It is not difficult to see that if $L\Gamma_\alpha$ is isomorphic to a (not necessarily unital) subfactor of M, then Γ_α is isomorphic to a subgroup of $\mathcal{U}(M)$. Therefore, it follows from Theorem 2 that

$$\{\alpha \in I : \mathbb{M}_n(M) \text{ contains a (not necessarily unital) subfactor isomorphic to } L\Gamma_\alpha\}$$

is at most countable for every $n \in \mathbb{N}$, and the conclusion follows. $\square$

Proof of Theorem 4. We only deal with the case where $\mathcal{C} = \{C^*\Gamma_\alpha : \alpha \in I\}$. To prove the theorem by reductio ad absurdum, suppose that there is a separable C^*-algebra A which is complementary universal for $\mathcal{C}$. We fix unital completely positive maps $\psi_\alpha \colon C^*\Gamma_\alpha \to A$ and $\varphi_\alpha \colon A \to C^*\Gamma_\alpha$ such that $\varphi_\alpha\psi_\alpha = \mathrm{id}_{C^*\Gamma_\alpha}$. Let $E \subset \Gamma$ be a finite set of generators of Γ containing 1 and let $u_\alpha(s)$ be the unitary element in $C^*\Gamma_\alpha$ corresponding to $s \in \Gamma$. Let $\varepsilon > 0$ be arbitrary. Since A is separable and I is uncountable, there are distinct α and β in I such that $\max_{s \in E} \|\psi_\alpha(u_\alpha(s)) - \psi_\beta(u_\beta(s))\| < \varepsilon$. It follows that denoting the left regular

representation of Γ_α by λ_α, we have

$$\|\frac{1}{|E|}\sum_{s\in E}\lambda_\alpha(u_\alpha(s))\otimes u_\beta(s)\|_{C^*_{\mathrm{red}}\Gamma_\alpha\otimes_{\max}C^*\Gamma_\beta}$$

$$\geq \|\frac{1}{|E|}\sum_{s\in E}\lambda_\alpha(u_\alpha(s))\otimes\varphi_\alpha\psi_\beta(u_\beta(s))\|_{C^*_{\mathrm{red}}\Gamma_\alpha\otimes_{\max}C^*\Gamma_\alpha}$$

$$\geq \|\frac{1}{|E|}\sum_{s\in E}\lambda_\alpha(u_\alpha(s))\otimes u_\alpha(s)\|_{C^*_{\mathrm{red}}\Gamma_\alpha\otimes_{\max}C^*\Gamma_\alpha}-\varepsilon$$

$$= 1-\varepsilon.$$

Since Γ has Kazhdan's property (T), if we choose $\varepsilon > 0$ sufficiently small, this implies that the trivial representation of Γ is weakly contained in $C^*_{\mathrm{red}}\Gamma_\alpha\otimes_{\max}C^*\Gamma_\beta$ (cf. Proposition 4.9 in [V]). Reasoning in the same way as the proof of Theorem 2, one can show that the trivial representation is weakly contained in $C^*_{\mathrm{red}}\Gamma_\alpha$. This is absurd. $\qquad\square$

Acknowledgment

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