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MINIMAL SPANNING TREES

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Abstract. The main result is that a recursive weighted graph having a minimal spanning tree has a minimal spanning tree that is Δ^0_2. This leads to a proof of the failure of a conjecture of Clote and Hirst (1998) concerning Reverse Mathematics and minimal spanning trees.

A *graph* is a pair $G = (V, E)$ for which V is its set of vertices and E its set of edges, an edge being a 2-element subset of V. A *weighted graph* is a pair $L = (G, w)$ (or a triple (V, E, w)) where G is a graph and $w : E \longrightarrow \mathbb{R}^+$ is a function assigning to each edge $e \in E$ a positive real number $w(e)$. If $H = (V', E')$ is a subgraph of G, then $W(H)$, the *weight* of H, is defined to be

$$W(H) = \sum_{e \in E'} w(e),$$

which is ∞ if the sum does not converge. A spanning tree $T = (V, E')$ of G is said to be *minimal* if $W(T) < \infty$ and $W(T) \leq W(T')$ for any other spanning tree T' of G.

In [1], Clote and Hirst study the recursion-theoretic and proof-theoretic strength of some statements of algorithmic graph theory. Theorem A was motivated by, and essentially disproves, a conjecture from [1]. Although this conjecture was from Reverse Mathematics (for which Simpson [2] is the recommended reference), Theorem A concerns just recursive graph theory.

A graph $G = (V, E)$ is *recursive* if $V \subseteq \omega$ and both V and E are recursive. A weighted graph $L = (G, w)$ is recursive if, in addition, the weight function $w : E \longrightarrow \mathbb{R}^+$ is recursive. In order to avoid making some further technical definitions as to what it means for w to be recursive, we will henceforth assume (as was done in [1]) that w takes values in $\mathbb{Q}^+$, the set of positive rationals. This restriction will not affect the results. Other terminology from recursion theory applied to graphs has the obvious meaning.

Theorem A. *Every recursive weighted graph having a minimal spanning tree has a minimal spanning tree that is* Δ^0_2.

By considering each spanning tree of a graph as a basis, we get that a connected graph is a matroid. A weighted matroid is a pair (M, w), where M is a matroid

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and $w : M \longrightarrow \mathbb{R}^+$. With other terms being given their most natural definitions, Theorem A can be generalized to matroids.

Theorem B. *Every recursive weighted matroid having a minimal basis has a minimal basis that is* Δ^0_2.

In fact, all the results and proofs in §§1 and 2 can be straightforwardly generalized to weighted matroids. These generalizations are left to the interested reader.

Minimal spanning trees and their generalization, pseudo-minimal spanning trees, are discussed in §1. In §2, we consider results concerning recursive weighted graphs. Examples in §3 show that some of these results are best possible. Finally, in §4, there are results of Reverse Mathematics.

1. Minimal spanning trees

The notion of a cycle of a graph is very familiar. To be definitive, we say that a *cycle* of the graph (V, E) is a nonempty finite subset $C \subseteq E$ such that each vertex of V is in exactly 0 or 2 edges in C and for each nonempty proper subset $C' \subseteq C$ there is a vertex that is in exactly 1 edge of C'. We now make a couple of crucial definitions.

Definition. Let $L = (V, E, w)$ be a weighted graph. We call a finite subset $B \subseteq E$ a *sprig* if for each cycle C of (V, E), there is $e \in C \backslash B$ such that $w(e) \geq w(e')$ for all $e' \in C$. A spanning tree $T = (V, E')$ is *pseudo-minimal* if each finite $B \subseteq E'$ is a sprig.

Lemma 1.1. *Every minimal spanning tree is pseudo-minimal.*

Proof. Let $L = (V, E, w)$ be a weighted graph with minimal spanning tree $T = (V, E')$. Let C be a cycle, and let r be the maximum weight of an edge in this cycle. For a contradiction, suppose that $e \in E'$ for each edge $e \in C$ for which $w(e) = r$. Let e be any such edge. The subgraph of T obtained by deleting the edge e has two components. So there is some $e' \in C$ such that $e' \neq e$ and e' joins these two components. Clearly, $e' \notin E'$, and so $w(e') < w(e)$. Then $T' = (V, (E' \backslash \{e\}) \cup \{e'\})$ is a spanning tree and $W(T') < W(T)$, which is a contradiction. □

Lemma 1.2. *Suppose B is a sprig of the weighted graph $L = (V, E, w)$, $e' \in E \backslash B$, and $C \subseteq B \cup \{e'\}$ is a cycle. If $e \in C \cap B$ is such that $w(e) \geq w(e')$, then $(B \backslash \{e\}) \cup \{e'\}$ is a sprig.*

Proof. Since B is a sprig, it follows that $w(e') = w(e) \geq w(c)$ for each $c \in C$. Let $B' = (B \backslash \{e\}) \cup \{e'\}$. To prove that B' is a sprig, we consider a cycle D, intending to show that there is $f \in D \backslash B'$ such that $w(f) \geq w(f')$ for all $f' \in D$. Because B is a sprig, there is $d \in D \backslash B$ such that $w(d) \geq w(d')$ for all $d' \in D$. We can then let f be such a d unless e' is the only such d. Thus, we can assume that $e' \in D$ and that $w(e') \geq w(f')$ whenever $f' \in D$. In particular, $e \notin D$.

There is a cycle $D' \subseteq C \cup D$ such that $e \in D'$ and $e' \notin D'$. Let $f \in D' \backslash B$ be such that $w(f) \geq w(f')$ for any $f' \in D'$. Then $w(f) \geq w(e)$, so that $w(f) \geq w(f')$ whenever $f' \in D$. Clearly, $f \in D \backslash B$; so the lemma is proved. □

Theorem 1.3. *T is a minimal spanning tree iff T is a pseudo-minimal spanning tree and $W(T) < \infty$.*

Proof. One direction is just Lemma 1.1. For the other direction, assume $T = (V, E_1)$ is pseudo-minimal and $W(T) < \infty$. It will suffice to prove, for any spanning tree $T' = (V, E_2)$, that if $r \in \mathbb{R}^+$, then

$$|\{e \in E_1 : w(e) > r\}| \leq |\{e \in E_2 : w(e) > r\}| \, .$$

We can assume that T' was chosen first to minimize $|\{e \in E_2 : w(e) > r\}|$ and second to maximize $|\{e \in E_1 \cap E_2 : w(e) > r\}|$. We then show that $\{e \in E_1 : w(e) > r\} \subseteq E_2$.

Suppose this is false, and let $e_1 \in E_1 \backslash E_2$ be such that $w(e_1) > r$. Then there is a cycle $C_2 \subseteq E_2 \cup \{e_1\}$. The subgraph of T obtained by deleting the edge e_1 has two components. So there is some $e_2 \in C_2 \cap E_2$ such that e_2 joins these two components. Then $e_2 \notin E_1$, and $w(e_2) \geq w(e_1) > r$ since T is a pseudo-minimal spanning tree. The spanning tree $(V, (E_2 \backslash \{e_2\}) \cup \{e_1\})$ contradicts the maximality condition on T'. □

If $L = (G, w)$ has a spanning tree of finite weight, then G is a countable graph and, by the previous theorem, the minimal and pseudo-minimal spanning trees are exactly the same. The following theorem characterizes those countable weighted graphs having pseudo-minimal spanning trees.

Theorem 1.4. *Let $L = (V, E, w)$ be a countable weighted graph. Then L has a pseudo-minimal spanning tree iff for every finite $X \subseteq V$ there is a sprig B such that X is a subset of a component of (V, B).*

Proof. One direction is trivial and does not depend on the countability of L. For the other direction, we will construct a pseudo-minimal spanning tree of $L = (G, w)$. Since G is countable, we might as well assume that $G = (\omega, E)$.

First, some terminology: Say that a sprig B is *connected* if any component of (ω, B) to which 0 does not belong is a singleton. For a connected sprig B, the component of (ω, B) to which 0 belongs is the *main* component. The construction will rely on the following claim:

> *Suppose $X \subseteq X' \subseteq \omega$, and B is a connected sprig having main component X. If there is a connected sprig B' having main component X', then there is such a sprig $B' \supseteq B$.*

To prove the claim, let B' be such a sprig that minimizes $|B \backslash B'|$. If this number is 0, we are done; so suppose it is not. Let $e \in B \backslash B'$. Let $C' \subseteq B' \cup \{e\}$ be a cycle. Since B is a sprig, there is $e' \in C' \backslash B$ such that $w(e') \geq w(e)$. By Lemma 1.2, $(B' \backslash \{e'\}) \cup \{e\})$ is a sprig, and this sprig is easily seen to be connected having the same main component as B'. This sprig contradicts the minimality condition on B', proving the claim.

Then there are a sequence $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ of finite subsets of ω and a sequence $B_0, B_1, B_2, \ldots$ of sprigs such that if $n < \omega$, then $n \in X_n$ and B_n is a connected sprig with main component X_n. By the claim, we can further require that $B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$. Then (ω, E'), where $E' = \bigcup_{n<\omega} B_n$, is a pseudo-minimal spanning tree. □

The question of whether, or to what extent, Theorem 1.4 can be extended to uncountable weighted graphs is open.

If T_1, T_2 are minimal spanning trees, then $W(T_1) = W(T_2)$. The following lemma shows that much more is true.

Lemma 1.5. *If $T_1 = (V, E_1)$ and $T_2 = (V, E_2)$ are pseudo-minimal spanning trees of L, $r \in \mathbb{R}^+$, and $\{e \in E_1 : w(e) = r\}$ is finite, then $|\{e \in E_1 : w(e) = r\}| = |\{e \in E_2 : w(e) = r\}|$.*

Proof. Consider T_1 and r as being fixed, and then assume T_2 is chosen first to minimize $|\{e \in E_2 : w(e) = r\}|$ and second to maximize $|\{e \in E_1 \cap E_2 : w(e) = r\}|$. To prove the lemma, we show that $\{e \in E_2 : w(e) = r\} \subseteq E_1$.

Suppose this is false, and that $e_2 \in E_2 \backslash E_1$ is such that $w(e_2) = r$. As in the proof of Lemma 1.3, there are $e_1 \in E_2 \backslash E_1$ and cycles $C_1 \subseteq E_1 \cup \{e_2\}$ and $C_2 \subseteq E_2 \cup \{e_1\}$. Since T_1, T_2 are pseudo-minimal , it follows that $w(e_1) = w(e_2)$. By Lemma 1.2, $(V, (E_2 \backslash \{e_2\}) \cup \{e_1\})$ is a pseudo-minimal spanning tree, contradicting the maximality condition on T_2. □

Lemma 1.6. *If $L = (V, E)$ has a pseudo-minimal spanning tree and $B \subseteq E$ is a sprig, then it has a pseudo-minimal spanning tree (V, E') such that $B \subseteq E'$.*

Proof. Let $T = (V, E')$ be a pseudo-minimal spanning tree that maximizes $|E' \cap B|$. Then $B \subseteq E'$. For, suppose not and let $e \in B \backslash E'$. There is a cycle $C \subseteq E' \cup \{e\}$. Then there is $e' \in C \backslash B$ such that $w(e') \geq w(e)$. It follows from Lemma 1.2 that $(V, (E' \backslash \{e'\}) \cup \{e\})$ is a pseudo-minimal spanning tree, contradicting the maximality condition on T. □

Theorem 1.7. *L has a minimal spanning tree iff L has a pseudo-minimal spanning tree and there is $r \in \mathbb{Q}^+$ such that $W(B) \leq r$ for each sprig B.*

Proof. This is an easy consequence of the previous results. □

2. Recursive weighted graphs

Theorems 1.4 and 1.7 have the following corollaries.

Corollary 2.1. *Let $\langle L_i : i < \omega \rangle$ be a recursive sequence of recursive weighted graphs. Then $\{i < \omega : L_i$ has a pseudo-minimal spanning tree$\}$ is a Π^0_3 set.* □

Corollary 2.2. *Let $\langle L_i : i < \omega \rangle$ be a recursive sequence of recursive weighted graphs. Then $\{i < \omega : L_i$ has a minimal spanning tree$\}$ is the difference of two Σ^0_3 sets.* □

The two previous corollaries will be shown in the next section to be best possible results.

By Theorem 1.3, Theorem A follows from the next corollary.

Corollary 2.3. *If L is a recursive weighted graph having a pseudo-minimal spanning tree, then it has a pseudo-minimal spanning tree that is Δ^0_2.*

Proof. The construction of the sequence $B_0, B_1, B_2, \ldots$ obtained in the proof of Theorem 1.4 is recursive in $\mathbf{0}'$, so that the spanning tree T is Σ^0_2. But then the complement of E' is also Σ^0_2, and so T is Δ^0_2. □

The previous corollary can be improved if the pseudo-minimal spanning tree is unique.

Corollary 2.4. *If L is a recursive weighted graph having a unique pseudo-minimal spanning tree, then this spanning tree is Π^0_1.*

Proof. An edge e is in the minimal spanning tree iff $\{e\}$ is a sprig. □

3. Examples

This section contains some examples which show that some of the results in §2 are best possible. Recall that a graph is n-regular iff each vertex has degree n.

Proposition 3.1. *Let $X \subseteq \omega$ be a Π^0_1 set. There is a recursive weighted 3-regular graph that has a unique minimal spanning tree T, and T and X are in the same m-degree.*

Proof. Let J be the graph having vertices $a_0, a_1, a_2, \dots, b_0, b_1, b_2, \dots$ and edges $\{a_i, a_{i+1}\}$, $\{b_i, b_{i+1}\}$, $\{a_i, b_i\}$ for $i < \omega$. For $n < \omega$, let J_n be the graph having vertices $a_0, a_1, \dots, a_n, a_{n+1}, b_0, b_1, \dots, b_n, b_{n+1}, c, d$ and edges $\{c, d\}$, $\{a_{n+1}, c\}$, $\{d, b_{n+1}\}$, $\{a_{n+1}, d\}$, $\{c, b_{n+1}\}$ and $\{a_i, a_{i+1}\}$, $\{b_i, b_{i+1}\}$, $\{a_i, b_i\}$ for $i \leq n$. Notice that all vertices of these graphs have degree 3 except for a_0 and b_0, which have degree 2. Call the edge $\{a_0, b_0\}$ the *bottom* edge.

These graphs are turned into weighted graphs by adjoining a *normal* weight function v, which we define as follows: $v(\{a_0, b_0\}) = 8$; $v(\{a_i, b_i\}) = 9$ if $i > 0$; $v(\{a_i, a_{i+1}\}) = v(\{b_i, b_{i+1}\}) = 2^{-i}$ for each i; and for J_n, $v(\{c, d\}) = v(\{a_{n+1}, c\}) = v(\{d, b_{n+1}\}) = 1$ and $v(\{a_{n+1}, d\}) = v(\{d, b_{n+1}\}) = 9$. Each of these weighted graphs has a unique minimal spanning tree (the weights of which are uniformly bounded by, say, 20). The edges of the minimal spanning tree for J are the bottom edge and $\{a_i, a_{i+1}\}, \{b_i, b_{i+1}\}$ for $i < \omega$. The edges of the minimal spanning tree for J_n are $\{c, d\}, \{a_{n+1}, c\}, \{d, b_{n+1}\}$ and $\{a_i, a_{i+1}\}, \{b_i, b_{i+1}\}$ for $i \leq n$. Notice that the bottom edge of J_n is not in its minimal spanning tree.

Now let X be a Π^0_1 set. If X is cofinite, then there are trivial examples. So let us assume that X is not cofinite. Let $k_0, k_1, k_2, \dots$ be a recursive, one-to-one enumeration of $\omega \backslash X$. For each $n < \omega$, let the graph G_n be J_i, if $n = k_i$, and J otherwise. Let G' be the graph formed as the disjoint union of the G_n's, making it connected by adding an edge from b^n_0, the b_0 of G_n, to a^{n+1}_0, the a_0 of G_{n+1}. Every vertex of G' has degree 3 with the exception of a^0_0 (which is the a_0 of G_0), which has degree 2. It is an easy matter to slightly augment G' to get 3-regular G. The details are unimportant, but for completeness, the following is provided: add to G' five new vertices c_0, c_1, c_2, c_3, c_4 and edges $\{c_0, c_1\}, \{c_1, c_2\}, \{c_1, c_3\}, \{c_2, c_3\}, \{c_2, c_4\}, \{c_3, c_4\}, \{c_0, c_4\}$ and $\{c_0, a^0_0\}$. We turn this graph into a weighted graph by adjoining the weight function w, where w restricted to G_n is 2^{-n} times the normal weight function of G_n, $w(\{b^n_0\}, a^{n+1}_0\}) = 2^{-n}$, $w(\{c_0, c_1\}) = w(\{c_1, c_2\}) = w(\{c_2, c_3\}) = w(\{c_3, c_4\}) = w(\{c_0, a^0_0\}) = 1$ and $w(\{c_1, c_3\}) = w(\{c_2, c_4\}) = w(\{c_0, c_4\}) = 9$.

It is clear that this is a recursive weighted 3-regular graph having a unique spanning tree T. Moreover, the bottom edge of G_n is in T iff $n \in X$. It is then easily seen that X and T are in the same m-degree. □

Corollary 3.2. *There is a recursive weighted 3-regular graph that has a unique minimal spanning tree T, and T is Π^0_1-complete.*

Proof. Let X in Proposition 3.1 be Π^0_1-complete. □

Theorem 3.3. *There is a recursive weighted graph that has a minimal spanning tree but has none that is Π^0_1.*

Proof. For brevity, this proof will be a little sketchy. Let $[V]^2$ be the set of 2-element subsets of V; thus, $(V, [V]^2)$ is the complete graph on V.

We begin with a construction of a weighted graph from a sequence from $[\omega]^2$. Given a sequence $e_0, e_1, e_2, \dots$ of edges in $[\omega]^2$, we construct the weighted graph $L = (G, w) = (V, E, w)$ such that $V \subseteq \omega$, L has a minimal spanning tree, and $(V, [\omega]^2 \backslash \{e_0, e_1, e_2, \cdots\})$ is not a minimal spanning tree. We first construct a sequence $L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$ of finite weighted graphs, where $L_n = (V_n, E_n, w_n) = (G_n, w_n)$ and G_n is an induced subgraph of G_{n+1}, and then let L be its union.

We will make use of the graph $H = (X, D)$ having vertex set $X = \{0, 1, 2, \dots, 7\}$ and edge set $C \cup \{\{0, 4\}\}$, where C is the cycle $\{\{0, 1\}, \{1, 2\}, \dots, \{6, 7\}, \{7, 0\}\}$. This graph is expanded to a weighted graph (H, v), where $v(\{0, 4\}) = 1$, $v(\{3, 4\}) = v(\{4, 5\}) = 1/2$, and $v(e) = 1/4$ for each other edge e. Each of the graphs G_n will have a preferred induced subgraph isomorphic to H with a preferred isomorphism. We will refer to the i-vertex of G_n, where $i \in X$, as the image of i under this preferred isomorphism, and similarly we will refer to the e-edge of G_n, where $e \in D$.

We start by letting $G_0 = (V_0, E_0)$ be isomorphic to H via the preferred isomorphism, where $V_0 \subseteq \omega$; the weight function w_0 is such that the preferred isomorphism is also an isomorphism of w_0 and v.

We will obtain L_{n+1} from L_n in one of three ways, which will depend on $\{e_0, e_1, \dots, e_n\}$.

(1) *The* $\{4, 5\}$*-edge of* G_n *is in* $\{e_0, e_1, \dots, e_n\}$. We then get G_{n+1} by letting the 1-vertex and 2-vertex of G_n be the 0-vertex and 4-vertex of G_{n+1} respectively, and then adjoining 6 new vertices (not occurring in any of the edges $e_0, e_1, \dots, e_n$) which will be the i-vertices for $i \in \{1, 2, 3, 5, 6, 7\}$. We let w_{n+1} agree with w_n on E_n. Each new edge f will be an e-edge for some $e \in C$, and then we let $w_{n+1}(f) = 2^{-2(n+1)} v(e)$.

(2) *The* $\{3, 4\}$*-edge of* G_n *is in* $\{e_0, e_1, \dots, e_n\}$ *and* (1) *fails.* We then get G_{n+1} by letting the 6-vertex and 7-vertex of G_n be the 0-vertex and 4-vertex of G_{n+1} respectively, and then continuing as in (1).

(3) *Both* (1) *and* (2) *fail.* Let $L_{n+1} = L_n$ with the same preferred isomorphism.

The weighted graph L has a minimum spanning tree: if L is finite, then there is no problem; if L is infinite, then for each $n < \omega$, delete the $\{0, 4\}$-edge of G_n and whichever of the $\{3, 4\}$-edge and $\{4, 5\}$-edge of G_n that does not occur first in the list $e_0, e_1, e_2, \dots$.

Let $E' = [V]^2 \backslash \{e_0, e_1, e_2, \cdots\}$. Then (V, E') is not a minimal spanning tree. For, as can be checked, if every finite $B \subseteq E'$ is a sprig, then the 0-vertex and 4-vertex of G_0 are in different components of (V, E').

We now construct the example by diagonalization. Consider a recursive, doubly-indexed sequence $\langle e_{ij} : i, j < \omega \rangle$ such that for any recursive sequence $e_0, e_1, e_2, \dots$ from $[\omega]^2$, there is $j < i$ such that $e_i = e_{ij}$ for each i. In a uniformly recursive way, obtain a weighted graph L_j from the sequence $e_{0j}, e_{1j}, e_{2j}, \dots$ as just described, but now arrange that the L_j's have pairwise disjoint vertex sets. Let L'_j be obtained from L_j by multiplying the weight function by 2^{-j}. Then let L be their disjoint union except that between L'_j L'_{j+1} an edge with weight 2^{-j} is adjoined. Since each L_j has a minimal spanning tree, it is clear that L also does.

But L does not have a Π^0_1 minimal spanning tree. For, suppose that (V, E) is one. Let $j < \omega$ be such that $E' = [\omega]^2 \backslash \{e_{0j}, e_{1j}, e_{2j}, \dots\}$, and let $T'_j = (V', E', w')$. Then $(V', E \cap E')$ is a minimal spanning tree of T_j, which is a contradiction. $\square$

This proof is easily modified so as to produce a recursive weighted graph having a minimal spanning tree but none that is the difference of two r.e. sets. Furthermore, analogous results up through the difference hierarchy can be obtained.

The next lemma gives an example showing that Corollary 2.1 cannot be improved. This lemma and the one following it are used to prove Theorem 3.6 showing that Corollary 2.2 is best possible. The proofs of these two lemmas have a couple of common features. One is that they both make use of the same standard example of Σ^0_3-complete set: If W_i is the i-th r.e. set, then $\{i < \omega : W_i \text{ is cofinite}\}$ is Σ^0_3-complete. The other is that they both involve the same graphs, which we now define. Given a set $X \subseteq \omega\backslash\{0\}$, we associate the graph $G = (V, E)$, where $V = X \cup \{0\}$ and $E = \{\{0, x\} : x \in X\} \cup \{\{x, x+1\} : x, x+1 \in X\}$.

Lemma 3.4. *There is a recursive sequence $\langle L_i : i < \omega\rangle$ of recursive weighted graphs such that $\{i < \omega : L_i$ has a pseudo-minimal spanning tree$\} = \{i < \omega : L_i$ has a unique minimal spanning tree$\}$, and this set is Π^0_3-complete.*

Proof. For each $i < \omega$, let G_i be the graph associated with $\{x+1 : x \in W_i\}$. Let L_i be obtained from G_i by adjoining the weight function w_i, where $w_i(\{0, x\}) = 2^{-x}$ and $w_i(\{x, x+1\}) = 2^{-(x+1)}$. It can be verified that, for each $i < \omega$, L_i has a unique minimal spanning tree iff L_i has a pseudo-minimal spanning tree iff W_i is not cofinite. (As defined, the weighted graphs L_i are not necessarily recursive. However, $\langle L_i : i < \omega\rangle$ is r.e., and so it can be replaced by an isomorphic sequence that is recursive.) □

Lemma 3.5. *There is a recursive sequence $\langle L_i : i < \omega\rangle$ of recursive weighted graphs each having a unique pseudo-minimal spanning tree such that $\{i < \omega : L_i$ has a minimal spanning tree$\}$ is Σ^0_3-complete.*

Proof. For each $i < \omega$, let G_i be the graph in the previous proof. Let L_i be obtained from G_i by adjoining the weight function w_i, where $w_i(\{0, x\}) = x$ and $w_i(\{x, x+1\}) = 2^{-(x+1)}$. It can be verified that each L_i has a unique pseudo-minimal spanning tree. This pseudo-minimal spanning tree is minimal iff W_i is cofinite. □

The next result shows that Corollary 2.2 is best possible.

Theorem 3.6. *There is a recursive sequence $\langle L_i : i < \omega\rangle$ of recursive weighted graphs such that $\{i < \omega : L_i$ has a minimal spanning tree$\}$ is complete among sets that are the difference of two Σ^0_3-sets.*

Proof. Let $D = \{i < \omega : W_i \text{ is cofinite}\}$. For each $i, j < \omega$, let $L_{i,j}$ be the weighted graph obtained by taking the disjoint union of the i-th graph that occurs in the sequence of Lemma 3.4 with the j-th graph that occurs in the sequence of Lemma 3.5 and adding an edge of weight 1 joining their copies of 0. Clearly, $L_{i,j}$ has a minimal spanning tree iff each of L_i and L_j do iff $(i, j) \in (\omega\backslash D) \times D$. The set $(\omega\backslash D) \times D$ is complete among sets that are the difference of two Σ^0_3-sets. □

4. Reverse Mathematics

This note was motivated by a conjecture of Clote and Hirst [1] in Reverse Mathematics. Corollary 2.1 already shows that this conjecture fails. Theorem 4.1 puts a more positive spin on its failure. The Clote-Hirst conjecture is this theorem but with statement (1) replaced with Π^1_1-CA_0. It is interesting to observe that the corresponding version of this conjecture for directed graphs (not undirected graphs, as we have been considering) is in fact proved in [1].

Theorem 4.1. *The following are equivalent over the base theory* RCA_0*:*

(1) ACA_0.

(2) *If* $\langle L_i : i \in \mathbb{N}\rangle$ *is a sequence of weighted graphs, then there is a function* $f : \mathbb{N} \longrightarrow \{0,1\}$ *such that for all* n, $f(n) = 1$ *iff* L_n *has a minimal spanning tree.*

Proof. For (1) $\Longrightarrow$ (2) it is enough to notice that the proof of Corollary 2.2 can be carried out in ACA_0. For the reverse implication, work in a model $\mathbb{N}$ of RCA_0. Let S be the successor graph whose vertex set is $\mathbb{N}$ with $\{k, k+1\}$ being a typical edge. Let $F : \mathbb{N} \longrightarrow \mathbb{N}$ be any function. Let $L_i = (S, w_i)$ be such that $w_i(\{k, k+1\}) = 1$ if $i \notin \{F(0), F(1), \dots, F(k)\}$ and $w_i(\{k, k+1\}) = 2^{-k}$ otherwise. Then L_n has a minimal spanning tree iff $W(L_n) < \infty$ iff n is in the range of F. It follows from Lemma III.1.3. of [2] that ACA_0 holds. □

A final reversal has to do with Theorem 1.4.

Theorem 4.2. *The following are equivalent over the base theory* RCA_0*:*

(1) ACA_0.

(2) *Let* $L = (V, E, w)$ *be a weighted graph. Then,* L *has a pseudo-minimal spanning tree iff for every finite* $X \subseteq V$ *there is a sprig* B *such that* X *is a subset of a component of* (V, B).

Proof. For (1) $\Longrightarrow$ (2) it is enough to notice that the proof of Theorem 1.4 can be carried out in ACA_0. For the reverse implication, notice that an appropriate version of Proposition 3.1 is provable in RCA_0, and then use Lemma III.1.3. of [2]. □

References

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