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MORAVA K-THEORY OF EXTRASPECIAL 2-GROUPS

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ABSTRACT. We compute the Morava K-theory of some extraspecial 2-groups and associated compact groups.

1. INTRODUCTION

Let G be a finite group and BG denote its classifying space. Not that many computations for the Morava K-theory of BG have been carried out, the most notable exception being Kriz's article [5] and its successor [6], where he calculates just enough about the 3-primary second Morava K-theory of the 3-Sylow subgroup of $GL_4(\mathbb{F}_3)$ to conclude that it cannot be concentrated in even degrees, the first such example known. Other computations can be found in [1], [3], [4], [8], [9], [10], and [11].

In this paper we present a few more calculations concerning extraspecial 2-groups. We mainly work with integral Morava K-theory for the prime 2, which shall be denoted $\tilde{K}(n)$. This is a complex oriented cohomology theory with coefficients $\tilde{K}(n)^* \cong W\mathbb{F}_{2^n}[v_n, v_n^{-1}]$, the ring of Laurent polynomials over the Witt ring $W\mathbb{F}_{2^n}$, with v_n of degree $-2(2^n-1)$. It has a complex orientation x such that the 2-series of the associated formal group law takes the form $[2](x) = 2x - v_n x^{2^n}$. Sometimes we switch to the mod 2 reduction $K(n)$.

In Section 2 we describe the groups we want to study and recall Quillen's computation of their mod 2 cohomology. As a corollary we consider a slight modification serving as motivation for our calculational approach. Section 3 contains the main technical result, Lemma 3.1, which under favourable circumstances computes the spectral sequence of an extension of $\mathbb{Z}/2 \times \mathbb{Z}/2$ by a "good" group in the sense of Hopkins-Kuhn-Ravenel, i.e., whose Morava K-theory is generated by transfers of Euler classes. The next two sections contain applications to extraspecials of order 8 and 32. Section 4 is a rehash of the already known computations for D_8 and Q_8 and serves mainly to set up notation for the next section, where we deal with the central products $D_8 \circ D_8$ and $D_8 \circ Q_8$. We need some of the multiplicative structure for D_8, and make repeated use of generalized characters à la Hopkins-Kuhn-Ravenel [3]. We also consider the associated compact groups that arise by replacing the centre $\mathbb{Z}/2$ by the circle group S^1. The last section contains calculations of the Euler

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characteristics of extraspecial groups (for any prime), due also to Brunetti [2]. We omit proofs, since they are now available in [2].

2. Extraspecial 2-groups

There are three types of (almost) extraspecial 2-groups, the so-called real, complex and quaternion types. These may be described as central products. Let D_8 and Q_8 denote the dihedral, respectively quaternion, group of order 8. The extraspecials of real type have order 2^{2m+1} for some $m > 0$ and correspond to m-fold central products of D_8 (for the quaternion type replace one of copy D_8 with a Q_8) whereas the complex type is obtained as the central product of a real extraspecial with a cyclic group of order four.

In this section we try to motivate our subsequent computations, and thus concentrate on the real case only. So let $D(m) := D_8 \circ \cdots \circ D_8$ (m copies); in Hall-Senior notation this group is known as 2_+^{1+2m}. Its mod 2 cohomology was computed by Quillen [7]: one has a central extension

$$1 \to \mathbb{Z}/2 \longrightarrow D(m) \longrightarrow E \to 1 \tag{2.1}$$

where $E \cong (\mathbb{Z}/2)^{2m}$ is a $2m$-dimensional vector space over $\mathbb{F}_2$. The Serre spectral sequence associated to this extension takes the form

$$E_2 = H^*(BE; H^*(B\mathbb{Z}/2)) \cong \mathbb{F}_2[u] \otimes \mathbb{F}_2[x_1, \ldots, x_{2m}] \tag{2.2}$$

with u and x_i in degree one; the extension class is $q := x_1x_2 + \cdots + x_{2m-1}x_{2m}$. Quillen's computation can be summarised as follows:

Theorem 2.1 (Quillen [7])**.** *The only differentials in the spectral sequence* (2.2) *are $d_2u = q$, $d_{2^k+1}u^{2^k} = Q_{k-1}q$ for $1 \le k < m$, where Q_i stands for Milnor's primitive operation in the Steenrod algebra. The sequence $(q, Q_0q, \ldots, Q_{m-2}q)$ is regular, and u^{2^m} is a permanent cycle since it represents the Euler class w_{2^m} of the spin representation Δ. Thus*

$$H^*(D(m); \mathbb{F}_2) \cong \mathbb{F}_2[w_{2^m}] \otimes \mathbb{F}_2[x_1, \ldots, x_{2m}]/(q, Q_0q, \ldots, Q_{m-2}q)\,.$$

The nontrivial Stiefel-Whitney classes of Δ are w_{2^m} and $w_{2^m-2^i}$, $0 \le i \le m$. □

Knowing the result, one can slightly rearrange the computation. $D(m+1)$ contains $D(m)$ as a normal subgroup with quotient $\mathbb{Z}/2 \times \mathbb{Z}/2$, i.e., one has an extension

$$1 \to D(m) \longrightarrow D(m+1) \longrightarrow V \to 1 \tag{2.3}$$

with $V \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ acting trivially on the kernel. The Serre spectral sequence corresponding to (2.3) has E_2-term given by

$$E_2 = H^*(BV; H^*(BD(m)) \cong \mathbb{F}_2[x_{2m+1}, x_{2m+2}] \otimes H^*(BD(m))\,. \tag{2.4}$$

Corollary 2.2. *The spectral sequence* (2.4) *collapses on the E_3-page. The only nontrivial differential is $d_2w_{2^m} = x_{2m+1}x_{2m+2} \otimes w_{2^m-1}$.*

Proof. Since the cohomology of extraspecial 2-groups of real type is detected on maximal elementary abelian subgroups, the action of d_2 can be worked out by looking at the restrictions to those subgroups. Each maximal elementary abelian W is of the form $C \times U$ where C is the centre and U a maximal isotropic subspace of the central quotient E. (Recall from [7] that q may be regarded as a quadratic form on E.) The corresponding extension is of the form

$$1 \to C \times U \longrightarrow D_8 \times U \longrightarrow V \to 1\,,$$

and the only differential is $d_2 u = x_{2m+1}x_{2m+2}$. Quillen tells us that Δ restricts to W as $\chi \otimes reg(U)$, where χ is the nontrivial character of C and $reg(U)$ the regular representation of U. Applying the formula expressing $w.(\chi \otimes reg(U))$ in terms of $w.(\chi)$ and $w.(reg(U))$ we obtain

$$w_i(\chi \otimes reg(U)) = \sum_{j=0}^{i} \binom{2^m - i + j}{j} w_1(\chi)^j w_{i-j}(reg(U))\,.$$

So w_{2^m} restricts to $\sum_{k=0}^{m} u^{2^k} w_{2^m - 2^k}(reg(U))$, since other Stiefel-Whitney classes of $reg(U)$ are zero, and w_{2^m-1} restricts to $w_{2^m-1}(reg(U))$. Thus d_2 is as claimed; the rest follows from a Poincaré series calculation. $\square$

Note that $w_{2^m}^2$ represents the Euler class of the spin representation of $D(m+1)$. Furthermore, there are extension problems in the E_∞-term. Let $q_m = x_1x_2 + \cdots + x_{2m-1}x_{2m}$ denote the extension class of $D(m)$. Then q_m drops in filtration to $x_{2m+1}x_{2m+2}$ (so we get the relation $q_{m+1} = 0$), and the other relations follow as solutions to extension problems related to $Q_i q_m = 0$ and $x_{2m+1}x_{2m+2}w_{2^m-1} = 0$.

The (additive) simplicity of the spectral sequence of this extension is what lets us believe it to be possible to emulate this computation in Morava K-theory. In the subsequent sections we shall try to prove that the Atiyah-Hirzebruch-Serre spectral sequence of (2.3) behaves analogously, meaning it has only two differentials (the second being $v_n \otimes Q_n$, see below).

3. Spectral sequence calculations

In this section we consider the Atiyah-Hirzebruch-Serre spectral sequence associated to extensions

$$1 \to G' \to G \to V \to 0$$

with $V \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, acting trivially on G'. The spectral sequence has E_2-term given by

$$E_2^{*,*} = H^*(\mathbb{Z}/2 \otimes \mathbb{Z}/2; \tilde{K}(n)^*(BG')) \Longrightarrow \tilde{K}(n)^*(BG)\,. \tag{3.1}$$

Lemma 3.1. *Let G be as above. Suppose $K(n)^{\mathrm{odd}}(BG') = 0$ for all $n \geq 1$, and moreover that all elements in $E_4^{0,*}$ are permanent cycles. Then $\tilde{K}(n)^{\mathrm{odd}}(BG) = 0$ and $\tilde{K}(n)^*(BG)$ has no p-torsion, and $K(n)^{\mathrm{odd}}(BG) = 0$.*

Proof. $K(n)^{\mathrm{odd}}(BG') = 0$ implies $\tilde{K}(n)^{\mathrm{odd}}(BG') = 0$ and $\tilde{K}(n)^*(BG')$ is p-torsion free. One has $H^*(BV;\mathbb{F}_2) \cong \mathbb{F}_2[x_1,x_2]$; setting $y_i = x_i^2$ and $\alpha = x_1^2x_2 + x_1x_2^2$, the E_2-page of the spectral sequence is

$$E_2^{*,*'} \cong \begin{cases} \tilde{K}(n)^*(BG') & \text{for } * = 0\,, \\ \tilde{K}(n)^*(BG') \otimes \mathbb{F}_2[y_1,y_2,\alpha]/(\alpha^2 = y_1^2y_2 + y_1y_2^2) & \text{for } * > 0\,. \end{cases}$$

We shall write π for the element $y_1^2y_2 + y_1y_2^2$. The first potentially nontrivial differential is d_3. Any even (respectively odd) degree element in $E_2^{*,*}$ is of the form $x \otimes f$ ($x \otimes f\alpha$) for some $x \in \tilde{K}(n)^*(BG')$ and $f \in \mathbb{F}_2[y_1,y_2]$. We shall first consider the case $n \geq 2$, the argument for $n = 1$ being similar (see the remark at the end). Note that d_3 is zero on any element of $\mathbb{F}_2[y_1,y_2,\alpha]/(\alpha^2=y_1^2y_2+y_1y_2^2)$ by comparison to the Atiyah-Hirzebruch spectral sequence for V and $n\geq 2$. Hence $d_3(x\otimes f)=x'\otimes f\alpha$ and $d_3(x\otimes f\alpha)=x'\otimes f\pi$ for some $x'\in K(n)^*(BG')$. Thus we obtain additive isomorphisms

$$\begin{cases} E_4^{0,*}\cong \tilde{K}, \\ E_4^{>0,*}\cong K\otimes \mathbb{F}_2[y_1,y_2]/(\pi)\ \oplus\ H\otimes\mathbb{F}_2[y_1,y_2]\{\alpha,\pi\}\end{cases}$$

where $\tilde{K}=\mathrm{Ker}(d_3|_{\tilde{K}(n)^*(BG')})$, $K=\mathrm{Ker}(d_3|_{K(n)^*(BG')})=\tilde{K}/(\tilde{K}\cap 2E_2^{0,*})$, and $H=H(K(n)^*(BG');d_3\otimes\alpha^{-1})$. As a $\tilde{K}(n)^*$-algebra, the E_4-page is generated by α, y_i, and the generators in $\tilde{K}$. By hypothesis, all but α are permanent cycles. So the next nonzero differential is

$$d_{2^{n+1}-1}(\alpha)=v_n\otimes Q_n\alpha=v_n\otimes(y_1^{2^n}y_2+y_1y_2^{2^n})=v_n\otimes q\pi$$

where $q=(y_1^{2^n}y_2+y_1y_2^{2^n})/\pi=(y_1^{2^n-2}+y_1^{2^n-3}y_2+\cdots+y_2^{2^n-2})$. Thus we get

$$\begin{aligned} E_{2^{n+1}}^{0,*} &\cong \tilde{K}\,, \\ E_{2^{n+1}}^{>0,*} &\cong K\otimes\mathbb{F}_2[y_1,y_2]/(\pi)\ \oplus\ H\otimes\mathbb{F}_2[y_1,y_2]/(q)\{\pi\}\,. \end{aligned}$$

This is concentrated in even degrees, whence $E_{2^{n+1}}\cong E_\infty$ and $\tilde{K}(n)^{\mathrm{odd}}(BG)=0$. It remains to prove that $\tilde{K}(n)^*(BG)$ has no 2-torsion. Let $0\neq x\in\tilde{K}(n)^*(BG)$. Represent x by $x'\in E_\infty$. If $x'\in E_\infty^{0,*}$ then it cannot be 2-torsion, since $\tilde{K}(n)^*(BG')$ is 2-torsion free. If x' is in $K\otimes\mathbb{F}_2[y_1,y_2]/(\pi)$, we may write $x'=\sum\bar{x}\otimes f$ with $\bar{x}\in K$, $f\in\mathbb{F}_2[y_1,y_2]/(\pi)$. Rewrite f as $y_1f_1+\lambda y_2^s$, $\lambda\in\mathbb{F}_2$. Since $2y_i=v_ny_i^{2^n}$ in $\tilde{K}(n)^*(BG)$ (this is immediate from the calculation for cyclic groups), $2x$ can be represented by

$$(2x)'=\sum v_n\bar{x}\otimes(y_1^{2^n}f_1+\lambda y_2^{2^n+s-1})\,.$$

We claim that the right-hand side of this expression is nonzero: if $\lambda\neq 0$, it does not lie in the ideal $(y_1y_2)\supset(\pi)$, and if $\lambda=0$, then $y_1^{2^n}f\in(\pi)$ implies $y_1f\in(\pi)$. Lastly suppose $x'\in H\otimes\mathbb{F}_2[y_1,y_2]/(y_1^{2^n-2}+\cdots+y_2^{2^n-2})\{\pi\}\subset H\otimes\mathbb{F}_2[y_1,y_2]/(Q_n\alpha)$. Write $x'=\sum\bar{x}\otimes f\pi$ and $f\pi=y_1f_1$. Then $(2x)'\neq 0$ if $v_n\otimes f_1y_1^{2^n}\neq 0$. But $f_1y_1^{2^n}\in(Q_n\alpha)$ implies $f_1y_1\in(Q_n\alpha)$: tensoring up with the finite field of 2^n elements $\mathbb{F}_{2^n}$ yields

$$Q_n\alpha=y_1^{2^n}y_2+y_1y_2^{2^n}=\prod_{\mu\in\mathbb{F}_{2^n}}(y_1+\mu y_2)\,.$$

Finally, for $n=1$ the differential d_3 is given by $v_1\pi$; the claim follows by filtering $E_2^{*,*}$ by powers of π and setting $q=1$. □

Since $\tilde{K}$ is 2-torsion free and the map defined by

$$ay_1^i\mapsto ay_1^{i+2^n-1}\quad\text{and}\quad y_2^i\mapsto y_2^{i+2^n-1}$$

on $E_\infty^{>0,*}$ is injective, one easily sees

Corollary 3.2. *Suppose G is as in Lemma 3.1. Then there is an additive isomorphism*

$$\begin{aligned} K(n)^*(BG) &\cong E_\infty^{0,*}/2\ \oplus\ E_\infty^{>0,*}/(y_1^{2^n},y_2^{2^n}) \\ &\cong \tilde{K}/2\ \oplus\ K\otimes\mathbb{Z}/2[y_1,y_2]^+/(y_1^{2^n},y_2^{2^n},\pi) \\ &\quad\oplus H\otimes\mathbb{Z}/2[y_1,y_2]/(y_1^{2^n-1},y_2^{2^n-1},q)\{\pi\}. \quad\square \end{aligned}$$

4. The cases D_8 and Q_8

The groups D_8 and Q_8 have presentations

$$\begin{aligned} D_8 &= \langle a_1, a \mid a_1^2 = a^4 = 1\,,\ [a_1, a] = a^2\rangle\ , \\ Q_8 &= \langle a_1, a_2 \mid a_1^4 = a_2^4 = 1\,,\ [a_1, a_2] = a_1^2 = a_2^2\rangle\,, \end{aligned}$$

respectively. Thus there are central extensions of the form $\mathbb{Z}/2 \to G \to V$ for G either D_8 or Q_8, i.e., we have $G' = \mathbb{Z}/2$ in the setup of Section 3. Setting $a_2 = aa_1$ in the case of D_8, the quotient V is generated by the cosets $\bar{a}_i$ for either group; let $x_i \in H^*(BV; \mathbb{F}_2)$ be dual to $\bar{a}_i$. Recall that $\tilde{K}(n)^*(B\mathbb{Z}/2) \cong \tilde{K}(n)^*[u]/(2u - v_n u^{2^n})$ where u is the Euler class of the nontrivial linear character η of $\mathbb{Z}/2$. In the spectral sequence (3.1), we get $d_3 u = \alpha$. Hence $H = \mathrm{Ker}(d_3)/\,\mathrm{Im}(d_3 \otimes \alpha^{-1}) = 0$, and u^2 is a permanent cycle, since it is the restriction of the Euler class of the irreducible two-dimensional complex representation ρ of G to the fibre. Thus

$$\begin{aligned} E_\infty^{0,*} &\cong \tilde{K}(n)^*[u^2]/((2u - v_n u^{2^n}) \cap \tilde{K}(n)^*[u^2]) \cong \tilde{K}(n)^*[u^2]\{1, 2u\} \\ E_\infty^{>0,*} &\cong \tilde{K}(n)^*[u^2]/(v_n u^{2^n}) \otimes \mathbb{F}_2[y_1, y_2]/(\pi)\,. \end{aligned}$$

It follows that $\tilde{K}(n)^*(BG)$ is concentrated in even degrees and has no 2-torsion, whence $K(n)^*(BG) \cong \tilde{K}(n)^*(BG)/(2)$. Choosing an element $\bar{c}_2 \in \tilde{K}(n)^*(BG)$ represented by u^2, one obtains

Theorem 4.1 ([9], [8])**.** *Let G be either D_8 or Q_8. Then there is an additive isomorphism*

$$K(n)^*(BG) \cong \big(K(n)^*\{\bar{c}_1\} \oplus K(n)^*[y_1, y_2]/(\pi, y_1^{2^n}, y_2^{2^n})\big)[\bar{c}_2]/(\bar{c}_2^{2^{n-1}})\,.$$

The multiplicative structure is given by

$$\bar{c}_1 y_1 = y_1^2\,, \quad \bar{c}_1 y_2 = y_2^2\,, \quad \bar{c}_1^2 = y_1^2 + y_1 y_2 + y_2^2 \tag{4.1}$$

identifying $\bar{c}_1 = v_n \bar{c}_2^{2^{n-1}} + y_1 + y_2$ for D_8 and $\bar{c}_1 = v_n \bar{c}_2^{2^{n-1}}$ for Q_8. □

The generators y_i can be identified with the Euler classes of the representations $\rho_i\colon G \to V \to \langle \bar{a}_i\rangle \xrightarrow{\eta} \mathbb{C}^*$. Switching from $\bar{c}_i$ to $c_i = c_i(\rho)$, we may write $c_2 = \bar{c}_2 \mod (y_1, y_2)^2$. Then $v_n c_2^{2^{n-1}} = v_n \bar{c}_2^{2^{n-1}} \mod (y_1, y_2)^{2^n}$. We also have $c_1 = \bar{c}_1 \mod (y_1, y_2)^2$, by considering restrictions to maximal abelian subgroups; see below. Hence relation (4.1) in the theorem holds modulo $(y_1, y_2)^3$ with $\bar{c}_i$ replaced by c_i.

We want to compute the restrictions of c_2 to the maximal subgroups of G. Consider $G = D_8$ first. Let $C = \langle a^2\rangle$ be the centre of D_8, and $A_i = \langle a_i\rangle$. The maximal subgroups are $A = \langle a\rangle \cong \mathbb{Z}/4$ and $C \times A_i \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. Let $\rho_A\colon A \to \mathbb{C}^*$ be a faithful representation of A. Then $c_1(\rho_A)$ restricts to the generator u of the centre, and identifying classes with their images under restriction, we may write

$$\begin{aligned} K(n)^*(BA) &\cong K(n)^*[u]/[4](u) \cong K(n)^*[u]/(u^{4^n})\,; \\ K(n)^*(BC \times A_i) &\cong K(n)^*[u, y_i]/([2](u), [2](y_i)) \cong K(n)^*[u, y_i]/(u^{2^n}, y_i^{2^n})\,. \end{aligned}$$

We have $\mathrm{Res}_A(\rho_i) = \rho_A \otimes \rho_A$, and since $\rho = \mathrm{Ind}_A^G(\rho_A)$, the double coset formula gives $\mathrm{Res}_A(\rho) = \rho_A \oplus \rho_A^{-1}$. The restrictions of the total Chern class are $\mathrm{Res}_A(c(\rho)) = (1 + u)(1 + [-1]u)$ and $\mathrm{Res}_{C\times A_i}(c(\rho)) = (1 + u)(1 + u +_{K(n)} y_i)$. Thus we obtain the following restrictions:

$$\mathrm{Res}_A(c_2) = ([-1](u))u = u^2 + v_n u^{2^n+1} \mod (u^{2^{n+1}})\,; \tag{4.2}$$

$$\mathrm{Res}_A(c_1) = \mathrm{Res}_A(y_i) = [2](u) = v_n u^{2^n}\,. \tag{4.3}$$

Similarly, we get

$$\mathrm{Res}_{C\times A_i}(c_2) = u(u+_{K(n)} y_i) = u^2 + uy_i + v_n u^{2^{n-1}+1} y_i^{2^{n-1}}\,; \tag{4.4}$$

$$\mathrm{Res}_{C\times A_i}(c_1) = u+(u+_{K(n)} y_i) = y_i + v_n u^{2^{n-1}} y_i^{2^{n-1}}\,. \tag{4.5}$$

Next consider the quaternion case. Here the maximal subgroups are $B_1 = \langle a_1\rangle$, $B_2 = \langle a_2\rangle$, and $B_3 = \langle a_1a_2\rangle$, all isomorphic to $\mathbb{Z}/4$. If $e_i\colon B_i \to \mathbb{C}^*$ is a faithful representation, we have $\rho \cong \mathrm{Ind}_{B_i}^{Q_8}(e_i)$ for each B_i, and, similar to the above, we can see that

$$\mathrm{Res}_{B_i}(c_2) = u_i^2 + v_n u_i^{2^n+1} \bmod (u^{2^{n+1}}) \text{ in } K(n)^*(BB_i) \cong K(n)^*[u_i]/(u_i^{4^n})\,. \tag{4.6}$$

To finish this section, we consider a compact group defined by Q_8 or D_8. When a group G has centre $C \cong \mathbb{Z}/2$, let us write $\tilde{G}$ for the central product $G\times_C S^1$, identifying C with $\{1,-1\}\subset S^1$. Then $\tilde{D}_8 \cong \tilde{Q}_8$. Using Lemma 3.1, we easily see:

Theorem 4.2. *There is an additive isomorphism*

$$K(n)^*(B\tilde{D}_8) \cong \big(K(n)^*\{\bar{c}_1\} \oplus K(n)^*[y_1,y_2]/(\pi, y_1^{2^n}, y_2^{2^n})\big)[\bar{c}_2].$$

The multiplicative structure is given by (4.1) *mod* $(y_1,y_2)^3$ *in Theorem 4.1.* □

5. Extraspecial groups of order 2^5

In this section we consider the central products $G = D_8\circ D_8$ and $G = D_8\circ Q_8$. In both cases, G is generated by elements $a_1,\dots,a_4$ of order 2, and we have an extension

$$1\to G'\to G\to V\to 0 \qquad \text{with} \quad G'\cong D_8\,,\ V\cong \mathbb{Z}/2\times\mathbb{Z}/2 \tag{5.1}$$

and trivial V-action on G'. Set $G_{ij} = \langle a_i, a_j\rangle \subset G$, numbering the generators a_i such that $G' = G_{12}$ and $A_i = \langle a_i\rangle$. Then $G_{34}\cong D_8$ or Q_8, and $G_{34}/C = V$ for C = centre of G. This allows us to keep the notation for $K(n)^*(BD_8)$ from the previous section. Furthermore, let $H^*(BV;\mathbb{F}_2) = \mathbb{F}_2[x_3,x_4]$, and $y_3, y_4, \alpha \in H^*(BV)$ correspond to x_3^2, x_4^2, and $x_3^2x_4 + x_3x_4^2$, respectively. We consider the spectral sequence

$$E_2^{*,*} = H^*(BV; \tilde{K}(n)^*(BD_8)) \Longrightarrow \tilde{K}(n)^*(BG)\,. \tag{5.2}$$

Lemma 5.1. *In the above spectral sequence, we have*

$$d_3c_2 = c_1\otimes\alpha \mod (y_1,y_2)^2\,.$$

Proof. For dimensional reasons, $d_3c_2 = (\lambda_1y_1+\lambda_2y_2+\lambda_3c_1)\otimes\alpha \mod (y_1,y_2)^2$ with $\lambda_i\in\mathbb{F}_2$. Consider the map of spectral sequences induced by

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A_1\times C & \longrightarrow & A_1\times G_{34} & \longrightarrow & V = G_{34}/C & \longrightarrow & 0\\
 & & \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle i} & & \big\| & & \\
1 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & V & \longrightarrow & 0.
\end{array}$$

Since $\mathrm{Res}_{A_1\times C}(c_2) = u^2+uy_1 \mod (y_1^2)$ and $d_3u = 1\otimes\alpha$, we get

$$i^*(d_3c_2) = d_3(u^2+uy_1) = y_1\otimes\alpha \mod (y_1^2)$$

and hence $\lambda_1+\lambda_3=1$. Similarly, replacing A_1 with A_2, we get $\lambda_2+\lambda_3=1$. Finally, consider the inclusion of A into G_{12}:

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] \arrow[d, "j"] & A\times_C G_{34} \arrow[r] \arrow[d, "j"] & V \arrow[r] \arrow[d, equal] & 0 \\ 1 \arrow[r] & G_{12} \arrow[r] & G \arrow[r] & V \arrow[r] & 0. \end{tikzcd}$$

Now modulo $u^{2^{n+1}}$, we have $\mathrm{Res}_A(c_2)=u^2+v_nu^{2^n+1}$ and thus $j^*(d_3c_2)=v_nu^{2^n}\otimes\alpha$. Since $\mathrm{Res}_A(c_1)=v_n(\mathrm{Res}_A(c_2))^{2^{n-1}}=v_nu^{2^n}$, we get $\lambda_1+\lambda_2+\lambda_3=1$, too. □

Therefore Theorem 4.1 gives

$$\begin{aligned} d_3(y_ic_2) &= y_ic_1\otimes\alpha=y_1^2\otimes\alpha \mod (y_1,y_2)^3\,,\\ d_3(c_1c_2) &= c_1^2\otimes\alpha=y_1y_2\otimes\alpha \mod (y_1,y_2)^3\,. \end{aligned}$$

Using these formulae, it is easy to see that $\tilde{K}=\mathrm{Ker}(d_3|_{\tilde{K}(n)^*(BD_8)})$ is generated as a $K(n)^*$-algebra by

$$\begin{aligned} &y_1\,,\ y_2\,,\ c_2^2 \text{ (which gives } c_1)\,,\ 2c_2,\\ &b_1=y_1^{2^n-1}c_2\,,\ b_2=y_2^{2^n-1}c_2\,,\ y_1b_2=y_1y_2^{2^n-1}c_2\,. \end{aligned} \tag{5.3}$$

The last three terms are in $\tilde{K}$ since $v_ny_i^{2^n}=0$ in $K(n)^*(BD_8)$. More precisely, we have

Lemma 5.2. *In the spectral sequence* (5.2)*, the kernel $\tilde{K}$ and the homology H with respect to $d_3\otimes\alpha^{-1}$ are given additively by*

$$\begin{aligned} \tilde{K} &\cong \begin{pmatrix} (\tilde{K}(n)^*[y_1,y_2]/(\tilde{\pi},[2](y_i))\ \oplus\ \tilde{K}(n)^*\{c_1\}\{1,2c_2\} \\ +\ \tilde{K}(n)^*\{b_1,b_2,y_1b_2\} \end{pmatrix}[c_2^2]/(c_2^{2^{n-1}})\,,\\ H &\cong K(n)^*\{1,y_1,y_2,b_1,b_2,y_1b_2\}[c_2^2]/(c_2^{2^{n-1}}) \end{aligned}$$

where $\tilde{\pi}=y_1y_2(y_1+_{\tilde{K}(n)}y_2)$. Note that $2b_i=2c_2y_i^{2^n-1}$. □

A similar statement holds for the associated compact group:

Lemma 5.3. *In the spectral sequence $1\to\tilde{G}'\to\tilde{G}\to V\to 1$, the kernel $\tilde{K}$ and the homology H are given additively by*

$$\begin{aligned} \tilde{K} &\cong (\tilde{K}(n)^*[y_1,y_2]/(\tilde{\pi},[2](y_i))\ \oplus\ \tilde{K}(n)^*\{c_1\})\{1,2c_2\}[c_2^2]\,,\\ H &\cong K(n)^*\{1,y_1,y_2\}[c_2^2]\,. \quad \square \end{aligned}$$

We want to show that all elements in $\tilde{K}$ are permanent. Let $t=\mathrm{Tr}_{G_{12}\times A_3}^G(c_2\otimes 1)$. By the double coset formula, we get

$$\mathrm{Res}_{G_{12}}^G(t) \quad = \sum_{g\in G_{12}\backslash G/G_{12}\times A_3} \mathrm{Tr}_{G_{12}\cap(G_{12}\times A_3)^g}^{G_{12}}\,\mathrm{Res}_{G_{12}\cap(G_{12}\times A_3)^g}^{(G_{12}\times A_3)^g}\, g^*(c_2\otimes 1)\,.$$

Here $G_{12}\backslash G/G_{12}\times A_3\cong A_4$ and $G_{12}\cap(G_{12}\times A_3)^g=G_{12}$ for all $g\in A_4$. Hence

$$\mathrm{Res}_{G_{12}}^G(t)=c_2+a_4^*c_2=2c_2\,.$$

Therefore, $t\in\tilde{K}(n)^*(BG)$ corresponds to the element $[2c_2]\in E_\infty^{0,*}$.

Next we look for elements corresponding to the b_i of Lemma 5.2. Let $A'=\langle a_3a_4\rangle$; this is cyclic of order 4. Let $\rho'_{A'}$ be a faithful one-dimensional representation of A'.

Set $\rho' = \mathrm{Ind}_{A'}^{G_{34}}(\rho_{A'})$ and $c_2' = c_2(\rho')$. Define $t_i = \mathrm{Tr}_{G_{34}\times A_i}^{G}(c_2' \otimes 1)$ for $i = 1, 2$. We claim the following identities:

$$v_n b_1 = \mathrm{Res}_{G_{12}}^{G}(t - t_2 + y_2^2 - y_1 y_2), \tag{5.4}$$

$$v_n b_2 = \mathrm{Res}_{G_{12}}^{G}(t - t_1 + y_1^2 - y_1 y_2). \tag{5.5}$$

It suffices to check them on the abelian subgroups of G_{12}, by [3]. Thus we need to compute the restrictions to $C \times A_i$ and A. Since ρ restricts to $\eta + \eta\lambda_i$ on $C \times A_i$ and to $\rho_A + \rho_A^3$ on A, we have

$$\mathrm{Res}_{C\times A_i}^{G}(t) = 2u(u +_{\tilde{K}(n)} y_i), \qquad \mathrm{Res}_{A}^{G}(t) = 2z[3](z),$$
$$\mathrm{Res}_{C\times A_1}^{G_{12}}(b_1) = u(u +_{\tilde{K}(n)} y_1) y_1^{2^n-1}, \qquad \mathrm{Res}_{A}^{G_{12}}(b_i) = z[3](z)([2]z)^{2^n-1},$$
$$\mathrm{Res}_{C\times A_2}^{G_{12}}(b_1) = 0.$$

Here $z = c_1(\rho_A)$ denotes the generator of $\tilde{K}(n)^*(BA) \cong \tilde{K}(n)^*[[z]]/[4](z)$; clearly y_1, y_2 restrict to $[2](z)$.

Now $C \times A_1 \backslash G / G_{34} \times A_2 \cong 1$ and $(C \times A_1) \cap (G_{34} \times A_2) = C$; the double coset formula then says

$$\begin{aligned} \mathrm{Res}_{C\times A_1}^{G}(t_2) &= \mathrm{Tr}_{C}^{C\times A_1} \mathrm{Res}_{C}^{G_{34}\times A_2)}(c_2' \otimes 1) = \mathrm{Tr}_{C}^{C\times A_1}(u^2) \\ &= u^2 \,\mathrm{Tr}_{\{1\}}^{A_1}(1) = u^2(2 - v_n y_1^{2^n-1}) \end{aligned}$$

where we used the fact (see, e.g., [3] or [5])

$$\mathrm{Tr}_{\{1\}}^{A_1}(1) = \frac{[2](y_1)}{y_1} = 2 - v_n y_1^{2^n-1}\,. \tag{5.6}$$

Similarly, we have $C \times A_2 \backslash G / G_{34} \times A_2 \cong A_1$ and $(C \times A_2) \cap (G_{34} \times A_2) = C \times A_2$, whence

$$\begin{aligned} \mathrm{Res}_{C\times A_2}^{G}(t_2) &= \mathrm{Res}_{C\times A_2}^{G_{34}\times A_2}(1 + a_1^*)(c_2' \otimes 1) \\ &= c_2(2\eta) + c_2(\eta \otimes \lambda_2) = u^2 + (u +_{\tilde{K}(n)} y_2)^2\,. \end{aligned}$$

By the double coset formula again,

$$\mathrm{Res}_{A}^{G}(t_i) = \mathrm{Tr}_{C}^{A}(u^2) = z^2\,\mathrm{Tr}_{C}^{A}(1) = z^2 \frac{[4](z)}{[2](z)}\,.$$

Thus

$$\mathrm{Res}_{C\times A_1}^{G}(t - t_2 + y_2^2 - y_1 y_2) - \mathrm{Res}_{C\times A_1}^{G_{12}}(v_n b_1) = (u(u +_{\tilde{K}(n)} y_1) - u^2)(2 - v_n y_1^{2^n-1})\,.$$

Let χ be a generalized character of $C \times A_1$. If $\chi(y_1) = 0$, then

$$\chi((u(u +_{\tilde{K}(n)} y_1) - u^2)(2 - v_n y_1^{2^n-1})) = 2(\chi(u)^2 - \chi(u)^2) = 0\,,$$

whereas if $\chi(y_1) \neq 0$, then $\chi(2 - v_n y_1^{2^n-1}) = [2](\chi(y_1))/\chi(y_1) = 0$. Secondly,

$$\mathrm{Res}_{C\times A_2}^{G}(t - t_2 + y_2^2 - y_1 y_2) - \mathrm{Res}_{C\times A_2}^{G_{12}}(v_n b_1) = 2u(u +_{\tilde{K}(n)} y_2) - (u +_{\tilde{K}(n)} y_2)^2 - u^2 + y_2^2\,.$$

Any generalized character χ with $\chi(u) = 0$ or $\chi(y_2) = 0$ clearly annihilates this expression. So assume, without loss of generality, that $\chi(u) = \pi$, where π is a uniformizing element. Any other nonzero root of the 2-series is of the form $\zeta\pi$ for a $(2^n - 1)$-st root of unity ζ. Then $[\zeta](\pi) = \zeta\pi$ and $\pi +_{\tilde{K}(n)} \zeta\pi = \pi +_{\tilde{K}(n)} [\zeta](\pi) = [1 + \zeta](\pi) = (1 + \zeta)\pi$, since $(1 + \zeta)^{2^n-1} \equiv 1 \mod 2$. Thus

$$\chi(2u(u +_{\tilde{K}(n)} y_2) - (u +_{\tilde{K}(n)} y_2)^2 - u^2 + y_2^2)) = 2\pi(1 + \zeta)\pi - (1 + \zeta)^2\pi^2 - \pi^2 + \zeta^2\pi^2 = 0\,.$$

Finally,

$$\begin{aligned}&\mathrm{Res}_A^G(t-t_2+y_2^2-y_1y_2)-\mathrm{Res}_A^{G_{12}}(v_nb_1)\\&\quad=2z[3](z)-z^2\frac{[4](z)}{[2](z)}-v_nz[3](z)([2](z))^{2^n-1}=(z[3](z)-z^2)\frac{[4](z)}{[2](z)}\end{aligned}$$

where we used $v_n([2](z))^{2^n-1}=2-[4](z)/[2](z)$. Let α denote the value of a character on z. Then either $[4](\alpha)/[2](\alpha)=0$, if $[2](\alpha)\neq 0$, or $[4](\alpha)/[2](\alpha)=2$, if $[2](\alpha)=0$, and in that case $\alpha[3](\alpha)-\alpha^2=\alpha(\alpha+_{\tilde{K}(n)}[2](\alpha))-\alpha^2=\alpha^2-\alpha^2=0$. This finishes the proof of equation (5.4). The other equation follows by exchanging the indices 1 and 2. Thus the assumptions of Lemma (3.1) hold, yielding

Theorem 5.4. *Let G be an extraspecial group of order* 32. *Then $K(n)^*(BG)$ is concentrated in even degrees and generated by transfers of Euler classes.* □

In the compact case it suffices to show that c_1 is a permanent cycle. Suppose that $d_rc_1=x\otimes f\alpha\neq 0$ for $3\leq r\leq 2^{n+1}-1$. Note that $x\otimes f\alpha^2=x\otimes f\pi\neq 0$ in $E_r^{*,*}$. But $d_r(c_1\otimes\alpha)$ must be zero in $E_r^{*,*}$, since it is so in $E_4^{*,*}$. This is a contradiction. The term $E_{2^{n+1}}^{*,*}$ is generated by even-dimensional elements, and c_1 is a permanent cycle.

From Lemma (5.3) and the formula in the proof of Lemma (3.1), we get

$$\begin{aligned}&\mathrm{gr}\,\tilde{K}(n)^*(B\tilde{G})\cong\tilde{K}\oplus K\otimes\mathbb{F}_2[y_3,y_4]^+/(\pi_{34})\oplus H\otimes\mathbb{F}_2[y_3,y_4]/(q_{34})\{\pi_{34}\}\\&\quad\cong(\tilde{K}(n)^*[y_1,y_2]/(\tilde{\pi}_{12},[2](y_i))\ \oplus\ \tilde{K}(n)^*\{c_1\})\{1,2c_2\}\\&\qquad\oplus((K(n)^*[y_1,y_2]/(\pi_{12},[2](y_i))\ \oplus\ K(n)^*\{c_1\})\otimes\mathbb{F}_2[y_3,y_4]^+/(\pi_{34})\\&\qquad\oplus(K(n)^*\{1,y_1,y_2\}\otimes\mathbb{F}_2[y_3,y_4]/(q_{34})\{\pi_{34}\})[c_2^2]\,.\end{aligned}\tag{5.7}$$

6. Euler characteristics of extraspecial p-groups

In this section we give the Euler characteristic of an extraspecial p-group. The result is not new; the same formula was obtained by Brunetti [2].

The Morava K-theory Euler characteristic $\chi_{n,p}(G)$ of a finite group G, i.e., the difference between the ranks of the even and odd degree parts of $K(n)^*(BG)$, can be computed using the formula from [3]:

$$\chi_{n,p}(G)=\sum_{A<G}\frac{|A|}{|G|}\mu_{\mathcal{A}(G)}(A)\chi_{n,p}(A)\tag{6.1}$$

where the sum is over all abelian subgroups $A<G$ and $\mu_{\mathcal{A}(G)}$ is a Möbius function defined recursively by

$$\sum_{A'<A}\mu_{\mathcal{A}(G)}(A')=1\tag{6.2}$$

where the sum is over all abelian subgroups $A'<G$ contained in A. In particular, $\mu_{\mathcal{A}(G)}(A)=1$ when A is maximal. It is easy to see that one only has to consider subgroups arising as intersections of maximal ones. Furthermore, one clearly has $\chi_{n,p}(A)=|A_{(p)}|^n$ where $A_{(p)}$ denotes the p-part of the abelian group A.

The abelian subgroups of an extraspecial p-group $D(m)=p_+^{1+2m}$ are in one-to-one correspondence with those subspaces W of the central quotient $V\cong\mathbb{F}_p^{2m}$ that are isotropic with respect to the bilinear form

$$b(x,y)=x_1y_2+x_2y_1+\cdots+x_{2m-1}y_{2m}+x_{2m}y_{2m-1}\,.$$

Let $\alpha_i^{(m)}$ denote the number of such subspaces of dimension i. Note that the maximal dimension of a b-isotropic subspace is m.

The following lemma is an easy exercise in counting:

Lemma 6.1. $\alpha_i^{(m)} = \prod_{j=1}^{i} \frac{p^{2(m-j+1)}-1}{p^j-1}$. □

The Möbius function on abelian subgroups can be computed via a Möbius function on b-isotropic subspaces defined as in (6.2). Let $\gamma_k^{(m)}$ denote its value on a subspace of dimension k: by symmetry, it is constant on subspaces of the same rank. Furthermore, it only depends on the *codimension* of a b-isotropic subspace in a maximal one, independent of m; this follows by considering $W^\perp/W$. The following formula can be proved inductively, see [2].

Lemma 6.2. $\gamma_k^{(m)} = (-p)^{(m-k)^2}$. □

Since a b-isotropic subspace W of dimension i gives rise to an abelian subgroup of index $2m-i$, we arrive at

Proposition 6.3 ([2])**.** *The Morava K-theory Euler characteristic of $G = p_+^{1+2m}$ is given by*

$$\chi_{n,p}(G) = \sum_{i=0}^{m} \frac{\alpha_i^{(m)}\gamma_i^{(m)}}{p^{2m-i}} p^{(i+1)n} = \sum_{i=0}^{m} (-1)^{m-i}\alpha_i^{(m)} p^{(m-i-1)^2+(n-1)(i+1)}$$

with α and γ as in the two lemmas above.

For example, for D_8 and $D(2) = 2_+^{1+4}$ we obtain

$$\begin{aligned} \chi_{n,2}(D_8) &= \frac{3}{2}4^n - \frac{1}{2}2^n\,, \quad \text{and} \\ \chi_{n,2}(D(2)) &= \frac{15}{4}(8^n - 4^n) + 2^n\,. \end{aligned}$$

This agrees with the Euler characteristics that we can compute using Corollary 3.2, as we shall now see. Let $Y_{i,j} = K(n)^*[y_i, y_j]^+/(\pi, y_i^{2^n}, y_j^{2^n})$, and denote by $\chi(-)$ the dimension of a $K(n)^*$-vector space. Then one easily computes $\chi(Y_{i,j}) = 3\cdot(2^n-1)$. We have $K(n)^*(BD_8) \cong (Y_{1,2} \oplus K(n)^*\{1, c_1\}) \otimes \mathbb{Z}/2[c_2]/(c_2^{2^{n-1}})$. Hence

$$\chi(K(n)^*(BD_8)) = (3\cdot(2^n-1)+2)2^{n-1} = 3\cdot 2^{2n-1} - 2^{n-1}\,.$$

Next consider $K(n)^*(BD(2))$. First note

$$\begin{aligned} \chi(\tilde{K}/2) &= \chi((Y_{1,2} + K(n)^*\{1, c_1\})\{1, 2c_2\} \otimes \mathbb{Z}/2[c_2^2]/(c_2^{2^{n-1}})) \\ &= (3\cdot(2^n-1)+2)\cdot 2\cdot 2^{n-2} = (6\cdot 2^n - 2)\cdot 2^{n-2}\,, \end{aligned}$$

where we used the fact that we can take either $y_i^j c_2$ or $2y_i^j c_2$ as a basis element and may neglect the summand $K(n)^*\{b_1, b_2, y_2 b_1\}$. Then

$$\begin{aligned} \chi(K \otimes Y_{3,4}) &= \chi((Y_{1,2} + K(n)^*\{1, c_1, b_1, b_2, y_2 b_1\}) \otimes Y_{34})\cdot 2^{n-2} \\ &= (3(2^n-1)+5)\cdot 3(2^n-1)\cdot 2^{n-2} = (9\cdot 2^{2n} - 3\cdot 2^n - 6)\cdot 2^{n-2}\,; \\ \chi(H \otimes \mathbb{Z}/2[y_3, y_4]/(q_{34}, y_3^{2^n-1}, y_4^{2^n-1})\{\pi\}) &= (6\cdot(2^n-1)(2^n-2))\cdot 2^{n-2} \\ &= (6\cdot 2^{2n} - 18\cdot 2^n + 12)\cdot 2^{n-2}\,. \end{aligned}$$

Therefore, we have $\chi(K(n)^*(BD(2)) = (15\cdot 2^{2n} - 15\cdot 2^n + 4)\cdot 2^{n-2} = \chi_{n,2}(D(2))$.

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