THE CHEEGER CONSTANT
OF SIMPLY CONNECTED, SOLVABLE LIE GROUPS

NORBERT PEYERIMHOFF AND EVANGELIA SAMIOU

(Communicated by Wolfgang Ziller)

ABSTRACT. We show that the Cheeger isoperimetric constant of a solvable simply connected Lie group G with Lie algebra $\mathfrak{G}$ is

$$h(G) = \max_{H \in \mathfrak{G}, ||H||=1} \operatorname{tr}(\operatorname{ad}(H)).$$

1. INTRODUCTION

The Cheeger isoperimetric constant $h(M)$ of a complete noncompact Riemannian manifold M is defined by

$$h(M) = \inf_{K \subset M} \frac{\operatorname{area}(\partial K)}{\operatorname{vol}(K)},$$

where K ranges over all connected, open submanifolds of M with compact closure and smooth boundary. For the bottom of the spectrum $\lambda(\Omega)$, there is Cheeger's inequality ([Che], [BPP], [Cha])

$$\lambda(\Omega) \geq \frac{h(\Omega)^2}{4}.$$

There is a converse of Cheeger's inequality due to Peter Buser, [Bu], [BPP]: There exist constants c_1 and c_2, depending on a lower bound on the Ricci curvature of M such that

$$\lambda \leq c_1 h + c_2 h^2.$$

Further relations between isometric and spectral properties (in particular, estimates of the heat kernel) can be found in the work of, e.g., Varopoulos, Coulhon, Saloff-Coste, Grigor'yan [Gr] and Pittet [Pi]. For connections between the Cheeger constant and Kazhdan's property T, see, e.g., [Br] or [Leu].

If $h(M) > 0$, one can easily see that M has exponential volume growth. The converse is not true. Hoke [Ho] has shown that $h(G) = 0$ for a simply connected Lie group with left-invariant metric if and only if G is unimodular and amenable. These Lie groups, however, have exponential volume growth if they are not of type R, see [Pa]. Examples of this class are horospheres in symmetric spaces of noncompact type and higher rank orthogonal to the barycenter of a Weyl chamber, see [Pe].

In this note we calculate the Cheeger constant for simply connected solvable Lie groups.

2. The Cheeger constant

We consider the following situation: Let $G = G_0 \rtimes \mathbb{R}$ be a semidirect product with Lie algebra $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathbb{R}$ endowed with a left-invariant metric such that $\mathfrak{G}_0$ is orthogonal to $\mathbb{R}$. We denote the unit vector in $\mathbb{R}$ by H_0 and obtain a diffeomorphism $\phi : G_0 \times \mathbb{R} \to G$, $\phi(g_0, t) = g_0 \exp(tH_0)$.

Lemma 1. *The left-invariant Haar measure μ on G is given by*

$$(2) \qquad d\mu(g_0, t) = \det(e^{\mathrm{ad}\,(-tH_0)})d\nu(g_0)dt = e^{-t\,\mathrm{tr}\,(\mathrm{ad}\,(H_0))}d\nu(g_0)dt,$$

where $d\nu$ is the left-invariant Haar measure on $G_0 = G_0 \times \{0\} \subset G$.

Proof. Note that $d\nu dt$ is left G_0-invariant and right $\mathbb{R}$-invariant. Consequently, the left-invariant Haar measure on G with respect to the above diffeomorphisms is given by $d\mu(g_0, t) = \delta(t)d\nu(g_0)dt$. We will calculate $\delta(t)$. Let $a = \exp(sH_0)$ and $f \in C_0^\infty(G)$. Then

$$
\begin{aligned}
\int_G f(ag)d\mu(g) &= \int_{\mathbb{R}} \int_{G_0} f(ag_0 \exp(tH_0))\delta(t)d\nu(g_0)dt \\
&= \int_{\mathbb{R}} \int_{G_0} f(\psi(g_0)a \exp(tH_0))\delta(t)d\nu(g_0)dt,
\end{aligned}
$$

where $\psi : G_0 \to G_0, \psi(g_0) = ag_0a^{-1}$. From the transformation formula this becomes

$$\int_{\mathbb{R}} \delta(t) \int_{G_0} f(g_0 \exp((s+t)H_0))|\det D\psi^{-1}(g_0)|d\nu(g_0)dt.$$

Let $X_1, \cdots, X_n$ be an orthonormal basis of $\mathfrak{G}_0$ with respect to the left-invariant metric. Then the $Y_j = \frac{d}{dt}\big|_{t=0} g_0 \exp(tX_j)$, $j = 1, \ldots, n$, form an orthonomal basis of $T_{g_0}G_0$, and

$$
\begin{aligned}
D\psi^{-1}(g_0)(Y_j) &= \frac{d}{dt}\Big|_{t=0} \psi^{-1}(g_0 \exp(tX_j)) = \frac{d}{dt}\Big|_{t=0} a^{-1}g_0 \exp(tX_j)a \\
&= dL_{a^{-1}g_0 a}(e^{\mathrm{ad}\,(-sH_0)}X_j).
\end{aligned}
$$

So we conclude that $|\det(D\psi^{-1}(g_0))| = \det(e^{\mathrm{ad}\,(-sH_0)})$. By left invariance of μ we get

$$
\begin{aligned}
\int_G f(ag)d\mu(g) &= \int_{\mathbb{R}} \delta(t)\det(e^{\mathrm{ad}\,(-sH_0)}) \int_{G_0} f(g_0 \exp((s+t)H_0))d\nu(g_0)dt \\
= \int_G f(g)d\mu(g) &= \int_{\mathbb{R}} \delta(s+t) \int_{G_0} f(g_0 \exp((s+t)H_0))d\nu(g_0)dt;
\end{aligned}
$$

hence $\delta(s) = \det(e^{\mathrm{ad}\,(-sH_0)})$. $\qquad \square$

The following theorem contains the main result of this section.

Theorem 3. *For the Cheeger constant of a Lie group G as above we have*

$$(4) \qquad\qquad\qquad h(G) \geq |\mathrm{tr}\,(\mathrm{ad}\,(H_0))|.$$

Moreover, if $h(G_0) = 0$, then formula (4) holds with equality.

Proof. We identify G with $G_0 \times \mathbb{R}$ via the diffeomorphism ϕ and denote by $\pi : G \to G_0$ the projection. Without loss of generality, we may assume that $\operatorname{tr}(\operatorname{ad}(H_0)) \geq 0$. For a connected, compact subset $K \subset G$ with smooth boundary ∂K and nonempty interior, let $U = \pi(K) \setminus \{\text{critical values of } \pi|_{\partial K}\}$.

On U we define functions $\delta^{\pm}$ by $\delta^+(u) = \max\{t \mid (u,t) \in K\}$ and $\delta^-(u) = \min\{t \mid (u,t) \in K\}$, $u \in U$. The functions $\delta^{\pm}$ are smooth, and by Sard's theorem the set U has full measure in $\pi(K)$. We denote by $\partial K^{\pm} = \{(u, \delta^{\pm}(u)) \mid u \in U\}$ the graphs of $\delta^{\pm}$. We estimate the volumes of K and ∂K. From (2) we get

$$
\begin{aligned}
\operatorname{vol}(K) &\leq \int_U \int_{\delta^-(u)}^{\delta^+(u)} e^{-t\operatorname{tr}(\operatorname{ad}(H_0))} dt \, d\nu(u) \\
&\leq \frac{1}{\operatorname{tr}(\operatorname{ad}(H_0))} \int_U \left(e^{-\delta^+(u)\operatorname{tr}(\operatorname{ad}(H_0))} + e^{-\delta^-(u)\operatorname{tr}(\operatorname{ad}(H_0))} \right) d\nu(u).
\end{aligned}
$$

Clearly $\operatorname{area}(\partial K) \geq \operatorname{area}(\partial K^+) + \operatorname{area}(\partial K^-)$. We have

$$
\operatorname{area}(\partial K^+) = \int_U \sqrt{\det((\langle D\varphi(u)e_i, D\varphi(u)e_j \rangle_{(u, \delta^+(u))})_{i,j=1...n-1}} \, d\nu(u),
$$

where $\varphi(u) = (u, \delta^+(u))$ and $e_1, \ldots, e_{n-1}$ is an orthonormal basis of $T_u G_0$. We estimate the integrand

$$
\begin{aligned}
\sqrt{\det((\langle D\varphi(u)e_i, D\varphi(u)e_j \rangle_{(u, \delta^+(u))})} &= \sqrt{\det((\langle e_i, e_j \rangle_{(u, \delta^+(u))} + vv^{\top})} \\
&\geq \sqrt{\det\left(\langle e_i, e_j \rangle_{(u, \delta^+(u))} \right)} \\
&= e^{-\delta^+(u)\operatorname{tr}(\operatorname{ad}(H_0))},
\end{aligned}
$$

(5)

where $v^{\top} = (e_1(\delta^+), \ldots, e_{n-1}(\delta^+))$ and the e_i's on the right-hand side are considered as elements in $T_{(u, \delta^+(u))}G$. The equality (5) follows from (2). This together with the analogous estimate for ∂K^- yields

$$
\tag{6} \operatorname{area}(\partial K) \geq \operatorname{tr}(\operatorname{ad}(H_0)) \operatorname{vol}(K).
$$

This proves inequality (4).

Finally, we prove equality in (4) in the case $h(G_0) = 0$. As before, we use the diffeomorphism $G \cong G_0 \times \mathbb{R}$. Let $K_0 \subset G_0$ be arbitrary and consider the set $K = K_0 \times [a,b] \subset G$ with boundary $\partial K = (K_0 \times \{a,b\}) \cup (\partial K_0 \times [a,b])$. Direct calculations yield

$$
\tag{7} \operatorname{vol}(K) = \operatorname{vol}(K_0) \frac{e^{-a\operatorname{tr}\operatorname{ad}H_0} - e^{-b\operatorname{tr}\operatorname{ad}H_0}}{\operatorname{tr}\operatorname{ad}H_0},
$$

as well as

$$
\operatorname{area}(K_0 \times \{a,b\}) = \operatorname{vol}(K_0) \left(e^{-a\operatorname{tr}\operatorname{ad}H_0} + e^{-b\operatorname{tr}\operatorname{ad}H_0} \right).
$$

Given $\epsilon > 0$, we can arrange for

$$
\tag{8} \frac{\operatorname{area}(K_0 \times \{a,b\})}{\operatorname{vol}(K)} \leq \operatorname{tr}\operatorname{ad}H_0 + \epsilon
$$

by choosing b sufficiently large. Note that (7) and the estimate (8) also make sense if $\operatorname{tr}\operatorname{ad}H_0 = 0$.

Let $n = \dim \mathfrak{G}_0$. Choosing an orthonormal basis $e_1, \ldots, e_{n-1}$ of $T_u \partial K_0 \subset \mathfrak{G}_0$ and abbreviating $e^{-t(\mathrm{ad}\, H_0)} = A_t$, we finally compute

$$\mathrm{area}(\partial K_0 \times [a, b]) = \int_a^b \mathrm{vol}_{n-1}(\partial K_0 \times \{t\})\, dt,$$

where the $(n-1)$-dimensional volume is given by

$$\mathrm{vol}_{n-1}(\partial K_0 \times t) = \int_{\partial K_0} \sqrt{\det\left(\langle A_t e_p, A_t e_q \rangle_{\mathfrak{G}_0}\right)_{p,q=1\ldots n-1}}\; d\,\mathrm{vol}_{\partial K_0}(u).$$

The integrand can be estimated from above by

$$M := \max\left\{\left.\sqrt{\det\left(P_U A_t{}^* A_t|_U\right)}\;\right|\; a \le t \le b,\; U \subset \mathfrak{G}_0,\; \dim U = n-1\right\},$$

where $P_U : \mathfrak{G}_0 \to U$ denotes orthogonal projection. Note that M is independent of the specific choice of K_0. Therefore, choosing K_0 such that $\mathrm{vol}_{n-1}\, \partial(K_0)/\mathrm{vol}(K_0)$ is sufficiently small, we may assume that

$$(9) \qquad \frac{\mathrm{area}(\partial K_0 \times [a, b])}{\mathrm{vol}(K)} \le \frac{M(\mathrm{tr\, ad}\, H_0)(b-a)}{e^{-a\,\mathrm{tr\, ad}\, H_0} - e^{-b\,\mathrm{tr\, ad}\, H_0}}\, \frac{\mathrm{vol}_{n-1}(\partial K_0)}{\mathrm{vol}(K_0)} \le \epsilon.$$

Putting (8) and (9) together yields $h(G) = \mathrm{tr\, ad}\, H_0$, since $\epsilon > 0$ was chosen arbitrarily small. $\qquad \square$

Now we consider the particular case of a simply connected solvable Lie group G with Lie algebra $\mathfrak{G}$ and $\mathfrak{N} = [\mathfrak{G}, \mathfrak{G}]$. In this case we obtain

Corollary 10. *The Cheeger constant of a simply connected solvable Lie group G with Lie algebra $\mathfrak{G}$ is*

$$(11) \qquad\qquad h(G) = \max_{H \in \mathfrak{G}, ||H||=1} \mathrm{tr}\left(\mathrm{ad}\,(H)\right).$$

Proof. Let $\mathfrak{G}_0$ denote the kernel of the 1-form $\alpha \colon \mathfrak{G} \to \mathbb{R}$, $X \mapsto \mathrm{tr\, ad}\, X$. Clearly $\mathfrak{G}_0$ contains $\mathfrak{N} = [\mathfrak{G}, \mathfrak{G}]$, since $\mathfrak{N}$ is nilpotent. Hence $\mathfrak{G}_0$ is an ideal, and the corresponding unimodular Lie group satisfies $h(G_0) = 0$ by Theorem 3, which proves the corollary in the case $\mathfrak{G}_0 = \mathfrak{G}$. Otherwise $\mathfrak{G}_0$ has codimension 1. Let H_0 be the maximum of α on the unit sphere of $\mathfrak{G}$, i.e.,

$$\max_{H \in \mathfrak{G}, ||H||=1} \mathrm{tr}\left(\mathrm{ad}\,(H)\right) = \mathrm{tr}\left(\mathrm{ad}\,(H_0)\right).$$

Then $H_0 \perp \mathfrak{G}_0$ and, again, Theorem 3 implies the statement of the corollary. $\qquad \square$

Remarks. Note that simply connected, solvable Lie groups with left-invariant metric may have curvature of both signs (e.g., horospheres in symmetric spaces). Corollary 10 in the particular case of simply connected strictly negatively curved homogeneous spaces (NCHS) was proved by Connell [Co]. He also showed for NCHS that the Cheeger constant coincides with the exponential volume growth rate.

Formula (11) also gives a lower bound for the topological entropy of compact Riemannian manifolds whose universal covering is a solvable Lie group with left-invariant metric. This follows from the lower estimate of the topological entropy by the exponential volume growth rate proved in [Ma]. In the case of a locally symmetric space M the topological entropy was calculated in [Sp] and agrees with both the exponential volume growth rate and the Cheeger constant of the universal covering $\tilde{M}$. It is given by $\|\rho\|$, where ρ is the sum of the positive roots with

multiplicities. More refined volume growth calculations in symmetric spaces were carried out in [Kn].

Acknowledgments

References

[Br] R. Brooks, *The spectral geometry of a tower of coverings*, J. Differential Geom. 23, 97-107, 1986. MR **87j:**58095

[BPP] R. Brooks, P. Perry, and P. Petersen, *On Cheeger's inequality*, Comment. Math. Helvetici 68, 599-621, 1993. MR **94k:**58150

[Bu] P. Buser, *A note on the isoperimetric constant*, Ann. Sci. École Norm. Sup. 15, 213-230, 1982. MR **84e:**58076

[Cha] I. Chavel, *Eigenvalues in Riemannian geometry*, Academic Press, Orlando, FL, 1984. MR **86g:**58140

[Che] J. Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian*, "Problems in Analysis", pp. 195-199, Princeton Univ. Press, Princeton, NJ, 1970. MR **53:**6645

[Co] Ch. Connell, *Asymptotic harmonicity of negatively curved homogeneous spaces and their measures at infinity*, Comm. Anal. Geom., vol. 8, no. 3, 575-633, 2000. MR **2001j:**53061

[Gr] A. Grigor'yan, *Estimates of heat kernels on Riemannian manifolds*, "Spectral theory and geometry", pp. 140-225, Cambridge Univ. Press, Cambridge, 1999. MR **2001b:**58040

[Ho] H. F. Hoke III, *Lie groups that are closed at infinity*, Trans. Amer. Math. Soc. 313, Number 2, 721-735, 1989. MR **89j:**58108

[Kn] G. Knieper, *On the asymptotic geometry of nonpositively curved manifolds*, Geom. Funct. Anal. 7, 755-782, 1997. MR **98h:**53055

[Leu] E. Leuzinger, *Kazhdan's property (T), L^2-spectrum and isoperimetric inequalities for locally symmetric spaces*, Comment. Math. Helv. 78, 116-133, 2003. MR **2004a:**53047

[Ma] A. Manning, *Topological entropy for geodesic flows*, Ann. of Math. (2) 110, 567-573, 1979. MR **81e:**58044

[Pa] A. Paterson, *Amenability*, Mathematical Surveys and Monographs, no. 29, Amer. Math. Soc., Providence, RI, 1988. MR **90e:**43001

[Pe] N. Peyerimhoff, *Isoperimetric and ergodic properties of horospheres in symmetric spaces*, in A. Katok, R. de la Llave, Y. Pesin, H. Weiss, editors, Smooth ergodic theory and its applications, Proc. Sympos. Pure Math. 69, Amer. Math. Soc., Providence, RI, 2001, pp. 793-808. MR **2002i:**37031

[Pi] Ch. Pittet, *The isoperimetric profile of homogeneous Riemannian manifolds*, J. Differential Geom. 54, 255-302, 2000. MR **2002g:**53088

[Sp] R. J. Spatzier, *Dynamical properties of algebraic systems*, Dissertation, Warwick and Maryland, 1983.

Mathematische Fakultät, Ruhr-Universität Bochum, Universitätsstrasse 150, 44780 Bochum, Germany

E-mail address: `peyerim@math.ruhr-uni-bochum.de`

Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537, 1678 Nicosia, Cyprus

E-mail address: `samiou@ucy.ac.cy`