PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 6, Pages 1839–1843
S 0002-9939(03)07234-4
Article electronically published on October 15, 2003

L^2-COHOMOLOGY OF BUILDINGS WITH FUNDAMENTAL CLASS

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(Communicated by Jozef Dodziuk)

ABSTRACT. For an infinite building whose Davis chamber is an orientable, gallery connected pseudomanifold (with boundary), we calculate the top-dimensional L^2-Betti number. This gives a complete determination of L^2-Betti numbers for 2-dimensional, right-angled hyperbolic buildings.

1. NOTATION

Basic facts and definitions concerning buildings and L^2-homology can be found in [5], [6], [7]. We denote the Davis realization of the building by X, the Coxeter system by (W, S), the Davis complex of W by Σ and the Davis chamber by D (for definitions, see [7], p. 589). Throughout we are dealing with simplicial chains and cochains on X, which can be thought of as L^2 functions on the set of simplices of X (after choosing orientations of all simplices). The cohomology that we want to calculate is the *reduced* L^2-cohomology, meaning that we divide the kernel of the coboundary map by the L^2-closure of its image. The chambers of the Davis complex are in canonical bijection with W; we denote the chamber corresponding to w by D_w. We assume throughout that the nerve L (the boundary of the Davis chamber D) is an orientable, gallery connected pseudomanifold of dimension $n-1$. Recall that an m-dimensional simplicial complex M is a *pseudomanifold* if every simplex is a face of some m-simplex, and every $(m-1)$-simplex is a face of exactly two m-simplices. An m-dimensional pseudomanifold is called *gallery connected* if for every two m-simplices σ, η there exists a sequence of m-simplices $\sigma_0, \dots, \sigma_k$ such that $\sigma_0 = \sigma$, $\sigma_k = \eta$, and $\sigma_i \cap \sigma_{i+1}$ is an $(m-1)$-simplex for each integer $i \in [0, k-1]$. We fix an orientation of L and induce an orientation on all n-simplices of D. Then we extend this induced orientation W-equivariantly to all n-simplices of Σ. Also, we orient n-simplices of X, pulling back orientations from Σ via a folding map. Notice that since $\dim X = n$, every n-cochain is a cocycle; hence, a harmonic n-cochain means the same thing as an n-cycle. If ψ is an n-cycle (or a harmonic n-cochain) on X, and if σ, τ are two n-simplices in X contained in the same chamber c, then $\psi(\sigma) = \psi(\tau)$ (since L is an orientable, gallery connected, pseudomanifold). We denote the common value by $\psi(c)$. All the n-cochains that we will consider will also have this property (we use $C^n(X)$ to denote the set of such cochains). For example, if b is a chamber of X, we denote by $\mathbf{1}_b$ the cochain whose

Received by the editors June 12, 2002 and, in revised form, February 10, 2003.
2000 *Mathematics Subject Classification.* Primary 58J22; Secondary 20F55, 20F65.
This work was partially supported by KBN grant 5 P03A 035 20.

value on simplices contained in b is 1, and whose value on all other simplices is 0. We renormalise the L^2-norm, multiplying it by a constant so as to have $\|\mathbf{1}_b\| = 1$.

We assume that for each $s \in S$ and every chamber b of X the number of chambers s-adjacent to b depends only on s and is finite. Then this number $q_{[s]}$ depends in fact only on the conjugacy class $[s]$ of s in W. We denote by $\mathbf{q}$ the collection of all $q_{[s]}$ and by $\mathbf{q}+\mathbf{1}$ the collection of integers $q_{[s]}+1$. The *thickness* of the building is defined to be $\mathbf{q}+\mathbf{1}$. We also put $\mathbf{q}^{-1} = (q_{[s]}^{-1})_{s\in S}$. For $w \in W$, let $\ell_{[s]}(w)$ be the number of occurrences of generators from a given conjugacy class $[s]$ in a reduced word representing w. Let $\mathbf{l}(w) = (\ell_{[s]}(w))_{s\in S}$. We define the generating function of the Coxeter group as follows: $W(\mathbf{t}) = \sum_{w\in W} \mathbf{t}^{\mathbf{l}(w)}$, where $\mathbf{t} = (t_{[s]})_{s\in S}$, $\mathbf{t}^{\mathbf{l}(w)} = \prod_{[s]} t_{[s]}^{\ell_{[s]}(w)}$. The building X (as well as the Davis complex) is equipped with a W-valued distance function d. Out of this function we produce a family of distances $d_{[s]} = \ell_{[s]} \circ d$. Thus, for two chambers b, c, we get a collection of numbers $\mathbf{d}(b,c) = (d_{[s]}(b,c))_{s\in S}$.

2. The standard n-cycle

The protagonist of this note is the harmonic n-cochain on X defined by: $\phi_b(c) = (-\mathbf{q})^{-\mathbf{d}(b,c)}$, where b is a chamber in X. Using the b-based folding $\pi_b : X \to \Sigma$ we calculate

$$\|\phi_b\|^2 = \sum_{c\in X} |\phi_b(c)|^2 = \sum_{w\in W} (-\mathbf{q})^{-2\mathbf{l}(w)} \mathbf{q}^{\mathbf{l}(w)} = \sum_{w\in W} \mathbf{q}^{-\mathbf{l}(w)}. \tag{2.1}$$

Therefore, $\phi_b \in L^2$ if and only if $\mathbf{q}^{-1}$ is in the convergence region of the power series $W(\mathbf{t})$. Also, if $\phi_b \in L^2$, then $\|\phi_b\|^2 = W(\mathbf{q}^{-1})$.

Theorem 2.1. *Suppose that (W,S) is an infinite Coxeter system such that the boundary of the Davis chamber is an $(n-1)$-dimensional, gallery connected, orientable pseudomanifold. Let X be a building with Weyl group W of thickness $\mathbf{q}+\mathbf{1}$. Then $L^2\mathcal{H}^n(X)$ (the space of square-summable harmonic n-cochains) is nonzero if and only if $\mathbf{q}^{-1}$ is in the convergence region of the series $W(\mathbf{t})$.*

Proof. $(\Rightarrow)$

Suppose $L^2\mathcal{H}^n(X) \neq 0$. Let ψ be a nonzero element of this space. Choose a chamber b such that $\psi(b) \neq 0$. Define an n-cochain $\overline{\psi}$ on Σ as follows: $\overline{\psi}(c)$ is the average of the values of ψ on $\pi_b^{-1}(c)$ (recall that $\pi_b : X \to \Sigma$ is the b-based folding). Now put $\psi_b = \overline{\psi} \circ \pi_b \in C^n(X)$. Notice that harmonicity of ψ implies $\mathbf{q}^{\mathbf{l}(w)}\overline{\psi}(D_w) = -\mathbf{q}^{\mathbf{l}(ws)}\overline{\psi}(D_{ws})$ or, equivalently,

$$\overline{\psi}(D_{ws}) = \begin{cases} -q_{[s]}^{-1}\overline{\psi}(D_w), & \text{if } l_{[s]}(ws) > l_{[s]}(w), \\ -q_{[s]}\overline{\psi}(D_w), & \text{if } l_{[s]}(ws) < l_{[s]}(w), \end{cases} \tag{2.2}$$

for all $s \in S$, $w \in W$. It follows that $\overline{\psi}(D_w) = (-\mathbf{q})^{-\mathbf{l}(w)}$, which gives $\psi_b = \psi(b)\phi_b$. However, by Schwartz's inequality, $\|\psi_b\| \le \|\psi\|$. Therefore $\phi_b \in L^2$ and, by (2.1), $\mathbf{q}^{-1}$ is in the convergence region of $W(\mathbf{t})$.

$(\Leftarrow)$

For every chamber b, ϕ_b is a nonzero element of $L^2\mathcal{H}^n(X)$. $\square$

3. Symmetric buildings

Now suppose that $G \subseteq \mathrm{Aut}(X)$ is a closed subgroup with a BN-pair adapted to X. In particular, B is the stabilizer in G of some chamber in X which we henceforth denote b. One can use G to measure von Neumann dimension $\dim_G$ of $L^2\mathcal{H}^i(X)$. We denote this dimension by $L^2b^i(X)$ (it turns out that it does not depend on the choice of G).

Theorem 3.1. *Under all the assumptions of Theorem* 2.1*, and the symmetry assumption stated above, we have:*

(a) *if* $\mathbf{q}^{-1}$ *is in the convergence region of* $W(\mathbf{t})$*, then* $L^2b^n(X) = \frac{1}{W(\mathbf{q}^{-1})}$*;*

(b) *if* $\mathbf{q}^{-1}$ *is not in the convergence region of* $W(\mathbf{t})$*, then* $L^2b^n(X) = 0$*.*

Proof. Part (b) follows from Theorem 2.1. Suppose then that $\mathbf{q}^{-1}$ is in the convergence region of $W(\mathbf{t})$. To find $L^2b^n(X)$ we need to embed the G-module $L^2\mathcal{H}^n(X)$ into $L^2(G)$. Recall that G/B can be identified with the set of all chambers in X. Using this identification and the quotient map $p : G \to G/B$ we can pull cochains on X back to G. Equip G with the Haar measure μ, normalised so that $\mu(B) = 1$ (G is unimodular, cf. [7]). Then the above pull-back gives an isometric G-equivariant embedding $p^* : L^2C^n(X) \to L^2(G)$.

We need to find the orthogonal projection $P : L^2(G) \to p^*(L^2\mathcal{H}^n(X))$. First, the orthogonal projection $L^2(G) \to p^*(L^2C^n(X))$ is given by the right convolution $*\mathbf{1}_B$. Put $\Phi = \phi_b \circ p \in L^2(G)$, and let $\Phi_{gB} = \Phi \circ g^{-1}$. We have to understand the restricted projection $P : p^*(L^2C^n(X)) \to p^*(L^2\mathcal{H}^n(X))$.

We will need the following:

Lemma 3.2. *In* $L^2C^n(X)$*, the orthogonal projection of* $\mathbf{1}_b$ *on* $L^2\mathcal{H}^n(X)$ *is* $\alpha\phi_b$*, where* $\alpha = W(\mathbf{q}^{-1})^{-1}$*.*

Proof. Since the cochain $\mathbf{1}_b$ is B-invariant, so is its projection on $L^2\mathcal{H}^n(X)$. But all B-invariant harmonic n-cochains are of the form $\alpha\phi_b$.

The coefficient α can be characterized as the unique number for which $\langle \phi_b, \mathbf{1}_b - \alpha\phi_b \rangle = 0$. This gives $\alpha = \langle \phi_b, \phi_b \rangle^{-1}$, and the lemma follows from (2.1). □

We continue with the proof of Theorem 3.1. Every function $f \in p^*(L^2C^n(X))$ can be written as $\sum_{g \in G/B} a_g \mathbf{1}_{gB}$. The lemma gives $P(\mathbf{1}_B) = \alpha\Phi$; by symmetry, we deduce $P(\mathbf{1}_{gB}) = \alpha\Phi_{gB}$. Therefore,

$$
\begin{aligned}
P(f)(u) = \sum_{g \in G/B} a_g P(\mathbf{1}_{gB})(u) &= \sum_{g \in G/B} a_g \alpha \Phi_{gB}(u) \\
&= \sum_{g \in G/B} a_g \alpha \Phi(g^{-1}u).
\end{aligned}
\tag{3.1}
$$

On the other hand,

$$
\begin{aligned}
(f*\Phi)(u) &= \int_G f(uh^{-1})\Phi(h)dh \\
&= \sum_{g\in G/B} a_g \int_G \mathbf{1}_{gB}(uh^{-1})\Phi(h)dh \\
&= \sum_{g\in G/B} a_g \int_G \mathbf{1}_{u^{-1}gB}(h^{-1})\Phi(h)dh \qquad (3.2)\\
&= \sum_{g\in G/B} a_g \int_G \mathbf{1}_{Bg^{-1}u}(h)\Phi(h)dh \\
&= \sum_{g\in G/B} a_g \Phi(g^{-1}u).
\end{aligned}
$$

Thus $\alpha(f*\Phi) = P(f)$.

Composing the two projections $L^2(G) \to p^*(L^2C^n(X)) \to p^*(L^2\mathcal{H}^n(X))$ we get $P = *(\mathbf{1}_B * \alpha\Phi) = *\alpha\Phi$.

The proof of part (a) goes as follows:

$$L^2b^n(X) = \dim_G L^2\mathcal{H}^n(X) = \mathrm{Tr}_G P = \alpha\Phi(e) = \alpha = W(\mathbf{q}^{-1})^{-1}.$$

□

4. Hyperbolic right-angled buildings of dimension 2

Let $X = X_{p,q}$ be the right-angled 2-dimensional hyperbolic building, with chamber a p-gon, $p > 4$, and thickness $q+1$ along every edge (cf. [2]). Because of the uniform thickness, we replace each $t_{[s]}$ by t; then $W(\mathbf{t})$ reduces to a one-variable power series $W(t)$. Both of these series represent rational functions, which were calculated by Bourdon [2]. In particular,

$$\frac{1}{W(t)} = 1 - \frac{p}{1+t} + \frac{p}{(1+t)^2}, \tag{4.1}$$

so that

$$W(t) = \frac{(1+t)^2}{1+(2-p)t+t^2}. \tag{4.2}$$

The smallest (in absolute value) zero of the denominator is $\frac{1}{2}(p-2-\sqrt{p^2-4p}) = 2(p-2+\sqrt{p^2-4p})^{-1}$; it lies between $(p-2)^{-1}$ and $(p-3)^{-1}$. This gives the following:

Lemma 4.1. *For $q \geq p-2$ the series* (2.1) *converges, while for $q < p-2$ it does not.*

We choose G as in section 2 (e.g., $G = \mathrm{Aut}(X)$), and we want to find the L^2-Betti numbers of X.

We have $L^2b^0(X) = 0$: a 0-cocycle is constant and the set of vertices of X is infinite, so that no 0-cocycle is in L^2. Standard arguments show that the L^2-Euler characteristic $L^2\chi(X) = L^2b^0(X) - L^2b^1(X) + L^2b^2(X)$ coincides with the Euler characteristic of the orbifold quotient X/G; the latter is $1 - \frac{p}{1+q} + \frac{p}{(1+q)^2} = \frac{1}{W(q)}$. Therefore $-L^2b^1(X) + L^2b^2(X) = \frac{1}{W(q)}$, and it suffices to find L^2b^2 to know the other Betti number.

Theorem 4.2. (a) *If $q < p-2$, then*

$$L^2b^0(X) = 0, \quad L^2b^1(X) = -\frac{1}{W(q)} = -1 + \frac{p}{1+q} - \frac{p}{(1+q)^2}, \quad L^2b^2(X) = 0.$$

(b) *If $q \geq p-2$, then*

$$L^2b^0(X) = 0, \quad L^2b^1(X) = 0, \quad L^2b^2(X) = \frac{1}{W(q)} = 1 - \frac{p}{1+q} + \frac{p}{(1+q)^2}.$$

Proof. (a) By the lemma, the series (1) diverges. Therefore, by Theorem 3.1(b), $L^2b^2(X) = 0$. Hence, $L^2b^1(X) = -\chi(X) = -\frac{1}{W(q)}$.

(b) By the lemma and Proposition 2(a), $L^2b^2(X) = \frac{1}{W(q^{-1})}$. However, it is easy to check using (4.1) or (4.2) that $W(q^{-1}) = W(q)$ (this is no accident, cf. [4]). It follows that $L^2b^2(X) = \frac{1}{W(q)} = \chi(X)$, which implies $L^2b^1(X) = 0$. □

Remarks. 1. The methods of this note are inspired by Borel's proof of irreducibility of the Steinberg representation, cf. [1].

2. We have not touched the question of unreduced cohomology; the L^2-cohomology of X is known to be reduced (and the L^2-Betti numbers are known) when all $q_{[s]}$ are sufficiently large, cf. [7].

3. In [3] Bourdon and Pajot determined the L^p-cohomology of right-angled hyperbolic buildings using more powerful methods.

References

1. Borel A., *Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup*, Invent. Math. **35** (1976), 233–259. MR **56:**3196
2. Bourdon M., *Sur les immeubles fuchsiens et leur type de quasi–isométrie*, Ergodic Theory Dynam. Systems **20** (2000), no. 2, 343–364. MR **2001g:**20056
3. Bourdon M., Pajot H., *Cohomologie L^p et espaces de Besov*, J. Reine Angew. Math. **558** (2003), 85–108.
4. Charney R., Davis M., *Reciprocity of growth functions of Coxeter groups*, Geom. Dedicata **39** (1991), no. 3, 373–378. MR **92h:**20067
5. Davis M., *Buildings are CAT(0)*, in Geometry and Cohomology in Group Theory (Durham, 1994), 108–123, London Math. Soc. Lecture Note Ser., 252, Cambridge Univ. Press, Cambridge, 1998. MR **2000i:**20068
6. Davis M., Okun B., *Vanishing theorems and conjectures for the L^2-homology of right-angled Coxeter groups*, Geom. Topol. **5** (2001), 7–74. MR **2002e:**58039
7. Dymara J., Januszkiewicz T., *Cohomology of buildings and of their automorphism groups*, Invent. Math. **150** (2002), 579–627. MR **2003j:**20052

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