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WEAK* PROPERTIES OF WEIGHTED CONVOLUTION ALGEBRAS

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ABSTRACT. Suppose that $L^1(\omega)$ is a weighted convolution algebra on $\mathbf{R}^+ = [0,\infty)$ with the weight $\omega(t)$ normalized so that the corresponding space $M(\omega)$ of measures is the dual space of the space $C_0(1/\omega)$ of continuous functions. Suppose that $\phi : L^1(\omega) \to L^1(\omega')$ is a continuous nonzero homomorphism, where $L^1(\omega')$ is also a convolution algebra. If $L^1(\omega)*f$ is norm dense in $L^1(\omega)$, we show that $L^1(\omega')*\phi(f)$ is (relatively) weak* dense in $L^1(\omega')$, and we identify the norm closure of $L^1(\omega') * \phi(f)$ with the convergence set for a particular semigroup. When ϕ is weak* continuous it is enough for $L^1(\omega) * f$ to be weak* dense in $L^1(\omega)$. We also give sufficient conditions and characterizations of weak* continuity of ϕ. In addition, we show that, for all nonzero f in $L^1(\omega)$, the sequence $f^n/||f^n||$ converges weak* to 0. When ω is regulated, $f^{n+1}/||f^n||$ converges to 0 in norm.

1. Introduction

Suppose that $\omega(x)$ is a positive Borel measurable function on $\mathbf{R}^+ = [0,\infty)$. When both ω and $1/\omega$ are locally bounded on $[0,\infty)$, we say that ω is a *weight*. When $\omega(x)$ is a weight, then $L^1(\omega)$ is the Banach space of (equivalence classes of) locally integrable functions f for which $f\omega$ is integrable. We give $L^1(\omega)$ the inherited norm

$$||f|| = ||f||_\omega = ||f\omega||_1 = \int_0^\infty |f(t)|\omega(t)dt.$$

Similarly, $M(\omega)$ is the analogous space of measures with the norm

$$||\mu|| = ||\mu||_\omega = \int_{\mathbf{R}^+} \omega(t)d|\mu|(t),$$

and $C_0(1/\omega)$ is the space of continuous functions h on $[0,\infty)$ for which

$$\lim_{x\to\infty} h(x)/\omega(x) = 0,$$

with the norm

$$||h(x)|| = ||h/\omega||_\infty = \sup\{h(x)/\omega(x)\}_{x\geq 0}.$$

We are particularly interested in the case in which the weight ω is an *algebra weight*; that is, ω is submultiplicative (i.e., $\omega(x+y) \leq \omega(x)\omega(y)$), is everywhere

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right continuous, and has $\omega(0) = 1$. The submultiplicativity implies that both $L^1(\omega)$ and $M(\omega)$ are Banach algebras under convolution, and that $L^1(\omega)$ is a closed ideal in $M(\omega)$ when we identify the function $f(t)$ with the measure $f(t)dt$. The other conditions guarantee that $M(\omega)$ is the dual space of the separable Banach space $C_0(1/\omega)$ under the natural duality $\langle \mu, h\rangle = \int h(t)d\mu(t)$, [Gr1, Theorem 2.2, p. 592] so that $M(\omega)$, and its subspace $L^1(\omega)$, are equipped with a natural weak*-topology. Requiring that ω be an algebra weight in our sense is just a normalization. Whenever $L^1(\omega)$ is an algebra, we can always replace ω by an equivalent algebra weight without changing the space $L^1(\omega)$ or its norm topology [Gr1, Theorem 2.1, p. 591].

In this paper we examine the structure of $L^1(\omega)$, and particularly homomorphisms between such algebras, in the weak* topology. The weak* topology is in many ways better behaved than the norm topology, and, as we shall see, can be used as a tool in proving results for normed topologies.

In section 2, we collect results characterizing weak* convergence of bounded nets in $M(\omega)$ and relating weak* convergence to convergence in various norms. In sections 3 and 4 we consider continuous nonzero homomorphisms $\phi : L^1(\omega) \to L^1(\omega')$. In previous papers, starting with [GGM], we considered characterizations of sufficient conditions for ϕ to be what we called a *standard homomorphism*; that is, $L^1(\omega) * f$ being norm dense in $L^1(\omega)$ implies $L^1(\omega') * \phi(f)$ is norm dense in $L^1(\omega')$. In section 3 we show that $L^1(\omega') * \phi(f)$ is always weak* dense, and we describe the norm closure of $L^1(\omega') * \phi(f)$. When ϕ is weak* continuous, we show that it is enough for $L^1(\omega) * f$ to be weak* dense. In section 4, we give characterizations and useful sufficient conditions for the weak* continuity of ϕ. In section 5 we give conditions on ω that guarantee that the sequences $f^{(n+1)}/||f^n||$ converge to 0 in norm. This is done by first showing that $f^n/||f^n||$ always converges to zero weak*, and then applying results collected in section 2.

2. Weak* convergence of nets

We first collect, mostly from earlier papers, a number of equivalent characterizations of weak* convergence of nets in $M(\omega)$.

Theorem 2.1. *Suppose that ω is an algebra weight on $\mathbf{R}^+$, and let $\{\lambda_n\}$ be a bounded net in $M(\omega)$. If $\{\lambda_n\}$ converges to λ weak* in $M(\omega)$, then we have:*

(a) *For all ν in $M(\omega)$, weak*-$\lim_n \lambda_n * \nu = \lambda * \nu$.*
(b) *For all continuous functions f on $\mathbf{R}^+$ with $f(0) = 0$, the net $\{\lambda_n * f\}$ converges pointwise to $\lambda * f$.*
(c) *If ω' is a weight with ω'/ω bounded and integrable and if f belongs to $L^1(\omega)$, then $\{\lambda_n * f\}$ converges to $\lambda * f$ in the norm of $L^1(\omega')$.*
(d) *For all locally integrable f and all $a > 0$, $\lim_n \int_0^a |\lambda_n * f(t) - \lambda * f(t)|dt = 0$.*

Conversely, if one of the conditions (a), (b), (c), or (d) holds for a single nonzero function (or measure), then $\{\lambda_n\}$ converges weak to λ in $M(\omega)$.*

The proof that (a) is equivalent to weak* convergence is in [Gr1, Lemma (2.2)]. Part (b) is proved in [GG1, Theorem (3.1)(a), p. 511]. The proof is given for sequences, but the same proof works for nets. Part (c), for sequences, is [GG1, Theorem (3.2), p. 512]; but [GG3, Theorem (1.3)] shows that if (c) holds for some f and all weak* convergent sequences, then convolution by f is a compact operator from $M(\omega)$ to $L^1(\omega')$. This then implies that (c) also holds for all bounded weak* convergent nets. Part (d) is an easy consequence of (c) [GG1, Cor. (3.3), p. 513].

Once we know that some type of convergence, as in (a), (b), (c), or (d), follows from weak* convergence, the proof of the converse follows from the fact that every bounded net has a weak* convergent subnet, together with the fact that the convolution of nonzero measures on $\mathbf{R}^+$ can never be zero. For the details see [Gr1, Lemma 3.2, p. 595], or [Gr3, Theorem (4.1), p. 183].

There are numerous other useful characterizations of weak* convergence. For instance [GG2, p. 52], it is enough that $\lim_n \langle \lambda_n, h\rangle = \langle \lambda, h\rangle$ for all continuous h with compact support, since the set functions with compact support are dense in $C_0(1/\omega)$.

The nicest results occur when all $\lambda_n * f$ converge to $\lambda * f$ in the norm of $L^1(\omega)$. Recall that the algebra weight $\omega(t)$ is *regulated* at $b \geq 0$ if $\lim_{x\to\infty} \omega(x+a)/\omega(x) = 0$ for all $a > b$. Recall also that if λ is a locally integrable function or a locally finite measure of $\mathbf{R}^+$, then $\alpha(\lambda)$ is the infimum of the support of λ $(\alpha(0) = \infty)$. The basic result relating convergence to weak* convergence in $M(\omega)$ is the following result, taken from [GGM, Theorem (3.2), p. 284] and [GG1, Theorem (2.3), p. 509].

Theorem 2.2. *Suppose that ω is an algebra weight on $\mathbf{R}^+$ and that $b \geq 0$. Then the following are equivalent:*

(a) *ω is regulated at b.*

(b) *Whenever $\{\lambda_n\}$ is a bounded net converging weak* to λ in $M(\omega)$ and g is a function in $L^1(\omega)$ with $\alpha(g) \geq b$, then $\lambda_n * g$ converges to $\lambda * g$ in the norm of $L^1(\omega)$.*

Condition (c) in Theorem (2.1) is the simplest condition on ω'/ω that guarantees convergence in norm in $L^1(\omega')$. A determination of precisely which ω'/ω work is given in [GG3]. The most important case, $\omega' = \omega$, is Theorem (2.2) above.

Theorem (2.1)(a) says that multiplication is weak* separately continuous on bounded subsets of $M(\omega)$. In fact, it is not hard to show [Gr1, Lemma 3.1, p. 595] that multiplication is weak* separately continuous on all of $M(\omega)$. The following result shows that on bounded subsets of $M(\omega)$, multiplication is actually jointly continuous in the weak* topology. Since the weak* topology restricted to bounded subsets of $M(\omega) = C_0(1/\omega)^*$ is metrizable [DS, Theorem V.5.1, p. 426], we need only consider sequences in the following result.

Theorem 2.3. *Suppose that ω is an algebra weight and that $\{\lambda_n\}$ and $\{\mu_n\}$ are sequences in $M(\omega)$. If weak*-$\lim \lambda_n = \lambda$ and weak*-$\lim \mu_n = \mu$, then weak*-$\lim \lambda_n * \mu_n = \lambda * \mu$.*

Proof. Choose some nonzero f in $L^1(\omega)$, and let ω' be as in Theorem (2.1)(c); for instance, we could let $\omega'(t) = e^{-t}\omega(t)$. Then $\lambda_n * f \to \lambda * f$ and $\mu_n * f \to \mu * f$ in norm in the Banach algebra $L^1(\omega')$. Hence the $L^1(\omega')$ norm limit of $(\lambda_n * \mu_n) * (f * f)$ is $(\lambda * \mu) * (f * f)$. It then follows from Theorem (2.1) that $\lambda_n * \mu_n \to \lambda * \mu$ in the weak* topology on $M(\omega)$. □

3. Weak*-standard homomorphisms

Throughout this section, ω and ω' are algebra weights and $\phi : L^1(\omega) \to L^1(\omega')$ is a continuous nonzero homomorphism. Then ϕ has a unique extension to a homomorphism from $M(\omega)$ to $M(\omega')$, and this extension is continuous with the same norm [Gr1, Theorem 3.4, p. 596]. Because of the uniqueness, we let ϕ denote both the original map and its extension. The homomorphism ϕ is said to be *standard* [GGM, p. 278] if $L^1(\omega') * \phi(f)$ is norm dense in $L^1(\omega')$ whenever $L^1(\omega) * f$ is norm dense in $L^1(\omega)$. In [GGM] we gave several equivalent characterizations of the standardness of homomorphisms, and we showed [GGM, Theorem (3.4), p. 284] that ϕ is standard if ω' is regulated at any $b \geq 0$; that is, if $\lim_{x\to\infty} \omega'(x+a)/\omega'(x) = 0$ for any $a > 0$. In this section we show that $L^1(\omega') * \phi(f)$ is always weak* dense when $L^1(\omega) * f$ is norm dense.

Let $\{\delta_t\}_{t\geq 0}$ be the convolution semigroup of point masses, so that $\delta_t * f(x) = f(x-t)$, the right translation of f; and let $\mu_t = \phi(\delta_t)$. Following [GGM, p. 280], we call

$$I = \{g \in L^1(\omega') : \lim_{t\to 0} \mu_t * g = g\}$$

the *convergence ideal* of ϕ (or of the semigroup $\{\mu_t\}$). Since $\{\mu_t\}$ is norm bounded near 0, the set I is easily seen to be a closed ideal. One of the characterizations of standardness of ϕ is that $I = L^1(\omega')$; that is, that (convolution by) μ_t is a strongly continuous semigroup on $L^1(\omega')$ [GGM, Theorem (2.2)(a), p. 280]. We are now ready for our main result.

Theorem 3.1. *Suppose that ω and ω' are algebra weights and that $\phi : L^1(\omega) \to L^1(\omega')$ is a continuous nonzero homomorphism. If $L^1(\omega) * f$ is norm dense in $L^1(\omega)$, then we have:*

(a) *The norm closure of $L^1(\omega') * \phi(f)$ is the convergence ideal of ϕ.*
(b) *$L^1(\omega') * \phi(f)$ is weak* dense in $L^1(\omega')$.*

Proof. Since we know [Gr2, Corollary (2.5), p. 162] that the convergence ideal of ϕ is weak* dense, it will be enough to prove (a). We also know [GGM, Theorem (2.4), pp. 281-282] that there exists f, specifically $f(t) = e^{-rt}$, with $L^1(\omega) * f$ dense and the norm closure $L^1(\omega') * \phi(f)$ equalling the convergence ideal of ϕ. It is also easy to see that all $L^1(\omega') * \phi(g)$ belong to the convergence ideal of ϕ [GGM, p. 282]. To complete the proof to the theorem, we just need the following lemma. $\square$

Lemma 3.2. *If $L^1(\omega) * f$ is norm dense in $L^1(\omega)$, then the norm closure of $L^1(\omega') * \phi(f)$ contains the norm closure of $L^1(\omega') * \phi(g)$ for all g in $L^1(\omega)$.*

Proof. Since $L^1(\omega) * f$ is dense, we can find a sequence $\{h_n\}$ in $L^1(\omega)$ for which $\lim(f * h_n) = g$, with the limit taken in the norm topology. By the continuity of ϕ, this implies that $\lim \phi(f) * \phi(h_n) = \phi(g)$. Hence $\phi(g)$, and therefore the norm closure of $L^1(\omega') * \phi(g)$ as well, belongs to the norm closure of $L^1(\omega') * \phi(f)$. This completes the proof of the lemma, and of Theorem (3.1). $\square$

Notice that not only the theorem, but also the lemma, show that the norm closure of $L^1(\omega') * \phi(f)$ is the same for all f in $L^1(\omega)$ with $L^1(\omega) * f$ norm dense. This greatly simplifies the formulas we were able to obtain in [GGM, p. 282].

The natural weak* analogue of standardness of homomorphisms should only assume that $L^1(\omega) * f$ is weak* dense in $L^1(\omega)$, rather than norm dense. The next result shows that this natural analogue does hold when the homomorphism is weak* continuous rather than just norm continuous. In section 4, we will study when homomorphisms are weak* continuous.

Theorem 3.3. *Suppose that the nonzero homomorphism $\phi : L^1(\omega) \to L^1(\omega')$ is weak* continuous. If $L^1(\omega) * f$ is weak* dense in $L^1(\omega)$, then $L^1(\omega') * \phi(f)$ is weak* dense in $L^1(\omega')$.*

Proof. Choose some g, say $g = e^{-rt}$, with $L^1(\omega) * g$ norm dense. Since we already know that $L^1(\omega') * \phi(g)$ is weak* dense, it will be enough to show that $\phi(g)$ is the weak* limit of a sequence in $L^1(\omega') * \phi(f)$. Since $L^1(\omega) * f$ is weak* dense, we can find a sequence $\{h_n\}$ in $L^1(\omega)$ with weak*-lim$(f * h_n) = g$. Since ϕ is a weak* continuous homomorphism, this shows that $\phi(f * h_n) = \phi(f) * \phi(h_n)$ converges weak* to $\phi(g)$ in $L^1(\omega')$. This completes the proof. □

The major unsolved question in the ideal theory of radical $L^1(\omega)$ is the standard ideal problem, which asks if $L^1(\omega) * f$ must be norm dense for all f in $L^1(\omega)$ with $\alpha(f) = 0$ ([D, p. 557], and [GG1, Question 1, p. 507]). The results in this section suggest the following, presumably easier, weak* analogue.

Question 3.4. *Suppose that $L^1(\omega)$ is a radical algebra and that f in $L^1(\omega)$ has $\alpha(f) = 0$. Must $L^1(\omega) * f$ be weak* dense in $L^1(\omega)$?*

When ω is regulated at any $b \geq 0$, $L^1(\omega) * f$ is norm dense if it is weak* dense ([GG3, Theorem (5.1)(b)] and [BD, Proposition 1.9, p. 72]). Hence an affirmative answer to Question (3.4) would solve the standard ideal problem for regulated weights. Of course a negative answer to Question (3.4) would also be a negative answer to the standard ideal problem.

4. Weak* continuous homomorphisms

As in the previous section, we let $\phi : L^1(\omega) \to L^1(\omega')$ be a continuous nonzero homomorphism, where ω and ω' are algebra weights. In this section we give sufficient conditions and characterizations of ϕ being weak* continuous. We start with some preliminary results. The following result is essentially a variant of the Krein-Smulian Theorem.

Lemma 4.1. *Let E and F be Banach spaces, and let $T : F^* \to E^*$ be a linear map. Then T is weak* continuous if weak*-lim $T(\lambda_n) = T(\lambda)$ whenever $\{\lambda_n\}$ is a bounded net with weak* limit λ. When F is separable, it is enough to consider only bounded sequences.*

Proof. It follows from the Krein-Smulian Theorem [DS, Theorem V.5.7, p. 429] that it is enough to show that the restriction of T to closed balls is weak* continuous. But this translates to the statement about bounded nets in the theorem. When F is separable, then the weak* topology on closed balls of F^* is metrizable [DS, Theorem V.5.1, p. 426]. So one only needs to consider sequences to prove continuity.

As one application of the above lemma we show that if ϕ is weak* continuous, then so is its extension to the corresponding measure algebras, just as with norm continuity. □

Lemma 4.2. *If $\phi : L^1(\omega) \to L^1(\omega')$ is a nonzero weak* continuous homomorphism, then so is its extension to a homomorphism from $M(\omega)$to $M(\omega')$.*

Proof. Let $\{\lambda_n\}$ be a bounded sequence (or net) in $M(\omega)$ with weak* limit λ. Let f be a nonzero element of $L^1(\omega)$ with $\phi(f) \neq 0$. Then, by Theorem (2.1), $\lambda_n * f$ converges weak* to $\lambda * f$ in $L^1(\omega)$. By the weak* continuity of ϕ on $L^1(\omega)$, this means that $\phi(\lambda_n * f) = \phi(\lambda_n) * \phi(f)$ converges weak* to $\phi(\lambda) * \phi(f)$. Since a weak* continuous map, like ϕ, must also be norm continuous, the sequence $\{\phi(\lambda_n)\}$ is bounded. It then follows, from Theorem (2.1) again, that weak*-$\lim \phi(\lambda_n) = \phi(\lambda)$. By Lemma (4.1), this implies that $\phi : M(\omega) \to M(\omega')$ is weak* continuous, and thus completes the proof. □

We will need the following simple result in both this and the next section.

Lemma 4.3. *Suppose that* $\{\lambda_n\}$ *is a net in* $M(\omega)$. *If* $\lim \alpha(\lambda_n) = \infty$, *then we have*

(a) $|\lambda_n|([0, a)) \to 0$ *for all* $a > 0$.

(b) *If* $\{\lambda_n\}$ *is bounded, then weak**-$\lim \lambda_n = 0$.

Proof. Part (a) is clear, since $|\lambda_n|([0, a)) = 0$ for all sufficiently large n. Similarly, if h is a continuous function with compact support, $\lim\langle \lambda_n, h\rangle = 0$. When $\{\lambda_n\}$ is bounded, the weak*-convergence of $\{\lambda_n\}$ then follows from the remarks after the proof of Theorem (2.1). □

The convergence in part (a) is much stronger than weak* convergence. For instance, $\delta_{1/n} - \delta_0$ converges to 0 weak* in every $M(\omega)$, but $|\delta_{1/n} - \delta_0|([0, a)) = 2$ for every $a > 1$.

We now give two different sufficient conditions for weak* continuity of $\phi : L^1(\omega) \to L^1(\omega')$. These results are in part motivated by Theorem (3.3) above, which essentially says that weak* continuous homomorphisms are weak* standard. First we give a simple proof of our earlier result [GGM, Theorem (3.5), p. 285], which says that ϕ is weak* continuous if ω is regulated at any $b \geq 0$. In this case Theorem (3.3) does not improve on Theorem (3.1) because, when ω is regulated, if $L^1(\omega) * f$ is weak* dense, then it must also be norm dense [GG3, Theorem (5.1)(b)].

Theorem 4.4. *Suppose that* ω *and* ω' *are algebra weights. If* ω *is regulated at some* $b \geq 0$, *then every continuous homomorphism* $\phi : L^1(\omega) \to L^1(\omega')$ *is weak** *continuous.*

Proof. Without loss of generality, we can assume that ϕ is not the zero homomorphism, so that we can apply Lemma (4.1). Let $\{\lambda_n\}$ be a bounded sequence (or net) in $M(\omega)$ with weak*-$\lim \lambda_n = \lambda$. Choose g in $L^1(\omega)$ with $\phi(g) \neq 0$ and $\alpha(g) \geq b$ (for instance, choose any f with $\phi(f) \neq 0$ and let $g = \delta_b * f$). By Theorem (2.2), $\lambda_n * g \to \lambda * g$ in the norm of $L^1(\omega)$. Since ϕ is norm continuous, this implies that $\phi(\lambda_n * g) = \phi(\lambda_n) * \phi(g)$ converges to $\lambda * g$ in norm and hence weak*. It then follows from Theorem (2.1) that $\phi(\lambda_n) \to \phi(\lambda)$ weak*. The theorem now follows from Lemma (4.1). □

For our other sufficient condition for weak* continuity, we will need to recall some terminology and results from [Gr1]. Suppose that $\phi : L^1(\omega) \to L^1(\omega')$ is a nonzero homomorphism, and let $\mu_t = \phi(\delta_t)$. Then there is a nonnegative number A for which $\alpha(\mu_t) = At$ [Gr1, Theorem (4.3)(a), p. 605], [GhM, Lemma 1, p. 344]. We call A the *character* of ϕ, and of $\{\mu_t\}$. When the character A is strictly positive, one also has [Gr1, Theorem 4.9, p. 607] $\alpha(\phi(\lambda)) = A\alpha(\lambda)$ for all λ in $M(\omega)$.

In our proof of the weak* continuity of homomorphisms of positive character, we will use Theorem (2.1)(d). The following notation will be convenient for this purpose. For each $a > 0$, we define the seminorm $||f||_a = \int_0^a |f(t)|dt$ on L^1_{loc}, the space of locally integrable functions on $\mathbf{R}^+$. Thus Theorem (2.1)(d) says that $\lambda_n * f$ converges to $\lambda * f$ in each of these seminorms. Similarly, Lemma (4.3)(a) says that

$\lambda_n \to 0$ in the analogous seminorms on $M_{loc}(\mathbf{R}^+)$, the space of locally finite Borel measures on $\mathbf{R}^+ = [0, \infty)$. We can now prove:

Theorem 4.5. *Let $\phi : L^1(\omega) \to L^1(\omega')$ be a continuous nonzero homomorphism. If the character A of ϕ is strictly positive, then ϕ is weak* continuous.*

Proof. We first show that for each $a > 0$ there is an $M = M(a) > 0$ for which

$$||\phi(f)||_{Aa} \leq M||f||_a \tag{4.1}$$

for each f in $L^1(\omega)$. Let $L^1(\omega)_a = \{f \in L^1(\omega) : \alpha(f) \geq a\} = \{f \in L^1(\omega) : ||f||_a = 0\}$, and define $L^1(\omega')_{Aa}$ analogously. Since $\phi(L^1(\omega)_a) \subseteq L^1(\omega')_{Aa}$, it follows that ϕ induces a continuous map from the quotient Banach algebra $L^1(\omega)/L^1(\omega)_a$ to the quotient algebra $L^1(\omega')/L^1(\omega')_{Aa}$. Since ω is bounded and bounded below on $[0, a)$, the quotient norm is equivalent to the norm induced by the seminorm $f \mapsto ||f||_a$, and similarly for the norm induced by the seminorm $g \mapsto ||g||_{Aa}$ on the quotient of $L^1(\omega')$. Formula (4.1) is now just the statement that the map induced by ϕ between the quotient spaces is bounded.

Now suppose that λ_n is a bounded sequence or net in $M(\omega)$ that converges weak* to λ. Choose $f \in L^1(\omega)$ with $\phi(f) \neq 0$ (actually ϕ has kernel $\{0\}$ [Gr1, Appendix, p. 613]). By Theorem (2.1)(d), each $||\lambda_n * f - \lambda * f||_a$ converges to 0. By formula (4.1), this implies that each $||\phi(\lambda_n) * \phi(f) - \phi(\lambda) * \phi(f)||_b$ converges to 0. By Theorem (2.1), this means $\phi(\lambda_n) \to \phi(\lambda)$ weak* in $M(\omega')$. By Lemma (4.1) this implies that ϕ is weak* continuous, and therefore completes the proof of the theorem. □

The algebra $L^1_{loc}(\mathbf{R}^+)$ of locally integrable functions on $\mathbf{R}^+$ is a Fréchet algebra under the seminorms $||\cdot||_a$ for $a > 0$. Thus Theorem (2.1)(d) says that $\lambda_n * f$ converges to $\lambda * f$ in the Fréchet topology on $L^1_{loc}(\mathbf{R}^+)$, and formula (4.1) says that ϕ is continuous in this topology, relativized by $L^1(\omega)$ and $L^1(\omega')$. The algebra $L^1_{loc}(\mathbf{R}^+)$, and particularly its automorphisms and derivations, is studied by Ghahramani and McClure in [GhM]. In a paper in preparation I will study the continuous homomorphisms of $L^1_{loc}(\mathbf{R}^+)$.

Homomorphisms of positive character seem to be better behaved than homomorphisms of character 0. Thus for positive character A we have $\alpha(\phi(\mu)) = A\alpha(\mu)$, which implies that ϕ is one-to-one [Gr1, Appendix, p. 613]. By the previous theorem and its proof we also know that ϕ is weak* continuous (so that Theorem (3.3) applies) and is continuous in the (relativized) Fréchet topology on $L^1_{loc}(\mathbf{R}^+)$. While much is known for character 0 [Gr1], the following natural question is open (for partial results see [Gr1, Theorem 4.11, p. 608]).

Question 4.6. *If $\phi : L^1(\omega) \to L^1(\omega')$ has character 0, must $\alpha(\phi(\mu)) = 0$ for all μ in $M(\omega)$?*

If the answer to Question (4.6) is yes, then ϕ would be one-to-one, and hence all continuous nonzero homomorphisms $\phi : L^1(\omega) \to L^1(\omega')$ would be one-to-one. For a discussion of known results on when ϕ is one-to-one, see [Gr3, Section 5].

We now give a relatively simple characterization of weak* continuity of homomorphisms.

Theorem 4.7. *Suppose that ω and ω' are algebra weights and that $\phi : L^1(\omega) \to L^1(\omega')$ is continuous, and let $\mu_t = \phi(\delta_t)$. Then ϕ is weak* continuous if and only if weak*-$\lim_{x\to\infty} \mu_x/\omega(x) = 0$.*

We first separate out the direction that assumes that ϕ is weak* continuous, and we determine the pre-adjoint of ϕ in this case.

Theorem 4.8. *Suppose that $\phi : L^1(\omega) \to L^1(\omega')$ is a weak*-continuous nonzero homomorphism. Then we have*

(a) *weak*-$\lim_{x\to\infty} \mu_x/\omega(x) = 0$;*
(b) *ϕ is the adjoint of the map $T : C_0(1/\omega') \to C_0(1/\omega)$ given by $Th(x) = \langle \mu_x, h\rangle$.*

Proof. We first observe that we always have weak*-$\lim_{x\to\infty} \delta_x/\omega(x) = 0$ in $M(\omega)$. For suppose that h belongs to the predual $C_0(1/\omega)$. Then

$$\langle \delta_x/\omega(x), h\rangle = h(x)/\omega(x) \to 0 \quad \text{as } x \to \infty,$$

by the definition of $C_0(1/\omega)$. Now when ϕ is weak* continuous we therefore have that $\phi(\delta_x/\omega(x)) = \mu_x/\omega(x)$ has weak* limit 0 as x goes to ∞. This proves (a).

Since ϕ is weak* continuous, there is some bounded linear map $L : C_0(1/\omega') \to C_0(1/\omega)$ with $\phi = L^*$. We just need to show that L equals T as defined in (b). Choose an h in $C_0(1/\omega')$. Then for all $x \geq 0$ we have

$$Lh(x) = \langle \delta_x, Lh\rangle = \langle L^*\delta_x, h\rangle = \langle \phi(\delta_x), h\rangle = Th(x),$$

as required. This completes the proof of Theorem (4.8). □

Proof of Theorem (4.7). Suppose that $\mu_x/\omega(x)$ approaches 0 weak* as $x \to \infty$. We need to show that ϕ is weak* continuous. We do this by first showing that the map T of Theorem (4.8)(b) is a bounded linear map, and then showing that $\phi = T^*$.

For h in $C_0(1/\omega')$ we define the function Th on $\mathbf{R}^+$ by $Th(x) = \langle \mu_x, h\rangle$. Since μ_x is weak* continuous [Gr1, Theorem 3.6(A), p. 599], Th is a continuous function. Since $Th(x)/\omega(x) = \langle \mu_x/\omega(x), h\rangle$, we have Th in $C_0(1/\omega)$ by our assumption that $\mu_x/\omega(x) \to 0$ weak*. We now need to show that the linear map $T : C_0(1/\omega') \to C_0(1/\omega)$ is bounded. Since all $\delta_x/\omega(x)$ are unit vectors in $L^1(\omega)$, we have, for each $x \geq 0$, that

$$|Th(x)/\omega(x)| \leq \frac{||\mu_x||}{\omega(x)}||h|| \leq ||\phi||\ ||h||,$$

where the norm of h is taken in $C_0(1/\omega')$. Thus

$$||Th|| = \sup |Th(x)/\omega(x)| \leq ||\phi||\ ||h||,$$

so that T is bounded.

Now, for each f in $L^1(\omega)$, we have $\phi(f) = \int_0^\infty f(t)\mu_t dt$ as a weak* integral on $L^1(\omega')$ [Gr1, formula (3.7), p. 599]. This means that for each h in $C_0(1/\omega')$, we have

$$\langle \phi f, h\rangle = \int_0^\infty f(t)\langle \mu_t, h\rangle dt = \int_0^\infty f(t)Th(t) = \langle f, Th\rangle = \langle T^*f, h\rangle.$$

Thus $\phi = T^*$; so ϕ is weak* continuous. This completes the proof of Theorem (4.7). □

Verifying the condition that $\mu_x/\omega(x) \to 0$ weak* should usually be easier to do than directly proving that ϕ is weak* continuous. For instance, if $\phi : L^1(\omega) \to L^1(\omega')$ is a continuous homomorphism with positive character, then $\mu_x/\omega(x)$ is a bounded net with $\lim_{x\to\infty} \alpha(\mu_x/\omega(x)) = 0$. It then follows from Lemma (4.3) that $\mu_x/\omega(x) \to 0$ weak* as x goes to ∞.

5. Normalized powers

There have been several papers which have considered the sequences $\omega_n = ||f^n||$ of norms of powers and the sequence $f^n/||f^n||$ of normalized powers of elements f of radical Banach algebras. See [A], [W], [S], [LRRW]. There are two extreme cases [LRRW, Corollary 2.5]. Solovej [S] showed that for f in the Volterra algebra $L^1[0,1)$ with $\alpha(f) = 0$, we always have $\lim \omega_{n+1}/\omega_n = 0$, so that the sequence $\{\omega_n\}$ is regulated at 1 in the sense of [BDL]. Loy, et al. [LRRW] construct and study f for which $f^n/||f^n||$ has a subsequence that is a bounded approximate identity. This is the key part of their construction of a weakly amenable commutative radical Banach algebra. We show that the situation in $L^1(\omega)$ is closer to the first extreme. We always have weak*-lim$(f^n/||f^n||) = 0$, and when $\omega(t)$ is regulated we also have $\lim(f^{n+1}/||f^n||) = 0$.

Theorem 5.1. *Suppose that ω is an algebra weight. For all nonzero f in $L^1(\omega)$, the sequence $f^n/||f^n||$ converges to 0 in the weak* topology of $L^1(\omega)$.*

Proof. To simplify the notation, we let $g_n = f^n/||f^n||$. If $\alpha(f) > 0$, then $\lim \alpha(g_n) = 0$; so $g_n \to 0$ weak* by Lemma (4.3)(b). Now suppose that $\alpha(f) = 0$. If $\omega(x) \geq C$ on $[0, a)$, then $||f^n||_\omega \geq C||f^n||_a$. Hence it follows easily from Solovej's result [S], and its obvious generalization to all $L^1[0, a)$ that $||g_n * f||_a = \int_0^a |g_n * f(t)|dt$ converges to 0 for all $a > 0$. Then $g_n \to 0$ weak*, by Theorem (2.1).

A direct application of Theorem (2.2) yields the following corollary. □

Corollary 5.2. *Suppose that $\omega(t)$ is an algebra weight that is regulated at $b \geq 0$. For all f in $L^1(\omega)$ with $\alpha(f) \geq b$, we have* $\lim \left(\frac{f^{n+1}}{||f^n||}\right) = 0$.

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