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THE POLYDISK NULLSTELLENSATZ

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Abstract. The Nullstellensatz for zeros in a closed polydisk of $\mathbf{C}^n$ is proved constructively with no appeal to choice axioms.

Let $\mathbf{C}$ be the complex numbers, K a subset of $\mathbf{C}^n$, and I a finitely generated ideal in the polynomial ring $\mathbf{C}[X_1, \ldots, X_n]$. Consider the following two conditions:

(1) for all $x \in K$ there exists $f \in I$ such that $f(x) \neq 0$;
(2) there exists $f \in I$ such that $f(x) \neq 0$ for all $x \in K$.

Obviously (2) implies (1). Let $N(K)$ be the proposition that (1) implies (2). The classical Nullstellensatz is $N(\mathbf{C}^n)$ because it is easy to show that a polynomial that vanishes nowhere on $\mathbf{C}^n$ must be a nonzero constant. We will show that $N(K)$ holds for K an arbitrary closed polydisk. There seems to be no obvious way to derive the case $K = \mathbf{C}^n$ from the polydisk cases, or vice versa.

For constructive purposes, conditions (1) and (2) are modified slightly when K is a closed polydisk.

Theorem 1 (the polydisk Nullstellensatz)**.** *Let $f_1, \ldots, f_m$ be generators of the ideal I in $\mathbf{C}[X_1, \ldots, X_n]$, and let K be a closed polydisk in $\mathbf{C}^n$. Then the following conditions are equivalent:*

- *There exists $\delta > 0$ such that $|f_1(x)| \vee \cdots \vee |f_m(x)| \geq \delta$ for all $x \in K$.*
- *There exist $f \in I$ and $\delta > 0$ such that $|f(x)| \geq \delta$ for all $x \in K$.*
- *There exist polynomials $f \in I$ that are arbitrarily close to 1 on K.*

Since K is compact, the first two conditions are readily seen to be classically equivalent to conditions (1) and (2) above. It is still easy to derive the first condition from the second. The third condition certainly implies the second. So the only issue is whether the first condition implies the third. That is what we will prove.

The positive contrapositive of Theorem 1 is an immediate consequence of Theorem 1:

Corollary 2. *Let $f_1, \ldots, f_m$ be generators of the ideal I in $\mathbf{C}[X_1, \ldots, X_n]$, and let K be a closed polydisk in $\mathbf{C}^n$. Then the following conditions are equivalent:*

- $\inf_{x \in K} (|f_1(x)| \vee \cdots \vee |f_m(x)|) = 0$*;*
- $\inf_{x \in K} |f(x)| = 0$ *for all $f \in I$;*
- 1 *is bounded away from I in the supremum norm on K.*

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Proof. Clearly the first condition implies the second. The second implies the third because $\sup_{x\in K}|f(x)-1| \geq 1-\inf_{x\in K}|f(x)|$. To prove that the third implies the first, suppose 1 is bounded away from I in the supremum norm on K. By Theorem 1, the number

$$\inf_{x\in K}\left(|f_1(x)| \vee \cdots \vee |f_m(x)|\right)$$

cannot be greater than 0; so it must be equal to zero. □

Since K is compact, the first condition of Corollary 2 says (classically) that the f_i have a common zero in K, and therefore there is a common zero of all the polynomials in I. What separates Corollary 2 from the Nullstellensatz is the lack of an easy classical proof that if I is proper, then 1 is bounded away from I on some closed polydisk. Similarly, what separates Theorem 1 from the Nullstellensatz is the lack of an easy classical proof that if, on each polydisk, there exist polynomials $f \in I$ that are arbitrarily close to 1, then $1 \in I$. Of course, neither of these easy classical proofs would be allowed to appeal to the Nullstellensatz!

In this paper, we will prove Theorem 1 constructively by using the central result from the chapter on commutative Banach algebras in [1]. The theory of approximate characters developed in that chapter depends on choice principles and exploits an enumeration of a dense subset of the Banach algebra. In the key Lemma 2.5, there is an appeal to (a result closely related to) the Hahn-Banach theorem to construct an approximate character. The approximate characters themselves are organized into a sequence of compact sets Σ_n via Theorem 4.9 of Chapter 4 of [1] which asserts that certain subsets of a compact space are compact for all but a countable number of values of a parameter, a theorem whose proof relies heavily on choice principles. Of the theory developed in the chapter on Banach algebras in [1], we will use only the crucial Theorem 2.1, which does not require the Hahn-Banach theorem, the enumeration of a dense subset, or the use of any choice principles.

Mortini and Rupp [3], building on von Renteln [4], give a proof of the one-variable Nullstellensatz for certain subalgebras of analytic functions on a compact set. This somewhat subsumes our polynomial result in the one-variable case, although they do not work within as strict a constructive framework as we do. Gunning and Rossi [2] give a classical proof of a local version of the Nullstellensatz for analytic functions of several variables. Their theorem deals with the behavior of convergent power series in a neighborhood of a point and does not address the essentially global issues involved in the polydisk Nullstellensatz.

1. Norms on polynomial rings

Let R be a normed ring and $r > 0$. We can norm $R[X]$ by setting $\left\|\sum a_i X^i\right\|_r = \sum |a_i|\, r^i$. Then

$$\|f+g\|_r = \sum |a_i + b_i|\, r^i \leq \sum |a_i|\, r^i + \sum |b_i|\, r^i = \|f\|_r + \|g\|_r$$

and

$$\|fg\|_r = \sum_n \left| \sum_{i+j=n} a_i b_j \right| r^n \leq \sum_n \sum_{i+j=n} |a_i|\, |b_j|\, r^n = \|f\|_r \, \|g\|_r \,.$$

For n variables, let r be the multireal $r_1, \ldots, r_n$ and let X be $X_1, \ldots, X_n$. Then $\|f\|_r = \sum |a_i|\, r^i$ as before, where i is a multi-index. We can do this inductively by considering $R[X_1, \ldots, X_{n-1}][X_n]$. When you complete in this norm, you get those

power series that converge absolutely on the closed polydisk $\overline{D}(0,r) = \{x : |x_i| \leq r_i$ for all $i\}$. For our purposes, we might as well take all the r_i equal, and call that r. We do that.

Now consider $\mathbf{C}[X_1, \ldots, X_n]$. Let $|f|_r$ be the supremum norm on the polydisk $\overline{D}(0,r)$ and let $\|f\|_r$ be as above. These norms are all equivalent on polynomials of total degree at most d.

Theorem 3. *For each polynomial f in $\mathbf{C}[X_1, \ldots, X_n]$ of total degree at most d,*

- $\|f\|_r \leq \|f\|_s \leq (s/r)^d \|f\|_r$ *if* $r \leq s$;
- $|f|_r \leq \|f\|_r \leq \left(\dfrac{s}{s-r}\right)^n |f|_s$ *if* $r < s$.

So all these norms are equivalent on the set of polynomials of total degree at most d.

Proof. All the inequalities are easy except the last. It follows from the n-variable Cauchy integral formula

$$a_m = \frac{f^{(m)}(0)}{m!} = \left(\frac{1}{2\pi i}\right)^n \int \frac{f(\xi)}{\xi^{m+1}} d\xi$$

for the coefficients of f, which shows that $|a_m| r^m \leq |f|_s (r/s)^m$. Now we have to sum $(r/s)^m$ over multi-indexes m. There are $\binom{n+k-1}{n-1}$ multi-indexes for each total index k, and $\sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k = (1-x)^{-n}$. □

The completion of $\mathbf{C}[X_1, \ldots, X_n]$ with respect to the norm $\|f\|_r$ consists of the power series that converge absolutely on the polydisk $\overline{D}(0,r)$. The completion of $\mathbf{C}[X_1, \ldots, X_n]$ with respect to the norm $|f|_r$ consists of the uniformly continuous functions on $\overline{D}(0,r)$ that are analytic in $D(0,r)$.

2. Bishop's Theorem 2.1

This is the crucial theorem in Bishop's theory of commutative Banach algebras in [1]. Its proof does not use choice principles, unlike many of the results that follow it. Bishop's Corollary 2.2 is an extension, and an immediate consequence, of his Theorem 2.1. The following special case of this corollary will be used to prove our Theorem 1.

Theorem 4. *Let $A = \mathbf{C}[X_1, \ldots, X_n]$, let $\widehat{A}$ be its completion under the norm $\|f\|_r$, and let $f_1, \ldots, f_m \in A$. If*

- *there exists $C > 0$ such that for each $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$, there exist $a_1, \ldots, a_m$, $b_1, \ldots, b_n \in \widehat{A}$, each of norm at most C, such that*

$$\sum_{i=1}^{m} a_i f_i + \sum_{j=1}^{n} b_j (X_j - \lambda_j) = 1,$$

then there exists $C' > 0$, depending only on C, m, n, and r, and there exist elements $a'_1, \ldots, a'_n \in \widehat{A}$, each of norm at most C', such that $\sum_{i=1}^{m} a'_i f_i = 1$.

To see the relevance of Theorem 4 to the Nullstellensatz, consider the latter's classical contrapositive: if $f_1, \ldots, f_m$ generate a proper ideal in $\widehat{A}$, then there exist $\lambda_1, \ldots, \lambda_n$ such that the ideal generated by $f_1, \ldots, f_m, X_1 - \lambda_1, \ldots, X_n - \lambda_n$ is proper. So $f_1, \ldots, f_m$ are in the (maximal) ideal generated by $X_1 - \lambda_1, \ldots, X_n - \lambda_n$, whence $f_i(\lambda_1, \ldots, \lambda_n) = 0$ for all i. As stated in [1],

In the classical theory of Banach algebras one considers arbitrary closed ideals of $\mathfrak{R}$, and extends them to maximal ideals. Instead, we shall work with finite linear combinations of given elements of $\mathfrak{R}$.

We will show that the hypothesis of Theorem 4 is equivalent to:

- There exists $\delta > 0$ such that for all $\lambda \in \overline{D}(0, r)$,

$$|f_1(\lambda)| \vee \cdots \vee |f_m(\lambda)| \geq \delta,$$

which is the first condition of Theorem 1. The third condition of Theorem 1, that we can find polynomials p_i such that $\sum p_i f_i$ is close to 1 on the polydisk $\overline{D}(0, r)$, follows from Theorem 4 upon approximating the analytic functions a_i' by polynomials.

We start by finding a bound on the quotient when dividing a polynomial in $R[X]$ by $X - \lambda$ using the division algorithm.

Lemma 5. *Let d be a positive integer, and t and r positive real numbers. Then there exists a positive real number K_1 such that, for any normed ring R, if*

(1) *$f \in R[X]$ has degree at most d,*
(2) *$\lambda \in R$ has norm at most t, and*
(3) *$f(X) = q(X)(X - \lambda) + f(\lambda)$ (which we get from the division algorithm),*

then $\|q\|_r \leq K_1 \|f\|_r$ and $|f(\lambda)| \leq K_1 \|f\|_r$.

Proof. Let $f(X) = a_d X^d + \cdots + a_1 X + a_0$ and $q(X) = q_{d-1} X^{d-1} + \cdots + q_1 X + q_0$. Looking at the coefficient of X^{d-i+1}, for $i = 1, \ldots, d$, on each side of (3), we see that $a_{d-i+1} = q_{d-i} - \lambda q_{d-i+1}$ and so

$$q_{d-i} = \sum_{j=0}^{i-1} a_{d-i+j+1} \lambda^j$$

for $i = 1, \ldots, d$. So

$$\begin{aligned} \|q\|_1 = \sum_{i=1}^{d} |q_{d-i}| &\leq \sum_{j<i\leq d} |a_{d-i+j+1}| \, t^j \\ &\leq (1 \vee t)^{d-1} \sum_{j<i\leq d} |a_{d-i+j+1}| \leq d \|f\|_1 (1 \vee t)^{d-1}. \end{aligned}$$

For $r \leq 1$,

$$\|q\|_r \leq \|q\|_1 \leq d \|f\|_1 (1 \vee t)^{d-1} \leq r^{-d} d \|f\|_r (1 \vee t)^{d-1}.$$

For $r \geq 1$,

$$\|q\|_r \leq r^d \|q\|_1 \leq r^d d \|f\|_1 (1 \vee t)^{d-1} \leq r^d d \|f\|_r (1 \vee t)^{d-1}.$$

So, for $r \in (-\infty, 1] \cup [1, \infty)$,

$$\|q\|_r \leq (r \vee r^{-1})^d (1 \vee t)^{d-1} d \|f\|_r .$$

We leave to the interested reader the constructive proof that a weak inequality (or any negative statement) that holds for all $r \in (-\infty, 1] \cup [1, \infty)$ holds for arbitrary r. In this case, the result follows naturally because $\|q\|_r$ and $\|f\|_r$ are uniformly continuous in r on any finite interval.

To bound $|f(\lambda)|$, note that $|f(\lambda)| \leq |f|_t \leq \|f\|_t$. The latter is at most $\|f\|_r$ if $t \leq r$, and at most $(t/r)^d \|f\|_r$ if $t \geq r$. So $|f(\lambda)| \leq (1 \vee t/r)^d \|f\|_r$. □

We need the several variables version of this lemma. Using the division algorithm and induction, one can show that if R is any ring whatsoever, and $f \in R[X_1, \ldots, X_n]$, then for any $\lambda_1, \ldots, \lambda_n \in R$, there exist unique polynomials $q_i \in R[X_1, \ldots, X_i]$ for $i = 1, \ldots, n$, such that

$$f(X_1, \ldots, X_n) = \sum_{i=1}^{n} q_i(X_1, \ldots, X_i)(X_i - \lambda_i) + f(\lambda_1, \ldots, \lambda_n).$$

Now we bound the q_i.

Theorem 6. *Let d and n be positive integers, and t and r positive real numbers. Then there exists a real number $K \geq 1$ such that, for any normed ring R, if*

- *$f \in R[X_1, \ldots, X_n]$ has total degree at most d,*
- *$\lambda_1, \ldots, \lambda_n \in R$ have norms at most t, and*
- *$f(X_1, \ldots, X_n) = \sum_{i=1}^{n} q_i(X_1, \ldots, X_i)(X_i - \lambda_i) + f(\lambda_1, \ldots, \lambda_n)$,*

then $\|q_i\|_r \leq K \|f\|_r$ for all i, and $|f(\lambda)| \leq K \|f\|_r$.

Proof. Consider $R' = R[X_1, \ldots, X_{n-1}]$ with the r-norm. Dividing $f(X_1, \ldots, X_n)$ by $X_n - \lambda_n$ in $R'[X_n]$ gives

$$f(X_1, \ldots, X_n) = q_n(X_1, \ldots, X_n)(X_n - \lambda_n) + f(X_1, \ldots, X_{n-1}, \lambda_n).$$

From Lemma 5 there is K_1 so that

$$\|q_n(X_1, \ldots, X_n)\|_r \leq K_1 \|f\|_r$$

and

$$\|f(X_1, \ldots, X_{n-1}, \lambda_n)\|_r \leq K_1 \|f\|_r .$$

Repeat with f replaced by $f(X_1, \ldots, X_{n-1}, \lambda_n)$. By induction we get $\|q_{n-i}\|_r \leq K_1^{i+1} \|f\|_r$ and

$$\|f(X_1, \ldots, X_{n-i-1}, \lambda_{n-i}, \ldots, \lambda_n)\|_r \leq K_1^{i+1} \|f\|_r .$$

So we can take $K = 1 \vee K_1^n$. □

Now we prove that the hypotheses of Theorems 1 and 4 are equivalent.

Theorem 7. *Let R be a normed ring and $r > 0$. Consider $A = R[X_1, \ldots, X_n]$ with the norm $\|f\|_r$. For $f_1, \ldots, f_m \in A$, the following conditions are equivalent:*

(1) *There is $C > 0$ such that for each $\lambda_1, \ldots, \lambda_n \in R$,*

$$\sum_{i=1}^{m} a_i f_i + \sum_{j=1}^{n} b_j (X_j - \lambda_j) = 1$$

for some $a_1, \ldots, a_m, b_1, \ldots, b_n \in \widehat{A}$, each of norm at most C.

(2) *There is $\delta > 0$ such that if each coordinate of $\lambda \in R^n$ has norm at most r, then $|f_i(\lambda)| \geq \delta$ for some i.*

Proof. Suppose (2) holds. Then, for some smaller δ, (2) also holds with r replaced by some $t > r$. Let $d \in \mathbf{N}$ exceed the formal total degree of each f_i and let K be as in Theorem 6 for these values of d, n, t and r. Let $B = 1 \vee \sup_i \|f_i\|_r$. We may assume that $\delta \leq 1$. Choose s so that $t > s > r$, and let

$$C = \frac{1}{s - r} \vee \frac{KB}{\delta}.$$

Given $\lambda \in R^n$, either $|\lambda_j| > s$ for some j or $|\lambda_j| < t$ for all j. In the case $|\lambda_j| > s$, let all the a's and b's be zero except for $b_j = -\lambda_j^{-1}(1 + X_j/\lambda_j + X_j^2/\lambda_j^2 + \cdots)$. Note that $\|X_j\|_r = r$; so $\|b_j\|_r < 1/(s-r) \leq C$. Clearly $b_j(X_j - \lambda_j) = 1$.

In the case $|\lambda_j| < t$ for all j, let i be such that $|f_i(\lambda)| \geq \delta$. Let $a_k = 0$ for $k \neq i$, set $a_i = 1/f_i(\lambda)$, and set $b_k = -a_i q_k(X_1, \ldots, X_k)$ where

$$f_i(X_1, \ldots, X_n) = \sum_{k=1}^{n} q_k(X_1, \ldots, X_k)(X_k - \lambda_k) + f_i(\lambda).$$

Now

$$a_i f_i + \sum_{k=1}^{n} b_k(X_k - \lambda_k) = a_i(f_i - \sum_{k=1}^{n} q_k(X_k - \lambda_k)) = 1.$$

Since $a_i = 1/f_i(\lambda_1, \ldots, \lambda_n)$, we have $|a_i| \leq 1/\delta \leq C$. Since $\|q_k\|_r \leq K\|f_i\|_r$ we have

$$\|b_k\|_r = |a_i|\,\|q_k\|_r \leq \frac{KB}{\delta} \leq C.$$

For the converse, pick $\delta < 1/(mC)$. If $|\lambda_j| \leq r$ for each j, then $|a_i(\lambda)| \leq \|a_i\|_r \leq C$. If (1) holds, then $\sum a_i(\lambda) f_i(\lambda) = 1$. So $mC \sup_i |f_i(\lambda)| \geq 1$, whence $|f_i(\lambda)| \geq \delta$ for some i. □

As a corollary we get Theorem 1, the polydisk Nullstellensatz.

Corollary 8. *Let $f_1, \ldots, f_m$ be polynomials in $\mathbf{C}[X_1, \ldots, X_n]$, and let I be the ideal they generate. Then for each $r > 0$, the following conditions are equivalent:*

(1) $\inf_{x \in D(0,r)} \sup_i |f_i(x)| > 0$;
(2) $\inf_{f \in I} \|f - 1\|_r = 0$.

Proof. If (2) holds, then there exists $f = \sum \alpha_i f_i$ that is bounded away from 0 by δ on $\overline{D}(0,r)$. Let B exceed the supremum of the sum of the $|\alpha_i|$ on $\overline{D}(0,r)$. Then $\sup_i |f_i(x)| \geq \delta/B$ on $D(0,r)$. Conversely, if (1) holds, then condition (2), and hence condition (1), of Theorem 7 holds. Theorem 4 then says that there exist functions a_i' in the completion of $\mathbf{C}[X_1, \ldots, X_n]$ such that $\sum_i a_i' f_i = 1$, hence, approximating a_i' by polynomials, elements of I that are as close as we please to 1. □

The first condition of Corollary 8 says that the f_i have no common zero on the polydisk of radius r. The second condition says that 1 is in the closure of I. The next corollary, which follows easily, is as close as we come to the Nullstellensatz.

Corollary 9. *Let $f_1, \ldots, f_m$ be polynomials in $\mathbf{C}[X_1, \ldots, X_n]$, and let I be the ideal they generate. Then the following two conditions are equivalent:*

(1) $\inf_{x \in D(0,r)} \sup_i |f_i(x)| > 0$ *for all r;*
(2) $\inf_{f \in I} \|f - 1\|_r = 0$ *for all r.*

The following two conditions are also equivalent:

(3) $\inf_{x \in D(0,r)} \sup_i |f_i(x)| = 0$ *for some r;*
(4) $\inf_{f \in I} \|f - 1\|_r > 0$ *for some r.*

Condition (1) of Corollary 9 says that the f_i have no common zero. Condition (2) says that 1 is in a strong closure of I. To get the (classical) Nullstellensatz from this, we have to pass from 1 being in that strong closure of I to $1 \in I$. The equivalence of conditions (3) and (4) is the positive contrapositive of the first equivalence.

Condition (3) says (classically) that the f_i have a common zero. Condition (4) says that 1 is bounded away from I on some polydisk. To get the (classical) Nullstellensatz from this, we have to pass from I being proper to 1 being bounded away from I on some polydisk.

3. An example

We cannot strengthen the conclusion of Theorem 1 to say that $1 \in I$. Consider the following example in the one-variable case. Let I be the ideal generated by the polynomials $bX-1$ and c, where b and c are complex numbers with the property that if $b \neq 0$, then $c \neq 0$. The ideal I clearly satisfies the three equivalent conditions of Theorem 1 for each closed polydisk K. However, we cannot conclude constructively that $1 \in I$, as the following lemma and theorem show.

Lemma 10. *If $g(X)(bX-1)+h(X)c=1$, and $n > \deg g$, then c divides b^n.*

Proof. Let $g(X) = \sum a_i X^i$, and note that $g(X)(bX-1) \equiv 1 \bmod c$. Thus $a_0 \equiv -1 \bmod c$ and $a_{i+1} \equiv a_i b \bmod c$ for $i \geq 0$, whereupon $a_i \equiv -b^i \bmod c$. But $a_n = 0$, and so c divides b^n. □

The next theorem constitutes a Brouwerian counterexample to the zero-variables strong Nullstellensatz. Together with the preceding lemma, it shows that you cannot prove constructively that $1 \in I$.

Theorem 11. *If the condition that c divides b^n for some n follows from the condition that $b \neq 0$ implies $c \neq 0$, then for each $c \geq 0$, either $c > 0$ or $c = 0$.*

Proof. Construct a strictly increasing continuous function λ from the nonnegative real numbers to the nonnegative real numbers such that $\lambda(0) = 0$ and $\lambda(y)^n/y \to \infty$ as $y \to 0$ for each positive integer n. This can be done by letting $y = 1/x^{x+1}$ and defining $\lambda(y)$ to be $1/x$. Let $c \geq 0$ and take $b = \lambda(c)$. Clearly, $b \neq 0$ if and only if $c \neq 0$. We will show that if $b^n = ca$ for some real number a, then $c \neq 0$ or $c = 0$. If $c \neq 0$, then $a = \lambda(c)^n/c$. But $\lambda(y)^n/y \to \infty$. So there exists $\varepsilon > 0$ so that $\lambda(y)^n/y > a$ if $y < \varepsilon$. Thus if $c > 0$, then $c \geq \varepsilon$. But either $c > 0$ or $c < \varepsilon$; so either $c > 0$ or $c = 0$. □

The conclusion of this theorem is equivalent to the statement that the real (or complex) numbers are discrete: for each pair of real numbers x and y, either $x = y$ or $x \neq y$. As is well known, this implies Bishop's Limited Principle of Omniscience (LPO) and the converse holds in the presence of countable choice.

An example along somewhat the same lines is the ideal in $\mathbf{C}[X, Y]$ generated by $XY - 1$ and Y. Of course this ideal contains 1. The observation here is that the values of the two generators can get arbitrarily small: choose y small and nonzero and let $x = y^{-1}$. So we cannot strengthen the first condition of Theorem 1 by making δ independent of K.

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