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SUBGROUP SEPARABILITY OF GRAPHS OF ABELIAN GROUPS

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Abstract. In the present paper we give necessary and sufficient conditions for the subgroup separability of the fundamental group of a finite graph of groups with finitely generated abelian vertex groups.

1. Introduction

A group G is called *subgroup separable* or *locally extended residually finite* (*LERF*) if every finitely generated subgroup of G is closed in the profinite topology, the topology whose open basis consists of the cosets of finite index subgroups of G.

There are other equivalent definitions of subgroup separability. The most commonly used is that G is subgroup separable if for every finitely generated subgroup H of G, H is the intersection of the finite index subgroups of G containing H. Or that for every finitely generated subgroup H of G and every $g \notin H$ there is a normal subgroup N of finite index in G such that $g \notin NH$. This is equivalent to saying that $gN \notin HN/N$ or that $\bigcap_{N \in \mathcal{N}} NH = H$ where $\mathcal{N}$ is the set of all normal subgroups of finite index in G.

Historically, subgroup separability was first shown to be a property of free groups by M. Hall, Jr. in [10]. The geometric motivation for studying subgroup separability of groups was given in [16] and [17]. More specifically, P. Scott proved that subgroup separability allows certain immersions to lift to an embedding in a finite cover. This result is of particular importance in 3-manifold topology. Since then many people have worked on the subject. For the families of groups that are known to satisfy subgroup separability the reader should consult the papers of Gitik [8] and Wise [19] and the references cited there.

On the other hand, there are very few known examples of finitely presented, residually finite, non-subgroup separable groups. The first example in the literature is the group $F_2 \times F_2$, shown by Mihailova in [13] to have an unsolvable generalised word problem, thus being non-subgroup separable.

The second example in the literature is the family BN of all groups with an HNN presentation of the form $\langle t, K \mid tKt^{-1} = A\rangle$ where $A \subsetneq K$. Such an HNN-extension is called ascending. The above groups are not subgroup separable by the

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results of Blass and Neumann in [5]. Notice that the BN groups include the residually finite, non-subgroup separable Baumslag-Solitar groups, namely the groups $BS_{n,m} = \langle x, a \mid xa^n x^{-1} = a^m \rangle$ with either $|n| = 1$ or $|m| = 1$ and $|m| \neq |n|$. Shalen recently showed [18] that non-subgroup separable Baumslag-Solitar groups cannot be subgroups of 3-manifold groups.

The third example in the literature (the first non-subgroup separable 3-manifold group) was discovered by Burns, Karrass and Solitar in [6] and is the group

$$BKS = \langle x, \alpha, \beta \mid x\alpha x^{-1} = \alpha, x\beta x^{-1} = \alpha\beta \rangle.$$

Furthermore, it was shown by Long and Niblo [11] and subsequently by Niblo and Wise [14] that the group BKS contains a subgroup of index two that is isomorphic to a subgroup of the group

$$\begin{aligned} L &= \langle x, y, \alpha, \beta \mid [x, \alpha] = [y, \beta] = [\alpha, \beta] = 1 \rangle \\ &\cong \langle \alpha, x \mid [\alpha, x] = 1 \rangle \underset{\alpha}{*} \langle \alpha, \beta \mid [\alpha, \beta] = 1 \rangle \underset{\beta}{*} \langle \beta, y \mid [\beta, y] = 1 \rangle. \end{aligned}$$

The above group L is a right-angled Artin group. For more details, terminology and a recent result on right-angled Artin groups the reader should consult the work of Bestvina and Brady in [4] and the references cited there. Niblo and Wise in [14] show that the fundamental group of a graph manifold is subgroup separable if and only if it does not contain a subgroup isomorphic to L. The group L seems to be the key point for the results of the present paper also. In fact, summarising the results of the present work together with the results of [12], we have shown the following.

Theorem. *The fundamental group of a finite graph of groups with finitely generated abelian vertex groups is subgroup separable if and only if it does not contain a subgroup isomorphic to any of the following:*

(1) L,
(2) BKS,
(3) *an ascending HNN-extension with finitely generated abelian base group,*
(4) *a non-residually finite HNN-extension with finitely generated abelian base group.*

This last category of groups was completely characterised in [3].

Furthermore, the non-subgroup separable example of Mihailova in [13] contains a subgroup isomorphic to L (see Corollary 1), and the examples of Gitik and Rips in [9] and of Allenby and Doniz in [1] are special cases of our main theorem (see Corollaries 3 and 2).

These give rise to the following interesting question.

Question. *Can we construct a finitely presented, residually finite, non-subgroup separable group that contains neither a finite index subgroup of BKS nor a finite index subgroup of any member of the family BN of groups?*

The present paper is structured as follows.

In Section 2 we give three lemmas which we need in the subsequent results.

In Section 3 we give necessary and sufficient conditions for the subgroup separability of groups that can be realised as fundamental groups of graphs of groups with finitely generated abelian vertex and edge groups.

In Section 4 we give some applications of our Theorem 1. In fact, we show that the non-subgroup separable example of [13] contains a subgroup isomorphic to L and also that the two examples of [9] and [1] are special cases of our main theorem.

2. Some intermediate results

In the sequel we shall make frequent use of the following lemma of Scott in [16].

Lemma 1 ([16])**.** *If G is a subgroup separable group, then any subgroup of G is subgroup separable and so is any group K that contains G as a subgroup of finite index.*

The following lemma is probably known, but we decided to include its proof since we could not find a reference for it.

Lemma 2. *Let K be a group and let G be the HNN-extension*

$$G = \langle t, K \mid tAt^{-1} = B \rangle$$

such that the associated subgroups A and B are finite. Then G is subgroup separable if and only if K is subgroup separable.

Proof. If G is subgroup separable, then the result follows from Lemma 1.

Conversely, assume that K is subgroup separable. Then K is residually finite, and we can easily show that G is also residually finite (see, for example, exercise 26, page 47 in [7]). Hence, for every $\alpha \in A$ we can find a finite index normal subgroup of G, say N_α, such that $\alpha \notin N_\alpha$. Similarly, for every $\beta \in B$ we can always find a finite index normal subgroup of G, say N_β, such that $\beta \notin N_\beta$. Hence, the group $N = (\bigcap_{\alpha \in A} N_\alpha) \cap (\bigcap_{\beta \in B} N_\beta)$ is a normal subgroup of finite index in G that meets A and B trivially.

Now let $(\mathcal{G}, X)$ be a graph of groups with a single vertex v with vertex group K and a single edge e (loop) with edge group A such that $\pi_1(\mathcal{G}, X) = G$. From Bass-Serre theory (see [7]) we know that the subgroup N is the fundamental group of a graph of groups with vertex groups $N \cap gKg^{-1}$ where g runs over a suitable finite set of (N, K) double coset representatives and edge groups $N \cap hAh^{-1}$ where h runs over a suitable finite set of (N, A) double coset representatives. Since K is subgroup separable, so is gKg^{-1}, and hence $N \cap gKg^{-1}$ is also subgroup separable. Since N is normal in G, and $N \cap A$ is trivial, it follows that $N \cap hAh^{-1} = \{1\}$, for any element $h \in G$. Hence, N is the free product of a finite number of subgroup separable groups and (possibly) a free group. But such a group is known to be subgroup separable. Furthermore, G is a finite extension of N; hence G is subgroup separable. $\square$

The following lemma is the second part of the proof of Lemma 2 in [12].

Lemma 3 ([12])**.** *Let G be a finitely generated group and H a finitely generated, abelian, normal subgroup of G. If G/H^n is subgroup separable for every $n \in \mathbb{N}$, then G is subgroup separable.* $\square$

Let us introduce some notation. Assume that X is a graph. On X we define as usual (see, for example, [7]) VX and EX, the sets of vertices and edges of X respectively, together with three functions $\iota : EX \to VX$, $\tau : EX \to VX$ and $\bar{\ }: EX \to VX$ such that for all $e \in EX$, $e \neq \bar{e}$, $e = \bar{\bar{e}}$ and $\tau(e) = \iota(\bar{e})$.

Suppose now that G is the fundamental group of a graph T of groups. Then, by Bass-Serre theory (see [7]), we know that there is a tree Γ on which G acts such that the quotient $G \setminus \Gamma$ equals the underlying graph T, and the group associated to a vertex w of T is the stabiliser of a vertex v of Γ that projects to w. Also, a change in the choice of v alters this group by conjugacy. So, in the sequel, we shall always choose a maximal subtree T_0 of T and a subtree Γ_0 of Γ whose projection to T induces a bijection from Γ_0 to T_0. This choice of Γ_0 gives a choice for a given vertex and edge group of T_0. Also, when necessary, we shall make choices of vertices and edges in $\Gamma \setminus \Gamma_0$ in order to extend Γ_0 to a subtree of Γ, say $\bar{\Gamma}$, such that there is a bijection between the edges of $\bar{\Gamma}$ and the edges of T. We will denote the stabiliser of f in Γ_0 by G_f. Assuming that the edge e projects in T_0 to e, we shall identify G_f with the edge group of f. Similarly, if a vertex w of Γ_0 projects to v in T_0, we shall identify the stabiliser of w in Γ, G_w, with the vertex group of v.

Now let T be a tree with three vertices v_i, $i = 1, 2, 3$ and two geometric edges e_i, $i = 1, 2$ such that v_1 and v_3 are the extremal vertices of T. Let $(\mathcal{G}, T)$ be the graph of groups with underlying graph T and finitely generated abelian vertex and edge groups. We denote by G_{v_i} the corresponding vertex groups and by G_{e_i} the corresponding edge groups. Since T is a tree, we have that $G_{e_i} = G_{\bar{e_i}}$ for every e_i, $i = 1, 2$. From the action of G on the tree Γ, it is automatic that $G_{e_i} \leq G_{\iota(e_i)}$ and that $G_{e_i} \leq G_{\tau(e_i)}$, $i = 1, 2$.

Lemma 4. *Let $(\mathcal{G}, T)$ be as above and assume that $e_1, e_2 \in ET$ such that $e_1 \neq \bar{e_2}$. Suppose also that all edge groups are proper subgroups of the vertex groups that these edges connect. Then the following statements are equivalent:*

(1) *$G = \pi_1(\mathcal{G}, T)$ is subgroup separable;*
(2) *$G_{e_1} \cap G_{e_2}$ is of finite index in G_{e_1} or in G_{e_2};*
(3) *G does not contain a subgroup isomorphic to L.*

Proof. $1 \Rightarrow 2$. From its definition, and standard Bass-Serre theory ([7]), $G = \pi_1(\mathcal{G}, T)$ is the free product with amalgamation of the form

$$\pi_1(\mathcal{G}, T) = G_{v_1} \underset{G_{e_1}}{*} G_{v_2} \underset{G_{e_2}}{*} G_{v_3}$$

where all groups are finitely generated abelian groups. Hence, all vertex and edge groups are subgroup separable.

Let G be subgroup separable, and assume that neither G_{e_1} nor G_{e_2} is finite. If $|G_{e_1} : G_{e_1} \bigcap G_{e_2}| = \infty = |G_{e_2} : G_{e_1} \bigcap G_{e_2}|$, then neither $|G_{v_2} : G_{e_1}| < \infty$ nor $|G_{v_2} : G_{e_2}| < \infty$. Indeed, if for example $|G_{v_2} : G_{e_1}| < \infty$, then $|G_{e_2} : G_{e_1} \bigcap G_{e_2}| < \infty$, a contradiction.

From the discussion before Lemma 4, we can choose Γ_0 to be a subtree of Γ whose projection to T induces a bijection from Γ_0 to T. Let w_i denote the vertex of Γ_0 that projects to v_i, and f_i the edge of Γ_0 that projects to e_i, such that $G_{w_i} = G_{v_i}$, $i = 1, 2, 3$ and $G_{f_i} = G_{e_i}$, $i = 1, 2$. Let H denote the subgroup of G generated by G_{w_2}, $aG_{w_2}a^{-1}$ and $bG_{w_2}b^{-1}$ where a and b are coset representatives of G_{w_2} in G with $a \in G_{w_1} \setminus G_{f_1}$ and $b \in G_{w_3} \setminus G_{f_2}$.

Then $aG_{w_2}a^{-1}$ is the stabiliser of aw_2 that is different from w_2 by the choice of a. Similarly, $bG_{w_2}b^{-1}$ is the stabiliser of bw_2 that is different from w_2 and aw_2, by the choice of b. The subgraph of Γ with vertices aw_2, w_2 and bw_2 is a tree with extremal vertices aw_2 and bw_2, and so H is the fundamental group of a graph of groups with graph as the above tree. Hence, H is of the form

$$aG_{w_2}a^{-1} \underset{H_1}{*} G_{w_2} \underset{H_2}{*} bG_{w_2}b^{-1}.$$

The amalgamating subgroup H_1 is the intersection $aG_{w_2}a^{-1} \cap G_{w_2}$ and so is the stabiliser of the segment joining w_2 and aw_2 in Γ. Since a stabilises w_1, and w_1 and w_2 are adjacent, this segment consists of the two edges f_1 and af_1. Thus, $aG_{w_2}a^{-1} \cap G_{w_2} = G_{f_1} \cap aG_{f_1}a^{-1}$. Since G_{f_1} and a lie in the abelian group G_{w_1} it follows that $G_{f_1} = aG_{f_1}a^{-1}$; so $H_1 = G_{f_1}$. Similarly, $H_2 = G_{f_2}$. Since G_{w_2} contains G_{f_1} and G_{f_2} with infinite index, we can choose $x \in aG_{w_2}a^{-1}$ of infinite order such that $\langle x\rangle \cap G_{f_1} = \{1\}$ and $y \in bG_{w_2}b^{-1}$ such that $\langle y\rangle \cap G_{f_2} = \{1\}$. This implies that $\langle x\rangle \cap \langle y\rangle = \{1\}$. Also, since $|G_{f_1} : G_{f_1} \cap G_{f_2}| = \infty$ we can choose $\beta_1 \in G_{f_1}$ such that β_1 has infinite order and $\langle \beta_1\rangle \cap G_{w_2} = \{1\}$. Similarly, since $|G_{f_2} : G_{f_1} \cap G_{f_2}| = \infty$, we can choose $\beta_2 \in G_{f_2}$ of infinite order with $\langle \beta_1\rangle \cap \langle \beta_2\rangle = \{1\}$. Let L be the subgroup of G generated by $\langle x, y, \beta_1, \beta_2\rangle$. From their choices we have that $x, \beta_1 \in aG_{w_2}a^{-1}$, $\beta_1, \beta_2 \in G_{w_2}$ and $\beta_2, y \in bG_{w_2}b^{-1}$. So, due to the form of H and the choices of x, y, β_1, β_2, L has a presentation

$$L = \langle x, \beta_1 \mid [x, \beta_1] = 1\rangle \underset{\beta_1}{*} \langle \beta_1, \beta_2 \mid [\beta_1, \beta_2] = 1\rangle \underset{\beta_2}{*} \langle \beta_2, y \mid [\beta_2, y] = 1\rangle.$$

But L was shown in [11] to be a non-subgroup separable group (containing a finite index subgroup of BKS), a contradiction to our assumption that every subgroup of G is subgroup separable by Lemma 1.

$2 \Rightarrow 1$. Suppose that G_{e_1} is finite. The group $M = G_{v_2} \underset{G_{e_2}}{*} G_{v_3}$ is a subgroup of G. Since G_{v_2} and G_{v_3} are abelian, the group $G_{e_2}^n$ is normal in G_{v_2} and G_{v_3} and hence in M for every integer $n \geq 1$. Also, $M/G_{e_2}^n$ is a free product of abelian groups with finite amalgamation, hence by the results of Allenby and Gregorac in [2], is subgroup separable for every $n \in \mathbb{N}$. Therefore, by Lemma 3, M is subgroup separable and so is G since it is a free product of subgroup separable groups with finite amalgamation, again by the results in [2]. A similar argument works if G_{e_2} is finite.

Finally, if neither G_{e_1} nor G_{e_2} is finite, assume that $|G_{e_1} : G_{e_1} \cap G_{e_2}| < \infty$. Let $H = G_{e_1} \cap G_{e_2}$. Then H^n is normal in each vertex group of $(\mathcal{G}, T)$ and so is normal in G for every $n \in \mathbb{N}$. Hence,

$$G/H^n = G_{v_1}/H^n \underset{G_{e_1}/H^n}{*} G_{v_2}/H^n \underset{G_{e_2}/H^n}{*} G_{v_3}/H^n.$$

Obviously, $|G_{e_1} : H^n| < \infty$. Moreover, $G_{v_2}/H^n \underset{G_{e_2}/H^n}{*} G_{v_3}/H^n$ is subgroup separable and so G/H^n is subgroup separable. Then, by Lemma 3, G is subgroup separable.

$1 \Rightarrow 3$. Since G is subgroup separable, then it cannot contain a subgroup isomorphic to L by Lemma 1.

$3 \Rightarrow 2$. In the proof that $1 \Rightarrow 2$, we showed that if (2) does not hold, then G contains a subgroup isomorphic to L. Thus, if G does not contain such a subgroup, it follows that (2) holds. □

3. Graphs of abelian groups

Let us assume now that $(\mathcal{G}, T)$ is a graph of groups with T a finite tree and such that every vertex and edge group of T is a finitely generated abelian group. Furthermore, assume that in $(\mathcal{G}, T)$, no edge group is equal to either of the vertex groups that this edge connects. Having in mind the discussion before Lemma 4, we can show the following.

Lemma 5. *Let $(\mathcal{G}, T)$ be as above. Then the following statements are equivalent:*

(1) *$G = \pi_1(\mathcal{G}, T)$ is subgroup separable;*
(2) *for every connected subgraph Y of T, the group $\bigcap_{e \in EY} G_e$ has finite index in G_e for at least one $e \in EY$;*
(3) *G does not contain a subgroup isomorphic to L.*

Proof. $1 \Rightarrow 2$. Let Γ be the G-tree and Γ_0 be the subtree of Γ whose projection to Y induces a bijection. Based on the discussion before Lemma 4, we shall identify the vertices and edges of Γ_0 to those of T.

Let us assume that G is subgroup separable. We shall show that for every connected subgraph Y of T, the group $\bigcap_{e \in EY} G_e$ has finite index in at least one G_e, $e \in EY$. The proof will be by induction on the number of geometric edges that Y contains.

If Y has one geometric edge, the result is trivial, and if Y has two geometric edges, the result follows from Lemma 4. Now let $m > 2$ be the number of the geometric edges of Y and let $e_1, \dots, e_m$ be a collection of edges of EY such that $e_i \neq \bar{e_j}$ for every $i, j \in \{1, \dots, m\}$. For simplicity, assume also that e_1 and e_m are edges that connect extremal vertices of Y to the rest of the tree. From the induction hypothesis, we have that $\bigcap_{i=1}^{m-1} G_{e_i}$ has finite index in G_{e_r} for some r with $1 \leq r \leq m-1$. Similarly, $\bigcap_{i=2}^{m} G_{e_i}$ has finite index in G_{e_s} for some s with $2 \leq s \leq m$. If $s = r$, then $\bigcap_{i=1}^{m} G_{e_i}$ has finite index in G_{e_s} and the result is proved. If $s \neq r$, then assume that $\bigcap_{i=1}^{m} G_{e_i}$ has infinite index in every G_{e_i}, $1 \leq i \leq m$.

Since $\bigcap_{i=1}^{m} G_{e_i}$ equals the intersection of $\bigcap_{i=2}^{m} G_{e_i}$ and $\bigcap_{i=1}^{m-1} G_{e_i}$, it follows that $\bigcap_{i=1}^{m} G_{e_i}$ has finite index in $G_{e_r} \cap G_{e_s}$. If $r \neq 1$, then $G_{e_r} \cap G_{e_s}$ has finite index in G_{e_s}, which is a contradiction. Thus $r = 1$, and similarly $s = m$.

Hence $\bigcap_{i=1}^{m-1} G_{e_i}$ has finite index in G_{e_1}, which implies that there is a $\kappa \in \mathbb{N}$ such that $G_{e_1}^{\kappa}$ is a subgroup of $\bigcap_{i=2}^{m-1} G_{e_i} = H$. Also $\bigcap_{i=2}^{m} G_{e_i}$ has finite index in G_{e_m}, which implies that there is a $\lambda \in \mathbb{N}$ such that $G_{e_m}^{\lambda}$ is a subgroup of $\bigcap_{i=2}^{m-1} G_{e_i} = H$.

On the other hand, we assumed that $\bigcap_{i=1}^{m} G_{e_i}$ has infinite index in every G_{e_i} and so $\bigcap_{i=1}^{m} G_{e_i}$ has infinite index in both G_{e_1} and G_{e_m}. Hence there is an element $x \in G_{e_1}$ of infinite order such that $\langle x \rangle \cap G_{e_m} = \{1\}$ and an element $y \in G_{e_m}$ of infinite order such that $\langle y \rangle \cap G_{e_1} = \{1\}$. Hence $\langle x \rangle \cap \langle y \rangle = \{1\}$. Since $G_{e_1}^{\kappa}$ and $G_{e_m}^{\lambda}$ are subgroups of H, we have that $\langle x^{\kappa} \rangle = \langle \alpha_1 \rangle \subset H$ and $\langle y^{\lambda} \rangle = \langle \alpha_2 \rangle \subset H$ with $\langle \alpha_1 \rangle \cap \langle \alpha_2 \rangle = \{1\}$.

Let L denote the subgroup of G that is generated by H, $\beta_1 H \beta_1^{-1}$ and $\beta_2 H \beta_2^{-1}$ where $\beta_1 \in G_{v_1}$ with $\beta_1 \notin G_{e_1}$ and $\beta_2 \in G_{v_m}$ with $\beta_2 \notin G_{e_m}$. Then, due to the choice of β_1 and β_2, L is the fundamental group of a graph of groups with graph a tree with three vertices corresponding to the subtrees of Γ, $\beta_1(e_2 \dots e_{m-1})$, $e_2 \dots e_{m-1}$ and $\beta_2(e_2 \dots e_{m-1})$. The vertices that correspond to $\beta_1(e_2 \dots e_{m-1})$ and $\beta_2(e_2 \dots e_{m-1})$ are extremal and so L is of the form

$$L = \beta_1 H \beta_1^{-1} \underset{A}{*} H \underset{B}{*} \beta_2 H \beta_2^{-1}.$$

The subgroup A is the intersection of $\beta_1 H \beta_1^{-1}$ and H. Since H is the stabiliser of the subtree $e_2 \dots e_{m-1}$, H fixes each of $e_2, \dots, e_{m-1}$ and so $\beta_1 H \beta_1^{-1}$ fixes each of $\beta_1 e_2, \dots, \beta_1 e_{m-1}$. Hence A fixes each edge in the union. Notice that the two subtrees $e_1 \dots e_{m-1}$ and $\beta_1(e_1 \dots e_{m-1})$ intersect in v_1. This implies that the minimal subtree of Γ that contains $e_2, \dots, e_{m-1}$ and $\beta_1 e_2, \dots, \beta_1 e_{m-1}$ contains e_1 and $\beta_1 e_1$. Since A must fix this minimal subtree, it follows that A fixes e_1 and $\beta_1 e_1$.

So $A = \beta_1 G_{e_1} \beta_1^{-1} \cap G_{e_1} \cap H \cap \beta_1 H \beta_1^{-1} = G_{e_1} \cap H \cap \beta_1 H \beta_1^{-1} = G_{e_1} \cap H$ since G_{v_1} is abelian. Similarly, $B = G_{e_m} \cap H$. Then $A \cap B = H \cap G_{e_1} \cap G_{e_m}$ and since $H \cap G_{e_1} \cap G_{e_m}$ has infinite index in both G_{e_1} and G_{e_m}, we have that $A \cap B$ has infinite index in both A and B. But Lemma 4 implies that L is not subgroup separable, a contradiction to the assumption that G is subgroup separable.

$2 \Rightarrow 1$. Let $Y = T$. Then the group $N = \bigcap_{e \in ET} G_e$ has finite index in G_e for at least one $e \in ET$. But N is normal in G because it is normal in each vertex group. So the group G/N has at least one edge with finite edge group.

Let $e_1, \ldots, e_m$ be the geometric edges of Y such that G_{e_i}/N is finite for every $i = 1, \ldots, m$. Then, the graph $Y \setminus \{e_1, \ldots, e_m\}$ is disconnected, consisting of connected components, some of them possibly single vertices. Notice that because of the hypothesis, for every connected component, the intersection of the edge groups that constitute the graph has finite index in some edge group, for a geometric edge in the graph. This group is again a normal subgroup of the fundamental group of the subgraph since it is normal in each vertex group of the subgraph. So we can again divide by this intersection of edge groups and repeat the above argument for every connected subgraph. The above procedure will terminate with a disconnected graph consisting of either single vertices or graphs with two edges that satisfy condition 2 of Lemma 4. Hence, all fundamental groups of the graphs of groups whose graphs are the components of the final graph are subgroup separable. Now, working backwards, we can reconstruct the group $\pi_1(\mathcal{G}, Y)$ and at each step use Lemma 3 along with the result of Allenby and Gregorac in [2] (that the free product of two subgroup separable groups with finite amalgamation is subgroup separable) to show that $\pi_1(\mathcal{G}, Y)$ is subgroup separable.

$1 \Rightarrow 3$. The result is immediate from Lemma 1.

$3 \Rightarrow 2$. Assume that G does not contain a subgroup isomorphic to L. Let Y be a subtree of T such that $H = \bigcap_{e \in EY} G_e$ has infinite index in every G_e, $e \in EY$. Then, it was shown above that the normal closure of H in G contains a subgroup isomorphic to L, a contradiction. □

Lemma 6. *Let $(\mathcal{G}, X)$ be a graph of groups where X consists of a single vertex v such that the vertex group is a finitely generated abelian group and n edge loops based at v such that neither of the two injections of each edge group into the vertex group is an isomorphism. Then the following are equivalent:*

(1) *$G = \pi_1(\mathcal{G}, X)$ is subgroup separable.*
(2) *For every subgraph Y of X, the group $\bigcap_{e \in EY} G_e$ has a subgroup H such that H is normal in G and H has finite index in at least one G_e, for some $e \in EY$.*
(3) *G does not contain a subgroup isomorphic to either L or BKS or an ascending HNN-extension with finitely generated abelian base group or a non-residually finite HNN-extension with finitely generated abelian base group.*

Proof. $1 \Rightarrow 2$. Assume that G is subgroup separable.

Let Y be any subgraph of X. Let $C = \pi_1(\mathcal{G}, Y)$. Then C is a k-fold HNN-extension with base group G_v and stable letters t_i, $i = 1, \ldots, k$. Let A_i and B_i be the associated subgroups under t_i.

Take T to be the G-tree such that Y is the quotient $G \backslash T$. Choose a vertex w in T such that $G_w = G_v$. For each oriented edge in Y there is exactly one edge in T that is incident to w and projects to this edge. Thus the subtree T_0 of T consisting of all edges incident to w projects down to Y inducing a 2-to-1 projection on edges. Fix an $l \in \{1, \ldots, k\}$, and take T' to be the tree consisting of T_0 and the translate of T_0 by t_l. Then T' is a tree with $4k$ vertices, since T_0 has $2k+1$ vertices and T_0 and $t_l T_0$ meet in two points w and $t_l w$. Let S be the fundamental group of the graph of groups with graph T'. Then S is generated by G_w, $t_i G_w t_i^{-1}$, $t_i^{-1} G_w t_i$, $t_l t_i G_w t_i^{-1} t_l^{-1}$ and $t_l t_i^{-1} G_w t_i t_l^{-1}$ for every $i = 1, \ldots, k$. Let us denote $D = \bigcap_{i=1}^{k} (A_i \cap B_i)$. The group

$$M_l = D \cap t_l D t_l^{-1}$$

is the intersection of all edge groups of T' and since G is subgroup separable, M_l has finite index in at least one edge group of T' by Lemma 5. If M_l has finite index in some $t_l A_i t_l^{-1}$, then it also has finite index in A_i since A_i and $t_l A_i t_l^{-1}$ have the same torsion free rank. So we can assume that M_l has finite index in some A_{λ_l} (and so in B_{λ_l}) for some $\lambda_l \in \{1, \ldots, k\}$. Then $t_l^{-1} M_l t_l$ has finite index in $t_l^{-1} A_{\lambda_l} t_l$. Since A_{λ_l} and $t_l^{-1} A_{\lambda_l} t_l$ have the same torsion free rank, we have that $t_l^{-1} M_l t_l$ has finite index in A_{λ_l}. Therefore, $M_l \cap t_l^{-1} M_l t_l$ has finite index in A_{λ_l} and so it also has finite index in both M_l and $t_l^{-1} M_l t_l$.

Let K be the subgroup of G generated by $\langle M_l, t_l^{-1} M_l t_l \rangle$. Notice that both M_l and $t_l^{-1} M_l t_l$ are subgroups of D; so both are contained in some edge group of Y. So, K is a finitely generated abelian group since it is a subgroup of some edge group of Y. Moreover, the group G_l generated by $\langle t_l, K \rangle$ is a subgroup of $\langle t_l, G_w \rangle$ and so has an HNN-extension presentation

$$G_l = \langle t_l, K \mid t_l (t_l^{-1} M_l t_l) t_l^{-1} = M_l \rangle.$$

But by Theorem 2 in [12] there is a finite index subgroup of M_l, say H_l, such that H_l is normalised by t_l for every $l \in \{1, \ldots, k\}$.

Since $t_l^{-1} M_l t_l$ has finite index in A_{λ_l} and $t_l^{-1} M_l t_l \subset D \subset A_{\lambda_l}$, we have that all A_{λ_l} have the same torsion free rank as D. Choose A_μ to be one of the above A_{λ_l}. Then all H_s are subgroups of finite index in A_μ and so we can find a finite index subgroup H of A_μ such that H is normal in G.

$2 \Rightarrow 1$. Use induction on the number n of edges X contains. If $n = 1$ the result is immediate from Theorem 2 in [12].

Let $Y = X$. Then, from hypothesis, there is a finite index subgroup H_i of some edge group G_{e_i} such that H_i is normal in $G = \pi_1(\mathcal{G}, X)$. Let A_i and B_i be the two isomorphic copies of G_{e_i} embedded in G_v. Then G/H_i is an HNN-extension of the form

$$\langle t_i, K \mid t_i (A_i/H_i) t_i^{-1} = B_i/H_i \rangle$$

where K is the $(n-1)$-fold HNN-extension of G_v/H_i and the groups A_i/H_i and B_i/H_i are finite. So, by the induction hypothesis, K is subgroup separable. But from Lemma 2, G/H_i^n is subgroup separable for every $n \in \mathbb{N}$. So, Lemma 3 shows that G is subgroup separable.

$1 \Rightarrow 3$. Immediate.

$3 \Rightarrow 2$. Assume that G does not contain any of the groups listed in the lemma. Take the normal closure of G_v in G and as before construct M_l. Since G does not contain a subgroup isomorphic to L, by Lemma 5, we have that M_l has finite index in some A_{λ_l}. Construct again the HNN-extension

$$G_l = \langle t_l, K \mid t_l (t_l^{-1} M_l t_l) t_l^{-1} = M_l \rangle$$

where K is the subgroup of G generated by $\langle M_l, t_l^{-1}M_lt_l\rangle$. As before, both M_l and $t_l^{-1}M_lt_l$ are contained in some edge group of Y and so K is a finitely generated abelian group. From the hypothesis that G does not contain a non-residually finite HNN-extension with finitely generated abelian base group we have that G_l is residually finite. Moreover, if $K = M_l$ and $t_l^{-1}M_lt_l \subsetneq K$ or $K = t_l^{-1}M_lt_l$ and $M_l \subsetneq K$, then G_l is an ascending HNN-extension with finitely generated abelian base group, contained in G, a contradiction. Hence, either $K = M_l = t_l^{-1}M_lt_l$ or $M_l, t_l^{-1}M_lt_l \subsetneq K$. In the first case, G_l is subgroup separable and K is a normal subgroup in G_l. In the second case, we can repeat the exact argument of the proof of Theorem 2 in [12] to show that either $M_l \cap t_l^{-1}M_lt_l$ contains a subgroup, say H_l, of finite index in both M_l and $t_l^{-1}M_lt_l$ such that H_l is normal in G_l or (if such a subgroup does not exist), to construct a subgroup of G_l isomorphic to BKS. The latter is a contradiction. So, in every case, there is a normal subgroup H_l of G_l with finite index in M_l for every $l \in \{1, \dots, k\}$. Notice that if $K = M_l = t_l^{-1}M_lt_l$, then $H_l = K$. But M_l has finite index in A_{λ_l}. So H_l has finite index in A_{λ_l}. Again, all A_{λ_l} have the same torsion free rank as D. Choose A_μ to be one of the A_{λ_l}. Then we can find a subgroup H of A_μ such that H has finite index in A_μ and is normal in G. □

We are now able to prove our main theorem.

Theorem 1. *Let $(\mathcal{G}, X)$ be a graph of groups such that X is a finite graph and every vertex group is a finitely generated abelian group. Then the following statements are equivalent:*

(1) *$G = \pi_1(\mathcal{G}, X)$ is subgroup separable;*
(2) *for every connected subgraph Y of X, there is a subgroup H of the group $\pi_1(\mathcal{G}, Y)$ such that H has finite index in at least one of the edge groups G_e, $e \in EY$ and H is normal in $\pi_1(\mathcal{G}, Y)$;*
(3) *G does not contain a subgroup isomorphic to L or BKS, nor does it contain an ascending HNN-extension with finitely generated abelian base group nor a non-residually finite HNN-extension with finitely generated abelian base group.*

Proof. Let $(\mathcal{G}, X)$ be as above and delete from X every edge with finite edge group. Let $(\mathcal{G}_\lambda, X_\lambda)$, $\lambda = 1, \dots, n$, be the connected components of the new graph of groups. In view of the results in [2] and Lemma 2, $G = \pi_1(\mathcal{G}, X)$ is subgroup separable if and only if $\pi_1(\mathcal{G}_\lambda, X_\lambda)$ is subgroup separable for every $\lambda = 1, \dots, n$.

$1 \Rightarrow 2$. Let Y be any subgraph of X_λ and T be a maximal tree of Y. Also let $G = \pi_1(\mathcal{G}, Y)$. Assume that $VY = \{v_1, \dots, v_r\}$, $\{e_1, \dots, e_{r-1}\} \subset ET$ such that $e_i \neq \bar{e_j}$ for every $i, j \in \{1, \dots, r-1\}$ and $\{\alpha_1, \dots, \alpha_k\} \subset E(Y \setminus T)$ such that $\alpha_i \neq \bar{\alpha_j}$ for every $i, j \in \{1, \dots, k\}$. Assume, finally, that A_i are the groups G_{α_i} and B_i are the groups $G_{\bar{\alpha}_i}$, $i = 1, \dots, k$.

Take Γ to be the G-tree such that Y is the quotient $G\backslash\Gamma$ and the group associated to a vertex v_i of Y is the stabiliser of a vertex of Γ that projects to v_i. Choose a subtree of Γ, say Γ_0, such that the projection of Γ to Y induces a bijection from Γ_0 to T. In this bijection, denote by w_i the vertex of Γ_0 that projects to v_i and by f_i the edge of Γ_0 that projects to e_i. Finally, for every edge α_i of $Y \setminus T$ incident to v_j, choose an edge g_i of $\Gamma \setminus \Gamma_0$ incident to w_j such that g_i projects to α_i. Let Γ' be the subtree of Γ with $V\Gamma' = \{w_1, \dots, w_r\}$ and $E\Gamma' = \{f_1, \dots, f_{r-1}, g_1, \dots, g_k\}$. Fix an $l \in \{1, \dots, k\}$ and take the subgroup of the normal closure of $H = \pi_1(\mathcal{G}, T)$, say S, that is generated by H, $t_iHt_i^{-1}$, $t_i^{-1}Ht_i$, $t_lt_iHt_i^{-1}t_l^{-1}$ and $t_lt_i^{-1}Ht_it_l^{-1}$ for every $i = 1, \ldots, k$. Then S is the fundamental group of a graph of groups with underlying graph consisting of Γ' and its translate by t_l. But S is a subgroup of G and so is subgroup separable.

Denote $N = \bigcap_{f \in E\Gamma_0} G_f$ and $D = \bigcap_{i=1}^{k}(A_i \cap B_i)$. Then, if we apply Lemma 5 to S we have that the group

$$M_l = N \cap D \cap t_lNt_l^{-1} \cap t_lDt_l^{-1}$$

has finite index in at least one of the edge groups it involves. So, M_l has finite index in either G_f or in $t_lG_ft_l^{-1}$ for some $f \in \{f_1, \ldots, f_{r-1}, g_1, \ldots, g_k\}$. But the torsion free rank of $t_lG_ft_l^{-1}$ is the same as that of G_f and so M_l has finite index in at least one G_f, $f \in \{f_1, \ldots, f_{r-1}, g_1, \ldots, g_k\}$.

Similarly, $t_l^{-1}M_lt_l \cap G_f$ has finite index in G_f and so $M_l \cap t_l^{-1}M_lt_l$ has finite index in both M_l and $t_l^{-1}M_lt_l$.

Since M_l and $t_l^{-1}M_lt_l$ are conjugate for every t_l, $l = 1, \ldots, k$ and commensurable, one can work as in Lemma 6 or as in the proof of Theorem 2 in [12] to obtain a finite index subgroup H of some edge group G_f such that H is normal G.

$2 \Rightarrow 1$. We use induction on the number of geometric edges of X_λ. If X_λ contains one edge, then $\pi_1(\mathcal{G}_\lambda, X_\lambda)$ is either an amalgamated free product of two abelian groups or an HNN-extension with abelian base group. In the first case, $\pi_1(\mathcal{G}_\lambda, X_\lambda)$ is always subgroup separable. In the second case, one can take the path in X_λ that meets successively the two copies of G_v, say A and B, associated by the stable letter. From the assumptions, $A \cap B$ has finite index in A (and necessarily in B) and there is a subgroup H of $A \cap B$ such that H is normal in A and B, so, by Theorem 2 in [12] is subgroup separable.

If X_λ has more edges, then condition (2) states that we can find a subgroup N of the group $\bigcap_{e \in EX_\lambda} G_e$ that is normal in G and has finite index in some edge group G_α, $\alpha \in EX_\lambda$. If $X_\lambda \setminus \{\alpha, \bar{\alpha}\}$ is disconnected, then $\pi_1(\mathcal{G}_\lambda, X_\lambda)/N$ is subgroup separable due to the inductive hypothesis and the results of [2]. If $X_\lambda \setminus \{\alpha, \bar{\alpha}\}$ is connected, then $\pi_1(\mathcal{G}, X_\lambda)/N$ is subgroup separable due to the inductive hypothesis and Lemma 2. Consequently, $\pi_1(\mathcal{G}_\lambda, X_\lambda)$ is subgroup separable from the results of Lemma 3.

$1 \Rightarrow 3$. If G is subgroup separable, then it cannot contain any of the above-mentioned groups by Lemma 1.

$3 \Rightarrow 2$. Let Y be any subgraph of X. Take T to be a maximal tree of Y and construct S as before. Using the results of Lemmas 5 and 6, one can show as before that G contains a normal subgroup of G, say H, such that H is of finite index in some edge group of Y. □

Notice here that the proof of the theorem mentioned in the introduction is an immediate consequence of the above theorem.

4. Applications

In this section we show that the three non-subgroup separable examples of [13], [1] and [9] can be treated using our techniques. More specifically, in Corollary 1 we use our method of proof to show that $F_2 \times F_2$ contains a subgroup isomorphic to L in [14]. In Corollaries 2 and 3 we show that the examples presented there are special cases of our theorem.

Corollary 1 ([13]). *The group $G = F_2 \times F_2$ is not subgroup separable.*

Proof. Take two copies of F_2 with generators $\langle x_1, x_2\rangle$ and $\langle y_1, y_2\rangle$. Then the subgroup of G, say H, generated by $H = \langle x_1, y_1\rangle$ is free abelian of rank two. The normal closure of H in G contains a subgroup of the form

$$y_2 H y_2^{-1} \underset{y_2 x_1 y_2^{-1} = x_1}{*} H \underset{y_1 = x_2 y_1 x_2^{-1}}{*} x_2 H x_2^{-1}.$$

But the above group is isomorphic to L and so it cannot be subgroup separable; hence G is not subgroup separable. □

Corollary 2 ([1]). *Let A be the group with presentation*

$$A = \langle a, d_1, d_2, d_3 \mid [d_i, d_j] = 1, a d_1 a^{-1} = d_1 d_2^2, [a, d_2] = [a, d_3] = 1\rangle$$

and let B be the group with presentation $B = \langle a, c \mid [a, c] = 1\rangle$. Then the group $G = A \underset{\langle a\rangle}{*} B$ is not subgroup separable.

Proof. The group G can be written as an HNN-extension of the form

$$G = \langle d_1, c, K \mid d_1^{-1} A_1 d_1 = B_1, c^{-1} A_2 c = B_2\rangle$$

where $K = \langle a, d_2, d_3 \mid [a, d_2] = [a, d_3] = [d_2, d_3] = 1\rangle$ and $A_1 = \langle a, d_2, d_3\rangle$, $B_1 = \langle d_2^2 a, d_2, d_3\rangle$, $A_2 = \langle a\rangle = B_2$. For G to be subgroup separable, Theorem 1 requires the existence of a subgroup H of K such that H has finite index in either A_1 or A_2 and such that H is fixed under the isomorphisms $\phi_i : A_i \to B_i$, $i = 1, 2$. One can easily see that the only candidate for H is $H = \langle a^k\rangle$, $k \in \mathbb{N}$. But H cannot be fixed by ϕ_1 and so such an H cannot exist. Hence, G is not subgroup separable. □

Corollary 3 ([9]). *Let D be a free abelian group with the basis d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8. Let $A = D \rtimes \langle a\rangle$, where $a^{-1} d_1 a = d_1 d_2$, $a^{-1} d_2 a = d_2$, $a^{-1} d_3 a = d_4$, $a^{-1} d_4 a = d_3$, $a^{-1} d_5 a = d_6$, $a^{-1} d_6 a = d_5$, $a^{-1} d_7 a = d_7$, $a^{-1} d_8 a = d_8$.*

Let B be any group containing an element b of infinite order. If B contains an element c such that $bc = cb$ and $c \notin \langle b\rangle$, then $A \underset{a=b}{*} B$ is not subgroup separable.

Proof. The subgroup of G, say H, generated by $\langle d_1, d_2, a, c d_2\rangle$ has a presentation of the form

$$H = \langle d_1, c d_2, K \mid d_1^{-1} A_1 d_1 = B_1, c d_2 A_2 (c d_2)^{-1} = B_2\rangle$$

where $K = \langle d_2, a \mid [d_2, a] = 1\rangle = A_1 = B_1$ and $A_2 = \langle a\rangle = B_2$. As before, for H to be subgroup separable, Theorem 1 requires the existence of a normal subgroup M such that either A_1/M or A_2/M is finite. So, the only possible candidate would be the groups $\langle a^k\rangle$, $k \in \mathbb{N}$. But the groups $\langle a^k\rangle$ cannot be fixed by the isomorphism between A_1 and B_1. So, such an M does not exist. Hence, H is not subgroup separable and neither is G. □

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