PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 9, Pages 2567–2575
S 0002-9939(04)07271-5
Article electronically published on April 8, 2004

THE S-ELEMENTARY FRAME WAVELETS ARE PATH CONNECTED

X. DAI, Y. DIAO, Q. GU, AND D. HAN

(Communicated by David R. Larson)

ABSTRACT. An s-elementary frame wavelet is a function $\psi \in L^2(\mathbb{R})$ which is a frame wavelet and is defined by a Lebesgue measurable set $E \subset \mathbb{R}$ such that $\hat{\psi} = \frac{1}{\sqrt{2\pi}}\chi_E$. In this paper we prove that the family of s-elementary frame wavelets is a path-connected set in the $L^2(\mathbb{R})$-norm. This result also holds for s-elementary A-dilation frame wavelets in $L^2(\mathbb{R}^d)$ in general. On the other hand, we prove that the path-connectedness of s-elementary frame wavelets cannot be strengthened to uniform path-connectedness. In fact, the sets of normalized tight frame wavelets and frame wavelets are not uniformly path-connected either.

1. INTRODUCTION

Let $L^2(\mathbb{R})$ be the set of Lebesgue square integrable functions on $\mathbb{R}$. The Fourier transform for $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ is defined by

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} f(t)dt.$$

This is denoted by $\mathcal{F}f$ or $\widehat{f}$. It is known that $\mathcal{F}$ can be uniquely extended to a unitary operator on $L^2(\mathbb{R})$. Let D and T be the dilation and translation operators on $L^2(\mathbb{R})$, namely $(Df)(x) = \sqrt{2}f(2x)$ and $(Tf)(x) = f(x-1)$ for any $f \in L^2(\mathbb{R})$. We will use $\widehat{D}, \widehat{T}$ for the product $\mathcal{F}D\mathcal{F}^{-1}$ and $\mathcal{F}T\mathcal{F}^{-1}$. It is known that $\widehat{D} = D^{-1}$ and $\widehat{T}f(t) = e^{-ist}f(t)$ [3]. A function $\psi \in L^2(\mathbb{R})$ is called a *frame wavelet* for $L^2(\mathbb{R})$ if there exist two positive constants $0 < a \leq b$ such that for any $f \in L^2(\mathbb{R})$,

$$a\|f\|^2 \leq \sum_{n,\ell \in \mathbb{Z}} |\langle f, D^n T^\ell \psi \rangle|^2 \leq b\|f\|^2. \tag{1}$$

If one can choose $a = b$ in (1), then ψ is called a *tight frame wavelet*. Furthermore, if $a = b = 1$, then ψ is called a *normalized tight frame wavelet*. Let E be a Lebesgue measurable set of finite measure, and χ_E the corresponding characteristic function. If the function $\psi_E \in L^2(\mathbb{R})$ defined by $\widehat{\psi_E} = \frac{1}{\sqrt{2\pi}}\chi_E$ is a frame wavelet, a tight frame wavelet or a normalized tight frame wavelet for $L^2(\mathbb{R})$, then the set E is called a *frame wavelet set*, a *tight frame wavelet set* or a *normalized tight frame wavelet set*

Received by the editors March 8, 2002 and, in revised form, February 20, 2003.
2000 *Mathematics Subject Classification.* Primary 46N99.
Key words and phrases. Frames, wavelets, frame wavelets, frame wavelet sets, Fourier transform.

for $L^2(\mathbb{R})$, respectively. The corresponding function ψ_E is called an *s-elementary*, a *tight s-elementary* or a *normalized tight s-elementary* frame wavelet. The name *s-elementary* is borrowed from [3], [8], where a wavelet whose Fourier transform is of the form $\frac{1}{\sqrt{2\pi}}\chi_E$ is called an *s-elementary wavelet*.

The topological property of various families of wavelets is an interesting topic in the study of wavelet theory. In [3], the question concerning the path-connectedness of the set of all orthonormal wavelets was raised. Similar questions were raised and studied in [7], [9] about the set of all MRA-wavelets, tight frame wavelets and MRA tight frame wavelets. In fact, discussions on such issues can be traced a few years back before the publication of [3]. One can ask similar questions on the families of normalized tight frame wavelets and frame wavelets. These turn out to be very hard questions, and all of them remain unsolved at this time. However, it has been proved that the family of s-elementary (orthonormal) wavelets is path-connected in [8], and it has since been shown that the set of all MRA-wavelets is also path-connected [6], [9]. In [2], the authors proved that the set of normalized tight s-elementary frame wavelets is path-connected.

In this paper, we are mainly concerned with the path-connectedness of the set of s-elementary frame wavelets. Showing the path-connectedness of the set of s-elementary frame wavelets can potentially lend a helping hand in proving the path-connectivity of the set of all frame wavelets, since one would only need to show that any frame wavelet is path-connected to an s-elementary frame wavelet. Since the set of normalized tight s-elementary frame wavelets is path-connected, it seems plausible that the set of s-elementary frame wavelets may also be path-connected. However, proving it is not trivial. The reason is that the proof in [2] relies on the characterization of the normalized tight frame wavelet sets (which is given in [1]). Yet, the characterization of frame wavelet sets is still an open question at this time. So, it is somewhat surprising that we are able to use the partial results about frame wavelet sets developed in [1] to prove that the set of s-elementary frame wavelets is indeed path-connected. This result can be generalized (by using a similar argument and some results from [2]) to s-elementary frame wavelets in higher dimensional cases with arbitrary expansive matrix dilations. This is done in Section 3. In the last section, we discuss the uniform path-connectivity of the sets of frame wavelets, normalized tight frame wavelets and s-elementary frame wavelets. We prove that none of these sets is uniformly path-connected.

We need to point out that our result only applies to the s-elementary frame wavelets so defined here. Since the result of [8] can easily be extended to the set of all MSF-wavelets, it is natural to ask whether our method can be extended to show the path-connectedness of the set of all frame wavelets whose Fourier transforms are supported on frame sets (the term MSF no longer applies in the frame wavelet case, since the support of a frequency frame wavelet can be as small as possible). At this time, we are unable to do this. Again, the major obstacle is the lack of a characterization of frequency frame wavelets.

2. Basic concepts and lemmas

Throughout this paper, we only deal with subsets of $\mathbb{R}$ that are Lebesgue measurable. Thus, in all lemmas and theorems, it is understood that all sets involved are Lebesgue measurable. Most definitions and the proofs of the lemmas in this section can be found in [1] and [2]. Please refer to these two papers for the details.

Let E be a Lebesgue measurable set in $\mathbb{R}$. A point $x \in E$ is said to have a dilation index $\delta_E(x) = k$ if there are exactly k points in the set $E \cap (\bigcup_{n\in\mathbb{Z}} 2^n x)$. For each fixed natural number k, the set $E(\delta, k) = \{x \in E : \delta_E(x) = k\}$ is Lebesgue measurable. Furthermore, each $E(\delta, k)$ is a disjoint union of k measurable sets $\{E^j(\delta, k) : 1 \le j \le k\}$ such that $\delta_{E^j(\delta,k)}(x) = 1$ for each point $x \in E^j(\delta, k)$. Similarly, a point $x \in E$ is said to have a 2π-translation index $\tau_E(x) = k$ if there are exactly k points in the set $E \cap (\bigcup_{n\in\mathbb{Z}}(2\pi n + x))$. For each fixed natural number k, the set $E(\tau, k) = \{x \in E : \tau_E(x) = k\}$ is Lebesgue measurable and is a disjoint union of k measurable sets $\{E^j(\tau, k) : 1 \le j \le k\}$ such that $\tau_{E^j(\tau,k)}(x) = 1$ for each point $x \in E^j(\tau, k)$. If there is a number M such that $E(\delta, k)$ and $E(\tau, k)$ are null sets for all $k > M$, the set E is called a *basic set.*

Assume further that E is of finite measure. For any $f \in L^2(\mathbb{R})$, define

$$(H_E f)(s) = \sum_{n,\ell\in\mathbb{Z}} \left\langle f, \widehat{D}^n \widehat{T}^\ell \frac{1}{\sqrt{2\pi}}\chi_E \right\rangle \widehat{D}^n \widehat{T}^\ell \frac{1}{\sqrt{2\pi}}\chi_E(s). \tag{2}$$

A set E is called a *Bessel set* if $H_E f$ converges in norm unconditionally for each $f \in L^2(\mathbb{R})$ and $\langle H_E f, f\rangle \le B\|f\|^2$ for some constant $B > 0$. Theorem 1 of [1] implies the following lemma.

Lemma 1. *A set E is Bessel if and only if it is a basic set. Moreover, if $\mu(E(\delta, m)) = \mu(E(\tau, m)) = 0$ for all $m > M$ (where μ is the Lebesgue measure), then $\langle H_E f, f\rangle \le M^{5/2}\|f\|^2$ for any $f \in L^2(\mathbb{R})$.*

On the other hand, the same argument used in the proof of Theorem 2 in [1] leads us to the following lemma.

Lemma 2. *Let E be a basic set. Assume that $\Omega = \bigcup_{k\in\mathbb{Z}} 2^k E(\tau, 1) = \bigcup_{k\in\mathbb{Z}} 2^k E$. Then*

$$\langle H_E f, f\rangle \ge \|f\|^2, \quad \forall f \in L^2(\mathbb{R}), \mathrm{supp}(f) \subset \Omega.$$

Lemma 3 below is obtained by using Lemma 1 and Lemma 2.

Lemma 3. *Let E be a basic set and $E(\tau, m) = E(\delta, m) = \emptyset, \forall m > M$. Let F be a measurable set such that $E \subset \bigcup_{k\in\mathbb{Z}} 2^k F$ and $F = F(\tau, 1)$. Then*

$$\langle H_E f, f\rangle \le M^{5/2}\langle H_F f, f\rangle, \quad \forall f \in L^2(\mathbb{R}).$$

Proof. Define $\Omega = \bigcup_{k\in\mathbb{Z}} 2^k F$ and $\Omega_1 = \mathbb{R}\backslash\Omega$. Let $f \in L^2(\mathbb{R})$. Define $f_1 = f\chi_\Omega$ and $f_2 = f\chi_{\Omega_1}$. Then we have $H_E f_2 = H_F f_2 = 0$ and $\langle H_E f_1, f_2\rangle = \langle H_F f_1, f_2\rangle = 0$. Hence $\langle H_E f, f\rangle = \langle H_E f_1, f_1\rangle \le M^{5/2}\|f_1\|^2$ by Lemma 1, and $\langle H_F f, f\rangle = \langle H_F f_1, f_1\rangle \ge \|f_1\|^2$ by Lemma 2. The result follows. $\square$

For any $E \subset \mathbb{R}$, let $\tau(E) = \bigcup_{\mathbb{Z}}(E + 2k\pi)$. Be careful not to confuse $\tau(E)$ with $\tau_E(x)$, the translation index of x in E. We say that two sets E and F are 2π-translation disjoint if $\tau(E) \cap \tau(F) = \emptyset$. The following lemma is obtained from Lemma 5 of [1].

Lemma 4. *If E and F are 2π-translation disjoint basic sets, then*

$$H_{E\cup F} f = H_E f + H_F f, \quad \forall f \in L^2(\mathbb{R}).$$

It is well-known that if $\psi = \psi_E$, then (1) is equivalent to

$$a\|f\|^2 \le \sum_{n,\ell\in\mathbb{Z}} \left|\left\langle f, \widehat{D}^n \widehat{T}^\ell \frac{1}{\sqrt{2\pi}}\chi_E \right\rangle\right|^2 \le b\|f\|^2. \tag{3}$$

Combining this with (2), we get

Lemma 5. *For $\psi = \psi_E$, (1) is equivalent to*

$$a\|f\|^2 \leq \langle H_E f, f\rangle \leq b\|f\|^2, \quad \forall f \in L^2(\mathbb{R}). \tag{4}$$

Finally, we will need the following lemma in proving our main theorem in the next section. This lemma is the one-dimensional case of Theorem 4 in [2].

Lemma 6. *The family of normalized tight s-elementary frame wavelets is path-connected under the $L^2(\mathbb{R})$ norm.*

3. The main theorem and its proof

In this section we prove our main result.

Theorem 1. *The family of s-elementary frame wavelets is path-connected under the $L^2(\mathbb{R})$ norm.*

Proof. We will prove that for a given frame wavelet set E, there is a continuous path of the form χ_{W_t} connecting χ_E to χ_F, where each W_t is a frame wavelet set and F is a normalized tight frame set. This implies that each s-elementary frame wavelet is connected by a continuous path (of s-elementary frame wavelets) to a normalized tight s-elementary frame wavelet. This in turn implies the theorem, by Lemma 6.

Let E be a frame wavelet set and ψ_E the corresponding s-elementary frame wavelet. E is a Bessel set, hence a basic set by Lemma 1. So there is a number M such that $E(\tau, m) = E(\delta, m) = \emptyset, \forall m > M$. Thus we can choose $B = M^{5/2}$ in (1) by Lemma 1. Let $a > 0$ be the lower frame bound of ψ_E. We have $a\|f\|^2 \leq \langle H_E f, f\rangle \leq M^{5/2}\|f\|^2$ for all $f \in L^2(\mathbb{R})$ by Lemma 5. Let m_0 be a positive integer large enough so that $M/2^{m_0} < 1/4$. Let

$$F = \left[-\frac{2\pi}{2^{m_0+1}}, -\frac{\pi}{2^{m_0+1}}\right) \cup \left[\frac{\pi}{2^{m_0+1}}, \frac{2\pi}{2^{m_0+1}}\right).$$

By Corollary 3 of [1], the set F is a normalized tight frame set. It is left to the reader to verify that $E \cup F$ is a basic set and every measurable subset of $E \cup F$ is a basic set.

For any $s \in E$, there is a unique integer $k(s)$ such that $s/2^{k(s)} \in F$. Thus $h(s) = s/2^{k(s)}$ defines a mapping from E to F. We leave it to our reader to prove that the image of each measurable subset in E under h is measurable. Furthermore, if E' is a subset of $E \cap \mathbb{R}\backslash[-\pi, \pi]$, then $\mu(h(E')) < \frac{1}{2^{m_0+1}}\mu(E')$. Define

$$\begin{aligned}
F_t^0 &= \left[-\frac{2\pi}{2^{m_0+1}}, -\frac{(2-t)\pi}{2^{m_0+1}}\right] \cup \left[\frac{\pi}{2^{m_0+1}}, \frac{(1+t)\pi}{2^{m_0+1}}\right], \\
F_t^1 &= h(\tau(F_t^0) \cap (E\backslash F_t^0)), \\
F_t^2 &= h(\tau(F_t^1) \cap (E\backslash F_t^1)), \\
&\cdots \\
F_t^n &= h(\tau(F_t^{n-1}) \cap (E\backslash F_t^{n-1})), \\
&\cdots \\
F_t &= \bigcup_{k\geq 0} F_t^k, t \in [0, 1].
\end{aligned}$$

Notice that the set F_t is a measurable subset of F; hence it is a basic set. Let $E_t = \tau(F_t) \cap E$. It is clear that any point in $\tau(E_t)$ must be in $\tau(F_t)$, hence cannot be in $\tau(E \backslash E_t)$. So the sets E_t and $E \backslash E_t$ are 2π-translation disjoint. By Lemma 4 we have

$$H_E f \quad = \quad H_{E_t} f + H_{E \backslash E_t} f.$$

Hence

$$\langle H_E f, f \rangle \quad = \quad \langle H_{E_t} f, f \rangle + \langle H_{E \backslash E_t} f, f \rangle \geq a \|f\|^2. \tag{5}$$

Similarly,

$$H_{F_t \cup (E \backslash E_t)} f \quad = \quad H_{F_t} f + H_{E \backslash E_t} f,$$

since F_t and $E \backslash E_t$ are also 2π-translation disjoint. It follows that

$$\langle H_{F_t \cup (E \backslash E_t)} f, f \rangle \quad = \quad \langle H_{F_t} f, f \rangle + \langle H_{E \backslash E_t} f, f \rangle. \tag{6}$$

Notice that $F_t = F_t(\tau, 1)$, since $F_t \subset F$ and $F = F(\tau, 1)$. Let $x \in E_t = E \cap \tau(F_t)$. If $x \notin F_t$, then $x \in \tau(F_t^n) \cup (E \backslash F_t^n)$ for some $n \geq 0$. So $h(x) \in F_t^{n+1} \subset F_t$. Hence we have

$$E_t \subset \bigcup_{k \in \mathbb{Z}} 2^k F_t. \tag{7}$$

By Lemma 3 we have

$$\langle H_{F_t} f, f \rangle \quad \geq \quad M^{-\frac{5}{2}} \langle H_{E_t} f, f \rangle. \tag{8}$$

Now define $W_t = F_t \cup (E \backslash E_t)$. Since $W_t \subset F \cup E$, it is a basic set. By Lemma 1, there is a positive number B (independent of t) such that

$$\langle H_{W_t} f, f \rangle \leq B \|f\|^2, \quad \forall f \in L^2(\mathbb{R}). \tag{9}$$

On the other hand, (5), (6) and (8) imply that

$$\begin{aligned}
\langle H_{W_t} f, f \rangle &= \langle H_{F_t} f, f \rangle + \langle H_{E \backslash E_t} f, f \rangle \\
&\geq M^{-\frac{5}{2}} \langle H_{E_t} f, f \rangle + \langle H_{E \backslash E_t} f, f \rangle \\
&> M^{-\frac{5}{2}} \big(\langle H_{E_t} f, f \rangle + \langle H_{E \backslash E_t} f, f \rangle \big) \\
&\geq a M^{-\frac{5}{2}} \|f\|^2.
\end{aligned}$$

Therefore, W_t is a frame wavelet set for each $t \in [0, 1]$. It is easy to verify that $W_0 = E$ (since $F_0 = E_0 = \emptyset$) and $W_1 = F \cup (E \backslash \tau(F))$. Notice that F, $E \backslash \tau(F)$ are 2π-translation disjoint. Thus, by Lemma 2, for any measurable subset G of $E \backslash \tau(F)$, $F \cup G$ is a frame set since $\bigcup_{k \in \mathbb{Z}} 2^k F = \mathbb{R}$. In particular, if we let $G_t = (-\tan(\frac{\pi}{2} t), \tan(\frac{\pi}{2} t)) \cap (E \backslash \tau(F))$, then $F \cup G_t$ is a frame set. We leave it to our reader to verify that the mapping $t \longrightarrow \chi_{F \cup G_t}$ is continuous in norm. Since $G_0 = \emptyset$ and $G_1 = E \backslash \tau(F)$, this defines a continuous path from χ_F to χ_{W_1}. Therefore, to complete the proof of Theorem 1, it suffices to show that the mapping $t \longrightarrow \chi_{W_t}$ is continuous in norm. We will achieve this in a few steps.

Step 1: We first show that the mapping $t \to \chi_{F_t}$ is continuous in norm. For $0 \leq t \leq 1$, we have $\mu(F_t^0) \leq \pi / 2^{m_0}$. By the property of E, for a point $s \in F_t^0$, the set $\{s + 2k\pi : k \in \mathbb{Z}\} \cap E$ has at most M points. This implies that

$$\mu(\tau(F_t^0) \cap (E \backslash F_t^0)) \quad \leq \quad M \mu(F_t^0). \tag{10}$$

Since $\tau(F_t^0)\cap(E\backslash F_t^0)\subset\mathbb{R}\backslash[-\pi,\pi]$, it follows from (10) that

$$\begin{aligned}\mu(F_t^1) &\le \frac{1}{2^{m_0}+1}\mu(\tau(F_t^0)\cap(E\backslash F_t^0))\\ &\le \frac{M}{2^{m_0+1}}\mu(F_t^0)\le\frac{1}{4}\mu(F_t^0).\end{aligned}$$

By induction, we have

$$\mu(F_t^n)\le\frac{M}{2^{m_0+1}}\mu(F_t^{n-1})\le\frac{1}{4^n}\mu(F_t^0). \tag{11}$$

Therefore, the convergence of $\chi_{\cup_{0\le k\le n}F_t^k}$ to χ_{F_t} is uniform with respect to $t\in[0,1]$. For any $\epsilon>0$, choose $N>0$ large enough so that $\pi/4^N<\epsilon/4$; then for any $t\in[0,1]$, we have

$$|\chi_{\cup_{0\le k\le N}F_t^k}-\chi_{F_t}|\le\sum_{k>N}\frac{1}{4^k}\mu(F_t^0)\le\frac{\mu(F_t^0)}{4^N}<\frac{\pi}{4^N}<\frac{\epsilon}{4},$$

since $\mu(F_t^0)\le\pi$ for any t. If the mapping $t\to\chi_{F_t^n}$ is continuous in norm for each n, then $\chi_{\cup_{0\le k\le N}F_t^k}$ is uniformly continuous on $[0,1]$. Thus, there exists $\delta(\epsilon)>0$ such that $|\chi_{\cup_{0\le k\le N}F_{t_2}^k}-\chi_{\cup_{0\le k\le N}F_{t_1}^k}|<\epsilon/2$ whenever $|t_2-t_1|<\delta(\epsilon)$. It follows that

$$\begin{aligned}|\chi_{F_{t_2}}-\chi_{F_{t_1}}| &\le |\chi_{\cup_{0\le k\le N}F_{t_2}^k}-\chi_{\cup_{0\le k\le N}F_{t_1}^k}|\\ &\quad+|\chi_{\cup_{0\le k\le N}F_{t_2}^k}-\chi_{F_{t_2}}|+|\chi_{\cup_{0\le k\le N}F_{t_1}^k}-\chi_{F_{t_1}}|\\ &\le\frac{\epsilon}{2}+\frac{\epsilon}{4}+\frac{\epsilon}{4}=\epsilon.\end{aligned}$$

That is, χ_{F_t} is also uniformly continuous on $[0,1]$. Therefore, it suffices for us to prove that the mapping $t\to\chi_{F_t^n}$ is continuous in norm for each n. We will prove this by induction. Clearly, the mapping $t\to\chi_{F_t^0}$ is continuous. Assume that it is true for n. We will show that it is true for $n+1$. For this purpose, we write $K\Delta L=(K\backslash L)\cup(L\backslash K)$ for any sets K and L, and let $D_t^n=\tau(F_t^n)\cap(E\backslash F_t^n)$. For any $t,t'\in[0,1]$, we claim that $D_t^n\Delta D_{t'}^n\subset\tau(F_t^n\Delta F_{t'}^n)\cap E$. Let $s\in D_t^n\Delta D_{t'}^n$. We can assume that $s\in D_t^n\backslash D_{t'}^n$. Then there is an integer k such that $s+2k\pi\in F_t^n$. However, $s\notin F_t^n$. It follows that $k\neq0$. Thus $s\notin F_{t'}^n$, for otherwise we would have both s and $s+2k\pi\in F_{t'}^n\cup F_t^n\subset F\subset[-\pi,\pi)$, which is impossible since $k\neq0$. Therefore $s\in E\backslash F_{t'}^n$. Since $s\notin D_{t'}^n=\tau(F_{t'}^n)\cap(E\backslash F_{t'}^n)$, it follows that $s\notin\tau(F_{t'}^n)$. Hence $s+2k\pi\in F_t^n\Delta F_{t'}^n$, and therefore $s\in\tau(F_t^n\Delta F_{t'}^n)\cap E$, as expected.

We now have

$$F_t^{n+1}\Delta F_{t'}^{n+1}\subset h(D_t^n\Delta D_{t'}^n)\subset h(\tau(F_t^n\Delta F_{t'}^n)\cap E). \tag{12}$$

Therefore,

$$\begin{aligned}\mu(F_t^{n+1}\Delta F_{t'}^{n+1}) &\le \mu(h((F_t^n\Delta F_{t'}^n)^+\cap E))\\ &\le \frac{M}{2^{m_0+1}}\mu(F_t^n\Delta F_{t'}^n). \qquad (13)\end{aligned}$$

(13) implies that the mapping $t\to\chi_{F_t^{n+1}}$ is continuous, since the mapping $t\to\chi_{F_t^n}$ is. This completes the proof that the mapping $t\to\chi_{F_t^n}$ is continuous in norm for all n. Hence the mapping $t\to\chi_{F_t}$ is continuous, as claimed.

Step 2: We now show that the mapping $t\to\chi_{E_t}$ is also continuous. In fact, this follows from the inclusion $E_t\Delta E_{t'}\subset\tau(F_t\Delta F_{t'})\cap E$, which implies that

$$\mu(E_t\Delta E_{t'})\le\mu(\tau(F_t\Delta F_{t'})\cap E)\le M\mu(F_t\Delta F_{t'}).$$

Step 3: Finally, the continuity of $t \to \chi_{W_t}$ follows from the continuity of the mappings $t \to \chi_{F_t}$ and $t \to \chi_{E\backslash E_t}$ and the fact that $F_t \cap (E\backslash E_t) = \emptyset$. This completes our proof of Theorem 1. □

In [1], it is shown that the frame bound of an s-elementary tight frame wavelet is a positive integer. If we use S_f to denote the set of all s-elementary frame wavelets, $S_f(j)$ to denote the set of all s-elementary tight frame wavelets of frame bound $j \geq 1$ (so $S_f(1)$ is the set of all s-elementary normalized tight frame wavelets), then it is not hard to see that each $S_f(j)$ is path-disconnected from $\bigcup_{k\neq j} S_f(k)$. For $j \neq 1$, it remains unclear whether $S_f(j)$ is path-connected. This situation is illustrated in Figure 1.

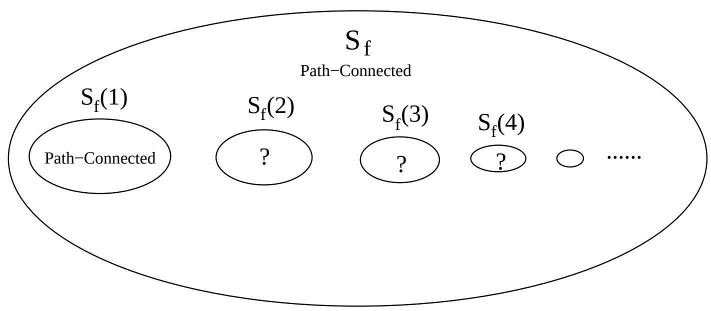

FIGURE 1. Illustration of the path-connectedness of s-elementary frame wavelets.

We now point out that Theorem 1 is also valid for higher dimensional cases. Let A be a $d \times d$ *expansive* matrix, that is, all eigenvalues of A have norm greater than 1. Let D_A be the unitary operator defined by $D_A f(s) = |\det A|^{\frac{1}{2}} f(As)$ and T_ℓ the unitary operator defined by $T_\ell f(s) = f(s-\ell)$, where $f \in L^2(\mathbb{R}^d)$ and $\ell \in \mathbb{Z}^d$. A function $\psi \in L^2(\mathbb{R}^d)$ is called an A-dilation frame wavelet for $L^2(\mathbb{R}^d)$ if there exist two positive constants $0 < a \leq b$ such that for any $f \in L^2(\mathbb{R}^d)$,

$$a\|f\|^2 \leq \sum_{n\in\mathbb{Z},\ell\in\mathbb{Z}^d} |\langle f, D_A^n T^\ell \psi\rangle|^2 \leq b\|f\|^2. \tag{14}$$

The Fourier-Plancherel transform $\mathcal{F}$ on $L^2(\mathbb{R}^d)$ is a unitary operator such that for $f \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$,

$$(\mathcal{F}f)(s) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i(s\circ t)} f(t)dm,$$

where $s \circ t$ denotes the real inner product. If the function $\psi_E \in L^2(\mathbb{R}^d)$ defined by $\widehat{\psi} = \frac{1}{(2\pi)^{d/2}}\chi_E$ for some measurable set E in $\mathbb{R}^d$ is a frame wavelet for $L^2(\mathbb{R}^d)$, then the function ψ_E is called an *s-elementary A-dilation frame wavelet*. We have

Theorem 2. *The family of s-elementary A-dilation frame wavelets is path-connected in the $L^2(\mathbb{R}^d)$ norm.*

4. Local commutant and uniform connectivity

Let ψ be a fixed orthonormal wavelet. The local commutant [3] at ψ is the set

$$C_\psi(D,T) = \{A \in B(L^2(\mathbb{R})) : AD^nT^m\psi = D^nT^mA\psi\}.$$

For each frame wavelet η, there is a unique operator $U_\eta \in C_\psi(D,T)$ such that $U_\eta\psi = \eta$, and U_η^* is injective and has closed range. Moreover, η is an orthonormal wavelet if and only if U_η is unitary, while η is a normalized tight frame wavelet if and only if U_η^* is an isometry [5].

Two frame wavelets η_0 and η_1 are said to be *uniformly path-connected* if there is a path of frame wavelets $\{\eta_t : t \in [0,1]\}$ such that U_{η_t} is a continuous path in the operator norm (and hence $\{\eta_t : t \in [0,1]\}$ is a continuous path in the L^2-norm). The uniform connectivity for certain classes of wavelets is related to the interpolation theory of wavelets and was investigated in several papers (cf. [3], [4]). We will prove that the path-connectedness of s-elementary frame wavelets cannot be strengthened to uniform path-connectedness. In fact, we will prove that the set of frame wavelets and the set of normalized tight frame wavelets are not uniformly path-connected either. We need the following simple lemma.

Lemma 7. *Let U be a unitary operator. If V is an isometry such that $\|U-V\| < 1$, then it must be unitary.*

Proof. Write $V = U + (V-U) = U(I + U^*(V-U))$. Since $\|U^*(V-U)\| \le \|V-U\| < 1$, it follows that $(I + U^*(V-U))$ is invertible. Thus V is invertible and hence unitary. $\square$

Theorem 3. *None of the following sets is uniformly path-connected:*
(i) *The set of all frame wavelets.*
(ii) *The set of all normalized tight frame wavelets.*
(iii) *The set of all s-elementary frame wavelets.*

Proof. We will only prove that set (i) is not uniformly path-connected. The other two cases are similar. Let η_0 be a Riesz wavelet (i.e., $\{D^nT^\ell\eta : n.\ell \in \mathbb{Z}\}$ is a Riesz basis for $L^2(\mathbb{R})$), and η_1 a frame wavelet which is not a Riesz wavelet. We claim that η_0 and η_1 can never be uniformly path-connected. In fact, if there exist $\{\eta_t : t \in [0,1]\}$ such that $\{U_{\eta_t}\}$ is a continuous path in the operator norm, write $U(t) = U_{\eta_t}$ and $S(t) = U(t)U(t)^*$, then it is obvious that $S(t)$ is also continuous in the operator norm. Since η_t is a frame wavelet, it follows that $S(t)$ (which is refereed as the *frame operator* in the literature; cf. [5]) is invertible for all t. By the continuity of the inverse operation, we have that $S(t)^{-1/2}$ must be continuous.

From the polar decomposition of $U(t)$, we have that $V(t) = U(t)^*S(t)^{-1/2}$ is an isometry for each t. Therefore $V(t)$ is a continuous path (in the operator norm) consisting of isometries. Since η_0 is a Riesz wavelet and η is not, we have that $V(0)$ is unitary but $V(1)$ is an isometry which is not unitary. This implies that $V(t)$ is a continuous path connecting a unitary and a non-unitary isometry, and contradicts Lemma 7. $\square$

Acknowledgement

The authors thank the referee for pointing out an error in an earlier version of the paper.

References

[1] X. Dai, Y. Diao and Q. Gu, *Frame Wavelet Sets in* $\mathbb{R}$, Proc. of Amer. Math. Soc., Vol **129** (2001), no. 7 , 2045-2055. MR **2002a:**46006

[2] X. Dai, Y. Diao, Q. Gu and D. Han, *Frame Wavelets in Subspaces of* $L^2(\mathbb{R}^d)$, Proc. of Amer. Math. Soc., Vol **130** (2002), no. 11: 3259–3267. MR **2003f:**42052

[3] X. Dai and D. Larson, *Wandering vectors for unitary systems and orthogonal wavelets*, Memoirs Amer. Math. Soc., **134** (1998), no. 640. MR **98m:**47067

[4] Q. Gu, *On interpolation families of wavelet sets*, Proc. of Amer. Math. Soc., Vol **128** (2000), no. 10, 2973-2979. MR **2000m:**42026

[5] D. Han and D. Larson, *Frames, bases and group representations,* Memoirs Amer. Math. Soc., **147** (2000), no. 697. MR **2001a:**47013

[6] Rufeng Liang, *Wavelets, their phases, multipliers and connectivity,* Ph.D. Thesis, December, 1998. University of North Carolina-Charlotte.

[7] M. Paluszynski, H. Sikic, G. Weiss and S. Xiao, *Tight frame wavelets, their dimension functions, MRA tight frame wavelets and connectivity properties,* Adv. in Comp. Math. **18** (2003), 297-327. MR **2004b:**42087

[8] D. Speegle, *The s-elementary wavelets are path-connected,* Proc. Amer. Math. Soc., Vol **127** (1999), no. 1, 223-233. MR **99b:**42045

[9] Wutam Consortium, *Basic Properties of Wavelets,* Journal of Fourier Analysis and Applications, Vol **4**, no. 4–5 (1998), 575-594. MR **99i:**42056

Department of Mathematics, University of North Carolina at Charlotte, Charlotte, North Carolina 28223

E-mail address: xdai@uncc.edu

Department of Mathematics, University of North Carolina at Charlotte, Charlotte, North Carolina 28223

Department of Mathematics, East China Normal University, Shanghai, People's Republic of China 200062

Department of Mathematics, University of Central Florida, Orlando, Florida 32816