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ON THE SINGULAR BRAID MONOID OF AN ORIENTABLE SURFACE

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ABSTRACT. In this paper we show that the singular braid monoid of an orientable surface can be embedded in a group. The proof is purely topological, making no use of the monoid presentation.

1. Introduction

The aim of this paper is proving, in an easy and fully topological way, that the singular braid monoid of an orientable surface can be embedded into a group.

The singular braid monoids appear naturally when studying braid groups of surfaces. They were introduced in [1] and [4] and they arise in some situations connected with the Vassiliev knot invariants ([2]).

A particular case of our theorem (that of the open disk) was proved by Fenn, Keyman and Rourke in [6], where it was pointed out that all the topological difficulties of the proof relied on the so-called diamond lemma. Their proof, although purely topological, was both quite involved and not adaptable (at least in an easy way) for general orientable surfaces. On the other hand, our approach results in a simpler proof, even for the open disk case.

An algebraic generalization of this result can be found in [3], which in fact may include the surface case if applied together with the results of [7] concerning the monoid presentation. However, we think that a topological and more down–to–earth proof of this result may shed some light and contribute towards a better understanding of the subject.

A final comment is in order: a different proof of this result, also based in the monoid presentation, has been announced to us by Bellingeri in a private communication.

2. The singular braid monoid and the singular braid group

Let U be any orientable surface. If we fix beforehand n distinct points $P_1, ..., P_n \in U$, a geometric braid is a set of n disjoint paths $b_1, ..., b_n$ inside the cylinder $U \times [0,1]$, such that b_i (also called the ith string) goes, monotonically in $t \in [0,1]$, from $(P_i, 0)$ to $(P_j, 1)$.

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A singular geometric braid is just the same as a geometric braid, except for the fact that we allow a finite number of transversal intersections between two strings. The intersection points will be called singular points.

If we consider braids modulo isotopies of $U \times [0,1]$ leaving fixed $U \times \{0,1\}$, we get the singular braid monoid (of U), noted SB_n, with the operation defined by concatenation.

We will note by $\alpha, \beta, \gamma, \ldots$ the geometric braids and by $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \ldots$ their corresponding classes in SB_n.

When U is the open disk D, SB_n is a very well-known monoid ([5]), generated by the braids $\widehat{\sigma_i}, \widehat{\sigma_i^{-1}}, \widehat{\tau_i}$, where $\sigma_i, \sigma_i^{-1}, \tau_i$ are the geometric braids shown in Figure 1. We will also speak of $\widehat{\sigma_i}, \widehat{\sigma_i^{-1}}, \widehat{\tau_i}$ in the general case, by means of the natural embedding of the singular braid monoid of the disk into SB_n.

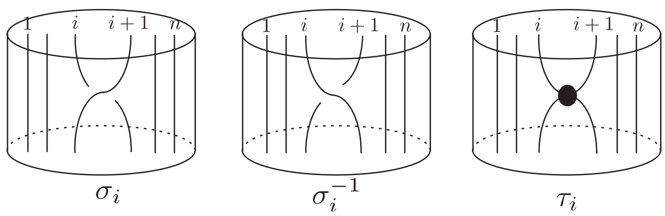

FIGURE 1.

In what follows, we will draw braids only in the cylinder $D \times [0,1]$ (for the sake of simplicity), although our arguments will work for $U \times [0,1]$.

We will introduce now the analogous setup to that of Fenn, Keyman and Rourke in [6] for the disk case.

We will call M the monoid where the elements are geometric singular braids (modulo isotopies) in which every singular point is assigned a colour, black or white.

We will then note by $\alpha, \beta, \gamma, \ldots$ the geometric braids (with coloured singular points) and by $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \ldots$ their corresponding classes in M.

Remark 2.1. There is a surjection from M to SB_n that consists of forgetting the colours of the singular points. On the other hand, SB_n can be embedded in M by assigning the black colour to every singular point. The image of $\widehat{\tau_i}$ under this injection will be noted also by $\widehat{\tau_i}$, while we will note by υ_i the braid obtained by assigning the white colour to the singular point of τ_i. We will call $\widehat{\upsilon_i}$ the opposite of $\widehat{\tau_i}$ (see Figure 2).

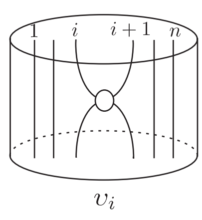

FIGURE 2.

Without explicit mention we will consider SB_n as a submonoid of M, using the above injection.

If we add to M the relations $\widehat{\tau_i}\widehat{\upsilon_i} = \widehat{\upsilon_i}\widehat{\tau_i} = 1$, we obtain a group, called the singular braid group (of U), noted SG_n (see Figure 3).

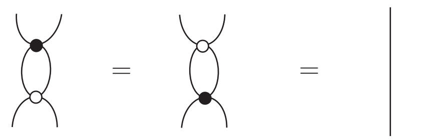

FIGURE 3. The relations $\hat{\tau}_i\hat{\upsilon}_i = \hat{\upsilon}_i\hat{\tau}_i = 1$ in SG_n

From now on, if a singular braid has the form, say $\widehat{\alpha} = \widehat{\alpha_1\tau_i\alpha_2}$, and we do not care about who α_1 and α_2 are, we will draw it as in Figure 4.

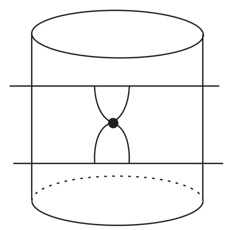

FIGURE 4.

3. THE EMBEDDING THEOREM

Theorem 3.1. *The natural map $SB_n \longrightarrow SG_n$ is one–to–one.*

Before proving the result we will introduce some notation. For α, β geometric singular braids, we will write $\alpha \nearrow \beta$ if there exist α_1, β_1 such that $\widehat{\alpha} = \widehat{\alpha_1}$, $\widehat{\beta} = \widehat{\beta_1}$ and we can obtain β_1 from α_1 by adding a pair of consecutive opposite singular points. In the same way, we will write $\alpha \searrow \beta$ if there exist α_1, β_1 such that $\widehat{\alpha} = \widehat{\alpha_1}$, $\widehat{\beta} = \widehat{\beta_1}$ and we can obtain β_1 from α_1 by erasing a pair of consecutive opposite singular points.

Remark 3.2. Note that $\widehat{\alpha}$ and $\widehat{\beta}$ define the same element on SG_n if and only if there exist $\alpha = \alpha_0, \alpha_1, ..., \alpha_k = \beta$ such that either $\alpha_i \nearrow \alpha_{i+1}$ or $\alpha_i \searrow \alpha_{i+1}$ for all $i = 0, ..., k-1$.

The following result is the key part of the proof of the theorem, as noted in [6], where it is proved for the case $U = D$. The proof will be given in the following section.

Lemma 3.3 (Diamond lemma)**.** *Let α, β, γ be geometric singular braids such that $\alpha \nearrow \beta \searrow \gamma$. Then, either $\widehat{\alpha} = \widehat{\gamma}$, or there exists η such that $\alpha \searrow \eta \nearrow \gamma$.*

Definition 3.4. We will say that $\widehat{\alpha} \in M$ is irreducible if there is no β with $\alpha \searrow \beta$.

Corollary 3.5. *If $\widehat{\alpha}, \widehat{\beta} \in M$ are irreducible and define the same element in SG_n, then $\widehat{\alpha} = \widehat{\beta}$ in M.*

The proof is immediate from the diamond lemma (see [6]) and, besides, since the elements of SB_n are irreducible (they only have black singular points), this proves that the map from SB_n to SG_n is an embedding. Hence the theorem is proved, up to the diamond lemma.

4. The diamond lemma

Given a geometric singular braid β, with a singular point p on it, we will denote by $\beta(p^+)$ and $\beta(p^-)$ the braid obtained from β by replacing p by a positive and negative crossing, respectively. That is, if $\beta = \beta_1\tau_i\beta_2$ (where p is the singular point in τ_i), we will have $\beta(p^+) = \beta_1\sigma_i\beta_2$ and $\beta(p^-) = \beta_1\sigma_i^{-1}\beta_2$. In a similar way, if we have $p_1, ..., p_m$ singular points on β, we will write $\beta(p_1^{s_1}, ..., p_m^{s_m})$ for the braid obtained by replacing each p_i by a crossing with sign $s_i \in \{+,-\}$.

If we have a singular braid β, with a singular point p on it, we will write $\beta(p^\bullet)$ and $\beta(p^\circ)$ for the braid obtained from β by replacing p (no matter which colour it has) by a black point and a white point, respectively.

Lemma 4.1. *Let β and β' be geometric singular braids, p a singular point on β. Assume $\widehat{\beta} = \widehat{\beta'}$, and let us call also p the point corresponding to p in β'. Then:*

(1) $\widehat{\beta(p^+)} = \widehat{\beta'(p^+)}$;

(2) $\widehat{\beta(p^-)} = \widehat{\beta'(p^-)}$.

Proof. Since both cases are analogous, we will do the first case only. Since $\widehat{\beta} = \widehat{\beta'}$, there exists an isotopy of the cylinder, H_t, with $H_0(\beta) = \beta$, $H_1(\beta) = \beta'$. Take a sphere S centered at p with radius small enough such that the only strings intersecting S are those forming the singular point (see Figure 5). We can assume that $\beta(p^+)$ and β coincide outside S.

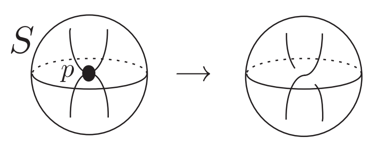

Figure 5.

We can suppose that $H_1(S)$ is also a sphere centered at p on β'. Hence, if we apply H_1 to $\beta(p^+)$, we obtain the braid β', except for the fact that the point p has been replaced by a positive crossing (since U is orientable). That is, we get $\beta'(p^+)$. So $H_1(\beta(p^+)) = \beta'(p^+)$. $\square$

Lemma 4.2. *Let $\widehat{\beta} \in M$, having two consecutive singular points, p and q. Then $\widehat{\beta(p^s, q^c)} = \widehat{\beta(p^c, q^s)}$, with $s \in \{+,-\}$, $c \in \{\bullet, \circ\}$.*

Proof. This result is a straightforward consequence of the well-known relation $\sigma_i\tau_i = \tau_i\sigma_i$ in the singular braid monoid of the disk (see Figure 6). $\square$

We can now proceed to prove the diamond lemma. As $\alpha \nearrow \beta$ we can consider, with no loss of generality, that $\alpha = \beta(p^+, q^-)$ for some pair p, q of opposite consecutive points in β. On the other side, as $\beta \searrow \gamma$, let us call β' the braid verifying $\widehat{\beta} = \widehat{\beta'}$ and, as above, $\gamma = \beta'(r^+, s^-)$ for a pair r, s of opposite consecutive points in β'.

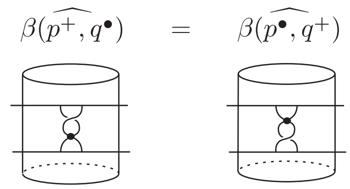

FIGURE 6.

Let us call H the cylinder isotopy taking β into β'. Then, using our previous notation, we can write $\alpha = \beta(p^+, q^-) \nearrow \beta \xrightarrow{H} \beta' \searrow \beta'(r^+, s^-) = \gamma$.

The technique used for the proof is heavily related to the proof of Lemma 4.1. In fact, we will make an extensive use of the fact that H brings together r and s, while possibly moving apart p and q, and H^{-1} acts the other way around. This fact, together with the use of the small spheres as in Lemma 4.1, will give us the result.

We will assume from now on that p is above q and that r is above s. Then we need to distinguish three cases:

(a) $p = r$, $q = s$;
(b) p, q, r, s are all distinct;
(c) $p = s$ or $q = r$.

Case (a). This is the easiest situation. Since $\widehat{\beta} = \widehat{\beta'}$, from Lemma 4.1 we have $\widehat{\beta(p^+)} = \widehat{\beta'(p^+)}$. Hence $\widehat{\alpha} = \beta(\widehat{p^+, q^-}) = \beta'(\widehat{p^+, q^-}) = \widehat{\gamma}$.

Case (b). In this case β has two pairs of opposite singular points, (p, q) and (r, s), of which (p, q) are consecutive. Hence β can be assumed to have the form $\beta_1 \tau_i \upsilon_i \beta_2$, and the analogous thing happens with β' and (r, s). See Figure 7.

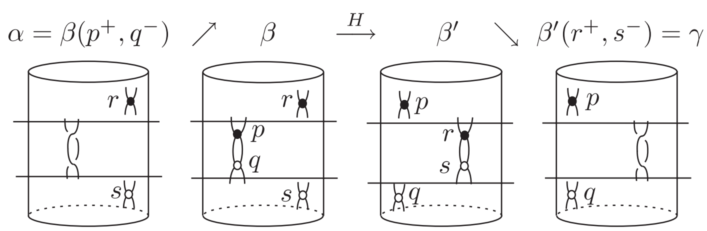

FIGURE 7. Case (b)

Then we can write

$$\alpha = \beta(p^+, q^-) \xrightarrow{H} \beta'(p^+, q^-) \searrow \beta'(p^+, q^-, r^+, s^-) \xrightarrow{H^{-1}}$$
$$\xrightarrow{H^{-1}} \beta(p^+, q^-, r^+, s^-) \nearrow \beta(r^+, s^-) \xrightarrow{H} \beta'(r^+, s^-) = \gamma.$$

This proves the existence of $\eta = \beta'(p^+, q^-, r^+, s^-)$ with $\alpha \searrow \eta \nearrow \gamma$. See Figure 8.

Case (c). We will do the subcase $q = r$, the other one being symmetric. We have the diagram shown in Figure 9.

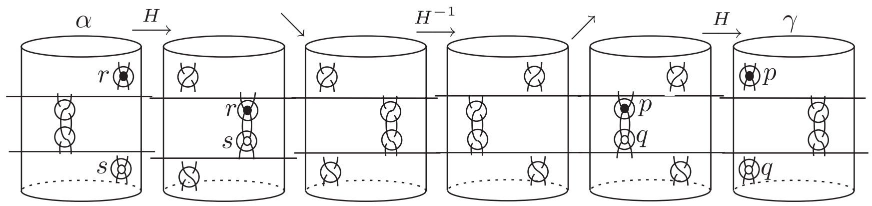

FIGURE 8.

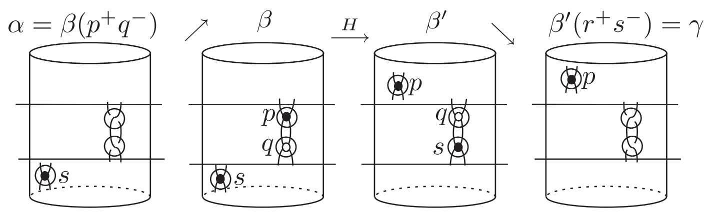

FIGURE 9. Case (c)

Then, from Lemma 4.2,

$$\alpha = \beta(p^+, q^-) \xrightarrow{H} \beta'(p^+, q^-) = \beta'(p^+, q^-, s^\bullet) \xrightarrow{\sim} \beta'(p^+, q^\bullet, s^-) \xrightarrow{H^{-1}}$$
$$\xrightarrow{H^{-1}} \beta(p^+, q^\bullet, s^-) \xrightarrow{\sim} \beta(p^\bullet, q^+, s^-) \xrightarrow{H} \beta'(p^\bullet, q^+, s^-) = \gamma,$$

where $\xrightarrow{\sim}$ stands for the (non-specified) isotopies that come from applying Lemma 4.2. This proves that $\widehat{\alpha} = \widehat{\gamma}$ and concludes the proof of the diamond lemma. See Figure 10. $\square$

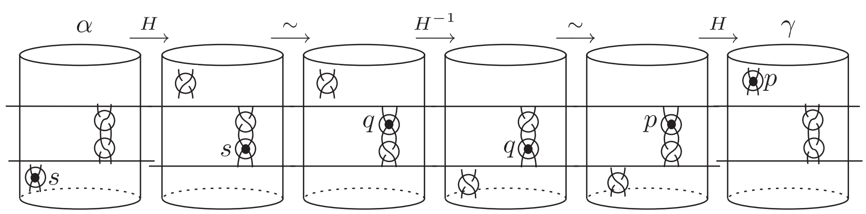

FIGURE 10.

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