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$E(2)$-INVERTIBLE SPECTRA SMASHING WITH THE SMITH-TODA SPECTRUM $V(1)$ AT THE PRIME 3

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Abstract. Let L_2 denote the Bousfield localization functor with respect to the Johnson-Wilson spectrum $E(2)$. A spectrum L_2X is called invertible if there is a spectrum Y such that $L_2X \wedge Y = L_2S^0$. Hovey and Sadofsky, *Invertible spectra in the $E(n)$-local stable homotopy category*, showed that every invertible spectrum is homotopy equivalent to a suspension of the $E(2)$-local sphere L_2S^0 at a prime $p > 3$. At the prime 3, it is shown, *A relation between the Picard group of the $E(n)$-local homotopy category and $E(n)$-based Adams spectral sequence*, that there exists an invertible spectrum X that is not homotopy equivalent to a suspension of L_2S^0. In this paper, we show the homotopy equivalence $v_2^3\colon \Sigma^{48}L_2V(1) \simeq V(1) \wedge X$ for the Smith-Toda spectrum $V(1)$. In the same manner as this, we also show the existence of the self-map $\beta\colon \Sigma^{144}L_2V(1) \to L_2V(1)$ that induces v_2^9 on the $E(2)_*$-homology.

Introduction

Let $\mathcal{S}_p$ denote the stable homotopy category of spectra localized away from the prime number p, and $E(n)$, the Johnson-Wilson spectrum such that $\pi_*(E(n)) = E(n)_* = v_n^{-1}\boldsymbol{Z}_{(p)}[v_1, \dots, v_n]$. We denote by $\mathcal{L}_n$ the full subcategory of $E(n)$-local spectra, and we have the Bousfield localization functor $L_n\colon \mathcal{S}_p \to \mathcal{L}_n \subset \mathcal{S}_p$ with respect to $E(n)$. We call a spectrum $X \in \mathcal{L}_n$ ($E(n)$-)*invertible* if there exists a spectrum $Y \in \mathcal{L}_n$ such that $X \wedge Y = L_nS^0$. In [4], Hovey and Sadofsky showed that every $E(n)$-invertible spectrum is homotopy equivalent to a suspension of L_nS^0 if $n^2 + n < 2p - 2$, and that every $E(1)$-invertible spectrum is homotopy equivalent to a suspension of L_1S^0 or L_1QM if $p = 2$. Here QM denotes the so-called question mark complex $S^0 \cup_2 e^1 \cup_\eta e^3$. In [5], Kamiya and the second author constructed an $E(2)$-invertible spectrum X such that $X \not\simeq \Sigma^k L_2S^0$ for any $k \in \boldsymbol{Z}$ and $X \wedge X \wedge X = L_2S^0$ at the prime 3. Unfortunately, we do not know whether X is an $E(2)$-localization of a finite spectrum. This case is different from the case where $p = 2$ and $n = 1$. These spectra L_1QM and X are, so far, the only known examples of $E(n)$-invertible spectra other than the sphere spectrum. For QM at the prime 2, there is a homotopy equivalence $v_1^{-2}\colon L_1V(0) \to \Sigma^4L_1V(0) \wedge QM$ (see Proposition 3.3), where $V(0)$ denotes the mod 2 Moore spectrum. This is deduced from the structure of the homotopy groups $\pi_*(L_1QM)$. We study here

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the invertible spectrum X in $\mathcal{L}_2$ at the prime 3. Consider the $E(2)$-based Adams spectral sequence $E_r^{s,t}(W)$ for a spectrum W converging to $\pi_*(L_2W)$. Then it is shown [5] that $d_5(g) = v_2^{-2}h_{11}b_{10}^2 g \in E_5^{5,4}(X)$ for the generator g of $E_2^{0,0}(X) = \boldsymbol{Z}_{(p)}$. We compute the E_∞-term $E_\infty^{*,*}(V(1)\wedge X)$, and then determine the homotopy groups $\pi_*(V(1)\wedge X)$ (see Corollary 2.13). This shows that $v_2^3 g$ detects a homotopy element $v_2^3 \in \pi_*(V(1)\wedge X)$. Furthermore, this extends to the map $\Sigma^{48}L_2V(1) \to V(1)\wedge X$.

Theorem A. *Here is a homotopy equivalence $v_2^3\colon \Sigma^{48}L_2V(1) \simeq V(1)\wedge X$.*

Here, the map v_2^k denotes a map f such that $E(2)_*(f) = v_2^k$. In the same manner as we obtain the map v_2^3, we also obtain the self-map v_2^9.

Theorem B. *There is a homotopy equivalence $v_2^9\colon \Sigma^{144}L_2V(1) \to L_2V(1)$.*

Recently, M. Behrens and S. Pemmaraju show the existence of the self-map $v_2^9\colon \Sigma^{144}V(1) \to V(1)$ [1].

The β-element $\beta_s \in {}^AE_2^2(S^0)$ for an integer $s > 0$ is defined to be the image of $v_2^s \in {}^AE_2^0(V(1))$ under the composite of the connecting homomorphisms ${}^AE_2^0(V(1)) \to {}^AE_2^1(V(0))$ and ${}^AE_2^1(V(0)) \to {}^AE_2^2(S^0)$. Here ${}^AE_r^*(W)$ denotes the E_r-term of the Adams-Novikov spectral sequence converging to $\pi_*(W)$. In [9], Oka showed how to prove the "if" part of the conjecture of Ravenel's: The element $\beta_s \in {}^AE_2^2(S^0)$ for an integer s survives to $\pi_*(S^0)$ if and only if $s \equiv 0,1,2,3,5,6$ mod 9. The "only if" part is shown in [11, Th. F]. In Oka's arguments, the self-map $v_2^9\colon \Sigma^{144}V(1) \to V(1)$ plays the principal role. Here we play the same game in $\pi_*(L_2S^0)$.

Corollary C ([13]). *The element $\beta_s \in E_2^2(S^0)$ for an integer s survives to $\pi_*(L_2S^0)$ if and only if $s \equiv 0,1,2,3,5,6$ mod 9.*

Note that it is, so far, not known whether $\beta_s \in {}^AE_2(S^0)$ for $s \equiv 3$ mod 9 survives to $\pi_*(S^0)$, even if there is the self-map $v_2^9\colon \Sigma^{144}V(1) \to V(1)$ in [1].

By the definition of β_s, Theorem A seems to indicate that if the element β_s survives to $\pi_*(L_2S^0)$, then β_{s+3} survives to $\pi_*(X)$.

Corollary D. *The element $\beta_{s+3} \in E_2^2(X)$ for an integer s survives to $\pi_*(X)$ if $s \equiv 0,1,2,5,6$ mod 9.*

Note that the non-existence of β_{s+3} for $s \equiv 4,7,8$ mod 9 follows from [11, Th. F] together with the equivalence v_2^3 in Theorem A. Here we do not conclude the case where $s \equiv 3$ mod 9 (see Remark 3.6).

In the next section, we consider the self-maps on $L_2V(1)$ using ring spectra $V(1)_k$ with $k > 1$ and show Theorem B and Corollary C. In section 2, we recall some facts on invertible spectra and show that $v_2^3 g\colon E_r^*(V(1)) \to E_r^*(V(1)\wedge X)$ is an isomorphism of spectral sequences, which induces an isomorphism of homotopy groups $\pi_*(L_2V(1)) \cong \pi_*(V(1)\wedge X)$. In the last section, we verify the equivalences $\Sigma^{-4}L_1V(0)\wedge QM = L_1V(0)$ for the invertible spectra QM at the prime 2. Then we construct a map $v_2^3 g\colon L_2V(1) \to V(1)\wedge X$ by the use of the result of the previous sections, which shows Theorem A. In the last section, we show Corollary D.

1. The self-maps on the spectrum $L_2V(1)_k$

Let $V(0)$ denote the mod 3 Moore spectrum and $V(1)_k$ be a cofiber of $\alpha^k\colon \Sigma^{4k}V(0) \to V(0)$ for the Adams map $\alpha\colon \Sigma^4V(0) \to V(0)$. Here $V(1)_1 = V(1)$ is the Smith-Toda spectrum. Then we have the cofiber sequences

$$S^0 \xrightarrow{3} S^0 \xrightarrow{i} V(0) \xrightarrow{j} V(1) \text{ and}$$
$$\Sigma^{4k}V(0) \xrightarrow{\alpha^k} V(0) \xrightarrow{i_k} V(1)_k \xrightarrow{j_k} \Sigma^{4k+1}V(0). \tag{1.1}$$

In [8, Th. 5.6], Oka showed that $V(1)_k$ is a ring spectrum for $k > 1$, in which a ring spectrum means a spectrum V equipped with a unit map $\iota\colon S^0 \to V$ and a multiplication $\mu\colon V \wedge V \to V$ such that $\mu(\iota \wedge 1_V) = 1_V = \mu(1_V \wedge \iota)$. In other words, a ring spectrum here is not assumed to satisfy the associative law. By a module spectrum, we also do not assume the associative law.

Lemma 1.2. *$V(1)$ is a $V(1)_k$-module spectrum with $\nu_k\colon V(1) \wedge V(1)_k \to V(1)$ for $k = 2, 4$.*

Proof. Consider the exact sequence

$$[V(1) \wedge V(1)_k, V(1)]_0 \xrightarrow{i_k^*} [V(1) \wedge V(0), V(1)]_0 \xrightarrow{(\alpha^k)^*} [V(1) \wedge V(0), V(1)]_{4k}$$

associated to the cofiber sequence $\Sigma^{4k}V(0) \xrightarrow{\alpha^k} V(0) \xrightarrow{i_k} V(1)_k$. It is shown in [16, Th. 6.11] that $[V(1), V(1)]_l = 0$ if $l = 8, 9, 16, 17$. Therefore, $[V(1) \wedge V(0), V(1)]_{4k} = 0$ if $k = 2, 4$, and so the $V(0)$-module structure $\nu \in [V(1) \wedge V(0), V(1)]_0$ is pulled back to a $V(1)_k$-module structure $\nu_k \in [V(1) \wedge V(1)_k, V(1)]_0$. $\square$

Let $\widetilde{i_k}$ and $\widetilde{j_k}$ be the maps in the cofiber sequence

$$\Sigma^{4k}V(1) \xrightarrow{\widetilde{\alpha^k}} V(1)_{k+1} \xrightarrow{\widetilde{i_k}} V(1)_k \xrightarrow{\widetilde{j_k}} \Sigma^{4k+1}V(1), \tag{1.3}$$

obtained from the 3×3 Lemma (Verdier's axiom), and let $i_0 = i_1 i\colon S^0 \to V(1)$ denote the inclusion to the bottom cell. Then

Lemma 1.4. *$\nu_2(i_0 \wedge 1) = \widetilde{i_1} + k\beta' i_0 j j_2$ for some $k \in \boldsymbol{Z}/3$.*

Proof. $\nu_2(i_0 \wedge 1)i_2 = \nu_2(1 \wedge i_2)(i_0 \wedge 1) = \nu(i_0 \wedge 1) = i_1$ and $\widetilde{i_1} i_2 = i_1$. It follows that $\nu_2(i_0 \wedge 1) - \widetilde{i_1} \in [V(1)_2, V(1)]_0$ is in the image of $(j_2)^*\colon [V(0), V(1)]_9 \to [V(1)_2, V(1)]_0$. Since $[V(0), V(1)]_9 = \boldsymbol{Z}/3\{\beta' i_0 j\}$ by [16, Prop. 6.9], we have the desired equation. $\square$

Lemma 1.5. *Let U be an invertible spectrum with $E(2)_*(U) = E(2)_*$, and ξ, a homotopy element of $\pi_*(V(1)_2 \wedge U)$ that induces v_2^k for $k \in \boldsymbol{Z}$ on $E(2)_*$-homology. Then ξ induces the map $\widehat{\xi}\colon V(1) \to V(1) \wedge U$ such that $E(2)_*(\widehat{\xi}) = v_2^k$.*

Proof. The map $\widehat{\xi}$ is defined as the composite $\Sigma^{|\xi|}V(1) = V(1) \wedge S^{|\xi|} \xrightarrow{1\wedge\xi} V(1) \wedge V(1)_2 \wedge U \xrightarrow{\nu_2\wedge 1} V(1) \wedge U$. Since $E(2)_*(j) = 0$, we have $E(2)_*(\nu_2(i_0 \wedge 1)) = E(2)_*(\widetilde{i_1})$ by Lemma 1.4. Then we compute $i_0^*(E(2)_*(\widehat{\xi})) = E(2)_*(\widehat{\xi} i_0) = E(2)_*((\nu_2 \wedge 1)(1 \wedge \xi)i_0) = E(2)_*((\nu_2 \wedge 1)(i_0 \wedge 1)\xi) = E(2)_*((\widetilde{i_1} \wedge 1)\xi) = \widetilde{i_1}_*(v_2^k) = v_2^k$. Noting that i_0^* is a monomorphism, we see that the lemma is proved. $\square$

For computing the homotopy groups $\pi_*(L_2W)$ for a spectrum W, we use the $E(2)$-based Adams spectral sequence $E_r^*(W)$ converging to $\pi_*(L_2W)$.

Lemma 1.6. *$v_2^{9t} \in E_2^*(L_2V(1)_3)$ for $t \in \boldsymbol{Z}$ is a permanent cycle.*

Proof. Recall [12] the spectrum C, which is defined to be a cofiber of the localization map $V(0) \to \alpha^{-1}V(0) = \mathrm{colim}_\alpha V(0)$. Then $E(2)_*(C) = E(2)_*/(3, v_1^\infty)$. Since we have a commutative diagram

$$\begin{tikzcd} V(1)_3 \arrow[r, "V_1^{j-3}"] \arrow[d, equal] & V(1)_j \arrow[r, "\pi'_j"] \arrow[d, "\iota_j"'] & V(1)_{j-3} \arrow[r, "\iota'_j"] \arrow[d, "\iota_{j-3}"] & V(1)_3 \arrow[d, equal] \\ V(1)_3 \arrow[r, "V_1^{j-2}"] & V(1)_{j+1} \arrow[r, "\pi'_{j+1}"] & V(1)_{j-2} \arrow[r, "\iota'_{j+1}"] & V(1)_3 \end{tikzcd}$$

of cofiber sequences for $j > 3$, we obtain a cofiber sequence $V(1)_3 \xrightarrow{f} C \xrightarrow{v_1^3} C$ by taking homotopy colimits. It is shown in [12] that $v_2^{9t}/v_1^3 \in E_2^*(C)$ is a permanent cycle. Furthermore, we read off from [12] that $v_1^3(v_2^{9t}/v_1^3) = 0 \in \pi_{144t}(L_2C)$, since $E_9^{s,144t+s}(L_2C) = 0$ for $s > 3$. Therefore, v_2^{9t}/v_1^3 is pulled back to $v_2^{9t} \in \pi_{144t}(V(1)_3)$. $\square$

Proof of Theorem B. By Lemma 1.6, we have a homotopy element $v_2^9 \in \pi_{144}(L_2V(1)_2)$ as the image of $v_2^9 \in \pi_*(L_2V(1)_3)$ under the map $\widetilde{i_2}\colon L_2V(1)_3 \to L_2V(1)_2$ of (1.3). Then this induces the desired self-map by Lemma 1.5, which induces an isomorphism on $E(2)_*$-homology. $\square$

As an application, we consider the β-elements in the homotopy groups $\pi_*(L_2S^0)$. In [7], the β-element β_s of the E_2-term $E_2^{2,16s-4}(S^0)$ is defined as the image of $v_2^s \in E_2^{0,16s}(V(1))$ under the composite of the connecting homomorphisms $E_2^{0,16s}(V(1)) \to E_2^{1,16s-4}(V(0))$ and $E_2^{1,16s-4}(V(0)) \to E_2^{2,16s-4}(S^0)$ associated to the cofiber sequences of (1.1). It is shown [13] that the β-element $\beta_s \in E_2^{2,16s-4}(S^0)$ survives to a homotopy element of $\pi_{16s-6}(L_2S^0)$ if $s \equiv 0, 1, 2, 3, 5, 6 \bmod 9$, which corresponds to one of Ravenel's conjectures in $\pi_*(S^0)$. Here we give another proof, which is what Oka showed in [9].

Proof of Corollary C. Since v_2 and v_2^5 are homotopy elements of $\pi_*(V(1))$ [9], we define β-elements as follows:

$$\beta_{9t} = j_0(v_2^9)^t, \quad \beta_{9t+1} = j_0(v_2^9)^t v_2, \quad \beta_{9t+2} = v_2^*(v_2^9)^t v_2,$$
$$\beta_{9t+5} = j_0(v_2^9)^t v_2^5 \quad \text{and} \quad \beta_{9t+6} = v_2^*(v_2^9)^t v_2^5,$$

where v_2^9 is the element in Theorem B, $j_0\colon V(1) \to S^6$ is the projection to the top cell, and $v_2^*\colon \Sigma^{10}V(2) \to S^0$ is the Spanier-Whitehead dual of v_2. It follows from the Geometric Boundary Theorem (*cf.* [10, Th. 2.3.4]) that each β-element β_s in the E_2-term survives to the homotopy element β_s defined above.

Since $V(1)_3$ is a ring spectrum and there is a map $v_2^{9t}\colon \Sigma^{144t}S^0 \to L_2V(1)_3$ by Lemma 1.6, we have the self-map v_2^{9t} as the composite $\Sigma^{144t}V(1)_3 = V(1)_3 \wedge \Sigma^{144t}S^0 \xrightarrow{1\wedge v_2^{9t}} V(1)_3 \wedge L_2V(1)_3 \to L_2V(1)_3$. Oka also showed $v_1^2v_2^3 \in \pi_{56}(V(1)_3)$ in [9, Lemma 4]. Then the composite $\Sigma^{144t+42}S^0 \xrightarrow{v_1^2v_2^3} \Sigma^{144t-14}V(1)_3 \xrightarrow{v_2^{9t}} \Sigma^{-14}L_2V(1)_3 \xrightarrow{j'_0} L_2S^0$ gives a homotopy element, which is shown to be detected by the element $\beta_{9t+3} \in E_2^{2,144t+44}(S^0)$ by the Geometric Boundary Theorem. Here j'_0 is the projection to the top cell. $\square$

2. The homotopy groups $\pi_*(V(1)\wedge X)$

Let $E(n)$ denote the Johnson-Wilson spectrum, and $\mathcal{L}_n$ the category of the $E(n)$-local spectra. Then we have the Bousfield localization functor $L_n\colon \mathcal{S}_p \to \mathcal{L}_n$ with respect to $E(n)$, where $\mathcal{S}_p$ denotes the category of (p)-local spectra. We call a spectrum $U \in \mathcal{L}_n$ invertible if there exists a spectrum $U' \in \mathcal{L}_n$ such that $U \wedge U' = L_nS^0$. Let $\mathrm{Pic}(\mathcal{L}_n)$ denote the collection of isomorphism classes of invertible spectra. Then in [4], Hovey and Sadofsky showed that $\mathrm{Pic}(\mathcal{L}_n)$ is a group with multiplication given by $[U]\cdot[V] = [U\wedge V]$ for invertible spectra U and V. Here $[U]$ denotes the isomorphism class of U. Since $[L_nS^k] \in \mathrm{Pic}(\mathcal{L}_n)$, $\mathrm{Pic}(\mathcal{L}_n)$ is the direct sum of $\boldsymbol{Z} = \{L_nS^k | k \in \boldsymbol{Z}\}$ and a subgroup $\mathrm{Pic}(\mathcal{L}_n)^0$. We write $E_r^{s,t}(W)$ for a spectrum W as the E_r-term of the $E(n)$-based Adams spectral sequence converging to $\pi_*(L_nW)$. It is further shown in [4] that $E(n)_*(U)$ for an invertible spectrum U is isomorphic to $E(n)_*$ as an $E(n)_*E(n)$-comodule. It follows that the E_2-term $E_2^{s,t}(U)$ is isomorphic to the E_2-term $E_2^{s,t}(S^0)$. In [5] (*cf.* [6]) it is shown that there is a descending filtration $\{F_r\}$ of $\mathrm{Pic}(\mathcal{L}_n)^0$ so that a monomorphism

$$\varphi_r\colon F_r/F_{r+1} \to E_r^{r,r-1}(S^0) \tag{2.1}$$

is defined for each $r > 1$ by assigning an isomorphism class $[U]$ to $d_r(g) \in E_r^{r,r-1}(U) = E_r^{r,r-1}(S^0)$, where g is the generator of $E_2^{0,0}(U) = E_2^{0,0}(S^0) = \boldsymbol{Z}_{(p)}$.

Now turn to the case where $p=3$ and $n=2$. Consider the chromatic comodules $N_0^0 = E(2)_*$, $M_0^0 = 3^{-1}E(2)_*$, $N_0^1 = E(2)_*/(3^\infty)$, $M_0^1 = v_1^{-1}N_0^1$ and $N_0^2 = M_0^2 = E(2)_*/(3^\infty, v_1^\infty)$ that fit in the exact sequences

$$0 \longrightarrow N_0^0 \longrightarrow M_0^0 \longrightarrow N_0^1 \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow N_0^1 \longrightarrow M_0^1 \longrightarrow M_0^2 \longrightarrow 0.$$

These associate the long exact sequences

$$\begin{gathered} 0 \longrightarrow H^0N_0^0 \longrightarrow H^0M_0^0 \longrightarrow H^0N_0^1 \xrightarrow{\delta} H^1N_0^0 \longrightarrow \cdots \quad \text{and} \\ 0 \longrightarrow H^0N_0^1 \longrightarrow H^0M_0^1 \longrightarrow H^0M_0^2 \xrightarrow{\delta'} H^1N_0^1 \longrightarrow \cdots, \end{gathered} \tag{2.2}$$

where $H^kM = \mathrm{Ext}^k_{E(2)_*E(2)}(E(2)_*, M)$ for an $E(2)_*E(2)$-comodule M. Then the universal Greek letter map $\eta\colon H^kM_0^2 \to H^{k+2}N_0^0 = E_2^*(S^0)$ is defined as the composite $\eta = \delta\delta'$. The E_2-term $E_2^*(S^0) = H^*N_0^0$ is given in [14] (*cf.* [13]) by using (2.2). In particular, $E_2^{5,4}(S^0) = \boldsymbol{Z}/3\{\eta(v_2^{-1}h_{11}b_{10}/3v_1), \eta(v_2^{-1}\xi\zeta_2/3v_1)\}$. In [5], we show that there is an invertible spectrum $X \in \mathcal{L}_2$ such that $\varphi_5([X]) = c\eta(v_2^{-1}h_{11}b_{10}/3v_1)$. In other words,

$$d_5(g) = c\eta(v_2^{-1}h_{11}b_{10}/3v_1)g \in E_5^{5,4}(X) \tag{2.3}$$

for the generator $g \in E_2^{0,0}(X)$. Here c is the non-zero element of $\boldsymbol{Z}/3$ that appears in the Toda differential

$$d_5(\beta_{3/3}) = c\alpha_1\beta_1^3, \tag{2.4}$$

where the elements $\beta_{3/3}$, α_1 and β_1 are defined by

$$\beta_{3/3} = \eta(v_2^3/3v_1^3), \quad \alpha_1 = \delta(v_1/3) \quad \text{and} \quad \beta_1 = \eta(v_2/3v_1).$$

The E_2-term $E_2^{*,*}(V(1))$ of the $E(2)$-based Adams spectral sequence converging to the homotopy groups $\pi_*(L_2V(1))$ is isomorphic to

$$(F \oplus F^*) \otimes K(2)_*[b_{10}] \otimes \Lambda(\zeta_2) \tag{2.5}$$

as a $K(2)_*$-module. Here, $K(2)_* = \boldsymbol{Z}/3[v_2^{\pm1}]$,

$$F = \boldsymbol{Z}/3\{1, h_{10}, h_{11}, b_{11}\}, \quad F^* = \boldsymbol{Z}/3\{\xi, \psi_0, \psi_1, b_{11}\xi\},$$

and the degrees of the generators are:

$$|1| = 0, \quad |h_{10}| = 3, \quad |h_{11}| = 11, \quad |b_{10}| = 10, \quad |b_{11}| = 34,$$
$$|\xi| = 6, \quad |\psi_0| = 13, \quad |\psi_1| = 21, \quad \text{and} \quad |\zeta_2| = -1.$$

Let $(i_0)_*\colon E_2^*(S^0) \to E_2^*(V(1))$ be the induced map from the inclusion $i_0\colon S^0 \to V(1)$. Then

$$(i_0)_*(\beta_{3/3}) = b_{11}, \quad (i_0)_*(\alpha_1) = h_{10} \quad \text{and} \quad (i_0)_*(\beta_1) = b_{10}.$$

Since $E(2) \wedge X = E(2)$, the generator $g \in E_2^0(X)$ induces an isomorphism $g\colon E_2^*(V(1)) \cong E_2^*(V(1) \wedge X)$. By the structure (2.5), we see the following.

Lemma 2.6. *$v_2^3 g$ induces an isomorphism $E_2^{s,t}(V(1)) \cong E_2^{s,t+48}(V(1) \wedge X)$.*

We will show that the map $v_2^3 g$ induces an isomorphism of the differential modules on E_5- and E_9-terms. Here $E_2^*(W) = E_5^*(W)$ and $E_6^*(W) = E_9^*(W)$ for a spectrum W are differential modules with differentials d_5 and d_9 of the $E(2)$-based Adams spectral sequence.

Lemma 2.7. *$\beta_{3/3}^2 g$ $(\neq 0) \in E_9^4(X)$.*

Proof. Recall [11, Prop. 5.9] the relations in the E_2-term $E_2^*(V(1))$:

$$v_2 h_{11} b_{10} = -h_{10} b_{11} \quad \text{and} \quad b_{11}^2 = -v_2^3 b_{10}^2. \tag{2.8}$$

Send $\beta_{3/3}^2$ to $E_2^*(V(1))$ under the map $(i_0)_*$, and we have $(i_0)_*(\beta_{3/3}^2) = b_{11}^2 = -v_2^3 b_{10}^2 \neq 0 \in E_2^*(V(1))$. It follows that $b_{11}^2 g \neq 0 \in E_2^*(V(1) \wedge X)$ and so $\beta_{3/3}^2 g \neq 0 \in E_2^*(X)$.

From the Toda differential (2.4), the derivation formula on d_5 induces

$$d_5(\beta_{3/3}^2) = -c\alpha_1 \beta_1^3 \beta_{3/3} \in E_5^9(S^0).$$

By the trivial pairing $S^0 \wedge X \to X$, we have the derivation formula on d_5. Furthermore, the universal Greek letter map η is the map of $E_2^*(S^0)$-modules. Therefore, by (2.3),

$$\begin{aligned} d_5(\beta_{3/3}^2 g) &= -c\alpha_1 \beta_1^3 \beta_{3/3} g + c\beta_{3/3}^2 \eta(v_2^{-1} h_{11} b_{10}/3v_1) g \\ &= -c\alpha_1 \beta_1^3 \beta_{3/3} g + c\eta(v_2^{-1} h_{11} b_{10} b_{11}^2/3v_1) g. \end{aligned}$$

Here $v_2^{-1} h_{11} b_{10} b_{11}^2/3v_1 = v_2 h_{10} b_{11} b_{10}^2/3v_1$ by (2.8). Thus,

$$\begin{aligned} d_5(\beta_{3/3}^2 g) &= -c\alpha_1 \beta_1^3 \beta_{3/3} g + c\eta(v_2 h_{10} b_{11} b_{10}^2/3v_1) g \\ &= -c\alpha_1 \beta_1^3 \beta_{3/3} g + c\alpha_1 \beta_{3/3} \beta_1^2 \eta(v_2/3v_1) g \\ &= 0. \end{aligned}$$

Since nothing hits $\beta_{3/3}^2 g$ under the differential d_5 by reason of degree, we obtain $\beta_{3/3}^2 g \neq 0 \in E_9^4(X)$. □

By the trivial pairing $V(1) \wedge X \to V(1) \wedge X$, we obtain the derivation formula:

$$d_r(xy) = d_r(x)y + (-1)^{t-s} x d_r(y) \tag{2.9}$$

for $x \in E_r^{s,t}(V(1))$ and $y \in E_r^*(X)$ (*cf.* [10, Th. 2.3.3]).

Lemma 2.10. *$v_2^3 g$ induces an isomorphism $E_9^{s,t}(V(1)) \cong E_9^{s,t+48}(V(1) \wedge X)$.*

Proof. By (2.9) and Lemma 2.7, we see that $d_5(xb_{11}^2g) = d_5(x\beta_{3/3}^2g) = d_5(x)\beta_{3/3}^2g = d_5(x)b_{11}^2g$ for $x \in E_2^*(V(1))$. Since $b_{11}^2 = -v_2^3b_{10}^2$, we obtain $d_5(xb_{10}^2v_2^3g) = d_5(x)b_{10}^2v_2^3g$. Since b_{10} is a polynomial generator, b_{10} acts monomorphically, and so we have

$$d_5(xv_2^3g) = d_5(x)v_2^3g.$$

This shows that v_2^3g is a map of differential modules and induces the desired isomorphism. □

Lemma 2.11. *v_2^3g induces an isomorphism $E_\infty^{s,t}(V(1)) \cong E_\infty^{s,t+48}(V(1) \wedge X)$.*

Proof. Since $E_2^{13,80}(V(1) \wedge X) = \boldsymbol{Z}/3\{v_2^{-1}b_{11}b_{10}^5\zeta_2g\}$ and $d_5(v_2^{-1}b_{11}b_{10}^5\zeta_2g) = d_5(v_2^{-4}b_{11}b_{10}^5\zeta_2)v_2^3g = cv_2^{-4}h_{10}b_{10}^8\zeta_2(v_2^3g) \neq 0$ by [11, Prop. 9.9, Cor. 10.4], we obtain $E_9^{13,80}(V(1) \wedge X) = 0$. Therefore, $(i_0)_*(d_9(\beta_{3/3}^2g)) = 0$. If we show that

$$d_9(xb_{10}) = yb_{10} \text{ implies } d_9(x) = y \text{ in } E_9^*(V(1) \wedge X), \tag{2.12}$$

then we see that v_2^3g induces an isomorphism $E_{13}^{s,t}(V(1)) \cong E_{13}^{s,t+48}(V(1) \wedge X)$ in the same way as Lemma 2.10. Since $E_{13}^s(V(1)) = 0$ if $s > 12$, $d_{13} = 0$ and so $E_{13}^*(V(1) \wedge X) = E_\infty^*(V(1) \wedge X)$.

Turn to (2.12). If we assume that $d_9(xb_{10}) = yb_{10}$, then there is an element $z \in \operatorname{Ker}\big(b_{10}\colon E_9^s(V(1) \wedge X) \to E_9^{s+2}(V(1) \wedge X)\big)$ such that $d_9(x) = y + z$. Note that $s \geq 9$. Since $zb_{10} = 0 \in E_9^{s+2}(V(1) \wedge X)$, there is an element $w \in E_5^{s-3}(V(1) \wedge X)$ such that $d_5(w) = zb_{10}$. By the structure (2.5) of the $E_5(= E_2)$-term, we see that there is an element $w' \in E_5^{s-5}(V(1) \wedge X)$ such that $w = w'b_{10}$, and $d_5(w') = z$, since b_{10} is a monomorphism on E_5-terms. It follows that $z = 0$ in the E_9-term, and we have $d_9(x) = y$ as desired. □

Since $\pi_*(V(1) \wedge X)$ is a $\boldsymbol{Z}/3$-vector space, there is no extension problem in the spectral sequence.

Corollary 2.13. *The homotopy groups $\pi_*(V(1) \wedge X)$ are isomorphic to the E_∞-terms for them.*

3. Invertible spectra and the Smith-Toda spectra

First we consider the case $p = 2$ and $n = 1$. Then it is shown in [4] (*cf.* [3]) that $\operatorname{Pic}(\mathcal{L}_1)^0 = \boldsymbol{Z}/2$, whose generator is represented by the $E(1)$-localization of the question mark complex $QM = V(0) \cup_\eta e^3$, where $V(0) = S^0 \cup_2 e^1$ is the mod 2 Moore spectrum. Let $E_r^*(W)$ for a spectrum W denote the E_r-term of the $E(1)$-based Adams spectral sequence for $\pi_*(L_1W)$. Since L_1QM is invertible, the E_2-term for $\pi_*(L_1V(0) \wedge QM)$ is isomorphic to that for $\pi_*(L_1V(0))$:

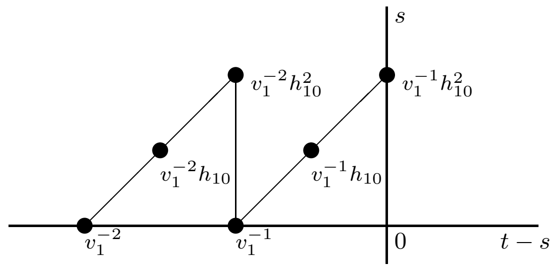
$$K(1)_*[h_{10}] \otimes \Lambda(\rho_1), \tag{3.1}$$

where $K(1)_* = \boldsymbol{Z}/2[v_1^{\pm 1}]$. The map $\varphi_3\colon \operatorname{Pic}(\mathcal{L}_1)^0 = F_3 \to E_3^{3,2}(S^0) = \boldsymbol{Z}/2\{\alpha_{-1}\alpha_1^2\}$ of (2.1) is an isomorphism such that $\varphi_3([L_1QM]) = \alpha_{-1}\alpha_1^2$. Here α_k for $k \in \boldsymbol{Z}$ denotes $\delta(v_1^k)$ for the connecting homomorphism $\delta\colon E_2^*(V(0)) \to E_2^{*+1}(S^0)$ associated to the cofiber sequence $S^0 \xrightarrow{2} S^0 \xrightarrow{i} V(0)$ and for $v_1^k \in E_2^0(V(0)) = K(1)_*$. Note that $i_*(\alpha_{-1}) = v_1^{-2}h_{10}$ and $i_*(\alpha_1) = h_{10}$ for the induced map $i_*\colon E_2^*(S^0) \to E_2(V(0))$ from the inclusion i. The E_2-term for QM is isomorphic to that for the sphere L_1S^0, and $d_3(g) = \alpha_{-1}\alpha_1^2g \in E_3^3(QM) = E_2^3(QM)$ by the definition of φ_3. Therefore, the Adams differentials on $E_2^*(V(0) \wedge QM)$ are computed by $d_3(i_*(g)) = v_1^{-2}h_{10}^3g \in E_3^{3,2}(V(0) \wedge QM)$ and the derivation formulas associated to the trivial pairing $L_1V(0) \wedge QM \to L_1V(0) \wedge QM$. In fact, $v_1^{-2}g$ induces the isomorphism $E_r^*(V(0)) \cong E_r^*(V(0) \wedge QM)$ of spectral sequences. Therefore, we have

$$E_\infty^{*,*}(V(0) \wedge QM) = N \otimes \boldsymbol{Z}/2[v_1^4, v_1^{-4}] \otimes \Lambda(\rho_1),$$

where N is the module given by

(3.2)



in which each dot denotes $\boldsymbol{Z}/2$ generated by the indicated element. Therefore, we see that $v_1^{-2}g \in \pi_{-4}(L_1V(0) \wedge QM)$ and $2v_1^{-2}g = 0$. Thus we obtain the map $v_1^{-2}g\colon \Sigma^{-4}V(0) \to L_1V(0) \wedge QM$, which induces an isomorphism on $E(1)_*$-homology.

Proposition 3.3. *The element $v_1^{-2}g \in \pi_{-4}(L_1V(0) \wedge QM)$ induces an equivalence $L_1\Sigma^{-4}V(0) \simeq L_1V(0) \wedge QM$.*

We will play the same game for the case where $p = 3$ and $n = 2$.

Lemma 3.4. *There is a homotopy element $v_2^3g \in \pi_{48}(V(1)_i \wedge X)$ for $i = 1, 2$ such that $E(2)_*(v_2^3g) = v_2^3$.*

Proof. We have seen that $v_2^3g \in \pi_{48}(V(1) \wedge X)$ in the previous section. We also see that $\pi_{43}(V(1) \wedge X) = \boldsymbol{Z}/3\{v_2^2h_{11}g\}$. Therefore, we obtain $v_2^3g \in \pi_{48}(V(1)_2 \wedge X)$ from the exact sequence $\pi_{48}(V(1)_2 \wedge X) \xrightarrow{\widetilde{i_1}_*} \pi_{48}(V(1) \wedge X) \xrightarrow{\widetilde{j_1}_*} \pi_{43}(V(1) \wedge X)$ associated to the cofiber sequence (1.3) with $k = 1$. Indeed, $v_2^2h_{11}g$ has the Adams filtration 1, while $\delta(v_2^3g) = 0$ for the connecting homomorphism δ corresponding to $\widetilde{j_1}$. $\square$

Now we have the similar results to Proposition 3.3.

Theorem 3.5. The element $v_2^3g \in \pi_{48}(V(1) \wedge X)$ induces an equivalence $\Sigma^{48}L_2V(1)_i \simeq V(1)_i \wedge X$ for $i = 1, 2$.

Proof. Since $V(1)_2$ is a ring spectrum, the element v_2^3g yields the self-map $\widetilde{v_2^3g}\colon V(1)_2 \to V(1)_2 \wedge X$, which induces an isomorphism on $E(2)_*$-homology. Therefore, the proposition for $i = 2$ follows. For $i = 1$, Lemmas 3.4 and 1.5 show the existence of the map $v_2^3g\colon \Sigma^{48}L_2V(1) \to V(1) \wedge X$, which is also an $E(2)_*$-equivalence. $\square$

Proof of Corollary D. In the same manner as the proof of Corollary C, the β-elements are defined as follows:

$$\beta_{9t+3}g = (j_0 \wedge 1_X)(v_2^3g)(v_2^9)^t, \quad \beta_{9t+4}g = (j_0 \wedge 1_X)(v_2^3g)(v_2^9)^t v_2,$$
$$\beta_{9t+5}g = (v_2^* \wedge 1_X)(v_2^3g)(v_2^9)^t v_2, \quad \beta_{9t+8}g = (j_0 \wedge 1_X)(v_2^3g)(v_2^9)^t v_2^5 \quad \text{and}$$
$$\beta_{9t+9}g = (v_2^* \wedge 1_X)(v_2^3g)(v_2^9)^t v_2^5.$$

$\square$

Remark 3.6. $v_2^3 g \in E_2^{0,48}(V(1)_3 \wedge X)$ cannot be a permanent cycle. In fact, we see that $d_9(\widetilde{j_3}_*(v_2^3 g)) = d_9(\delta(v_2^3 g)) = d_9(v_2^2 h_{10} g) = v_2^{-1} b_{10}^5 g \neq 0$, where $\widetilde{j_3}$ is the map of (1.3) with $k = 3$. Therefore, we cannot tell whether or not $\beta_{9t+6} g$ survives to a homotopy element of $\pi_*(X)$ different from Corollary C. Indeed, we do not know if $v_1^2 v_2^6 g \in \pi_*(V(1)_3 \wedge X)$.

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