PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 10, Pages 2899–2909
S 0002-9939(04)07482-9
Article electronically published on May 20, 2004

ON EXTENSION OF ISOMETRIES BETWEEN UNIT SPHERES OF AL_p-SPACES $(0 < p < \infty)$

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(Communicated by N. Tomczak-Jaegermann)

Abstract. In this paper, we study the extension of isometries between unit spheres of atomic AL_p-spaces $(0 < p < \infty,\ p \neq 2)$. We find a condition under which an isometry T between unit spheres can be linearly isometrically extended. Moreover, we prove that every onto isometry between unit spheres of atomic AL_p-spaces $(0 < p < \infty,\ p \neq 2)$ can be linearly isometrically extended to the whole space.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and of reals. We use the standard notation of vector lattices, and for all other notation we follow the definitions of the papers [1], [2], and [12]. This paper mainly discusses the atomic AL_p-spaces on $\mathbb{R}$ with positive values of p that is not equal to two; therefore, E and F are used to denote these spaces unless otherwise stated.

The unit spheres of atomic AL_p-spaces are frequently discussed; we use $S(E)$ and $S(F)$ to denote the unit spheres of E and F. Moreover, T denotes an isometry from $S(E)$ to $S(F)$.

A mapping $T\colon E \to F$ is said to be an isometry if

$$\|Tx - Ty\| = \|x - y\| \quad (\forall x, y \in E). \tag{1.1}$$

For $1 \leq p < +\infty$, an AL_p-space E is a Banach lattice in which

$$\|x + y\|^p = \|x\|^p + \|y\|^p \quad (\forall x, y \in E \,\text{with}\, |x| \wedge |y| = \theta).$$

For the cases that $0 < p < 1$, an AL_p-space E means a p-homogeneous F-lattice, and the last equality turns into

$$\|x + y\| = \|x\| + \|y\| \quad (\forall x, y \in E \,\text{with}\, |x| \wedge |y| = \theta).$$

A complete disjoint system $\{e_\lambda\colon \lambda \in \Lambda\}$ of E is a pairwise disjoint collection of elements of E_+, the positive cone of E. That is, a complete disjoint system satisfies

Received by the editors April 15, 2002 and, in revised form, June 24, 2002.
2000 *Mathematics Subject Classification.* Primary 46B04; Secondary 46A40.
Key words and phrases. Atomic AL_p-space, isometric extension, atom.
This work belongs to the Doctoral Programme Foundation of the Institution of Higher Education (20010055013) and the Programme of National Science Foundation of China (10271060). It was supported by the National Science Foundation of China (10171014) and the Foundation of Fujian Educational Committee (JA02166).

the following two conditions:

(1) for each pair $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$, $e_\lambda \wedge e_\mu = \theta$, and

(2) for each $u \in E$ with $u \wedge e_\lambda = \theta$ for every $\lambda \in \Lambda$, the equality $u = \theta$ holds.

An atom e of a vector lattice E is a nonzero element of E with the property that the conditions $\theta \leq y, z \leq |e|$ and $y \wedge z = \theta$ imply $y = \theta$ or $z = \theta$.

If an element e is an atom of an Archimedean vector lattice E, then the principal ideal E_e generated by e is a projection band B_e (see [1, Theorem 2.16]).

An atomic vector lattice is a vector lattice E having a complete disjoint system consisting of atoms of E. In this case, we have $x = \sup_{\lambda \in \Lambda} \{P_{e_\lambda}(x)\}$, where P_{e_λ} is a band projection with respect to B_{e_λ} (see [1, Theorem 2.15]). An atomic AL_p-space is in fact linearly isometric to $l_p(\Gamma)$, where Γ may be any complete disjoint system of atoms of norm one.

In the following sections, we use $\{e_\gamma\}_{\gamma \in \Gamma}$ to denote a norm-one complete disjoint system of E, B_{Te_γ} to denote a principal projection band generated by Te_γ, and P_γ to denote a principal band projection from E onto B_{Te_γ} $(\gamma \in \Gamma)$.

P. Mankiewicz ([9]) proved in 1972 that an isometry mapping an open connected subset of a normed space E onto an open subset of another normed space F can be extended to an affine isometry from E onto F. In 1987, D. Tingley in [13] raised the following problem:

> *Let $S(E)$ be the unit sphere of a Banach space E, and let $S(F)$ be the unit sphere of another Banach space F. Let $T\colon S(E) \to S(F)$ be isometric. Does there exist any isometry $\widetilde{T}\colon E \to F$ such that $\widetilde{T}|_{S(E)} = T$?*

In the same paper, Tingley proved that if E and F are finite-dimensional Banach spaces and $T\colon S(E) \to S(F)$ is a *surjective* isometry, then $T(-x) = -T(x)$ for all $x \in S(E)$. Ma Yumei proved the same result for the infinite-dimensional space l_1 in [10]. During the past decade, Professor Ding Guanggui ([3]-[5]) and a group of his students including Wang Risheng ([14]-[20]), Xiao Yuhui ([21, 22]), and Zhan Dapeng ([23, 24]) had been working on this topic and had obtained many significant results. A number of these works only considered the *surjective* mappings between two spaces of the same type. However, Professor Ding Guanggui had discussed the extension of isometries between unit spheres of spaces of different type in [3]. Moreover, he also first discussed the mapping extension problem on Hilbert spaces without assuming the surjectivity of T in [4], where he showed that a 1-Lipschitz mapping between the unit spheres of two Hilbert spaces can be extended to a real linear isometry of the whole space.

For real AL_p-spaces E and F with $p > 1$, Xiao Yuhui proved in [21, Theorem 1] that every surjective isometry between unit spheres $S(E)$ of E and $S(F)$ of F can be linearly isometrically extended to the whole space. For complex AL_p-spaces E and F with $p > 1$, he proved in [21, Theorem 2] that for every surjective isometry between unit spheres $S(E)$ and $S(F)$ there exist an additive isometry $\widetilde{T}$ from E into F and two mutually complementary subspaces X and Y of E such that the restriction of $\widetilde{T}$ to $S(E)$ is T, that to X is linear and that to Y is conjugate linear.

However, a crucial result, Corollary 3, which stated that T maps every atom in $S(E)$ to an atom in $S(F)$, was not proved in the paper [21]. For the cases $p > 1$, Professor Ding Guanggui proved the assertion in [5], and gave the representation of a surjective isometry $T\colon S(E) \to S(F)$ by a method that is different from what Xiao Yuhui has in [21]. In our study in this paper, we shall drop the *surjectivity*

assumption for the mappings, and mainly study the case that $T\colon S(E) \to S(F)$ is an *injective* isometry. We find a condition under which an isometry $T\colon S(E) \to S(F)$ can be extended to a linear isometry $\widetilde{T}\colon E \to F$. Therefore, we generalize the results of Xiao Yuhui in [21].

2. Some important lemmas

Lemma 2.1 ([11]). *Let ξ and η be two real numbers. Then:*

(i) $|\xi+\eta|^p + |\xi-\eta|^p \geq 2\left(|\xi|^p + |\eta|^p\right)$ *for every real number* $p \geq 2$;
(ii) $|\xi+\eta|^p + |\xi-\eta|^p \leq 2\left(|\xi|^p + |\eta|^p\right)$ *for every real number* p, $0 < p \leq 2$;
(iii) $|\xi-\eta|^p \leq |\xi|^p + |\eta|^p$ *for every real number* p, $0 < p < 1$.

Moreover, if $p \neq 2$, equality can only hold in the above inequalities if $\xi = 0$ or $\eta = 0$.

Corollary 2.1. *Let f and g be two elements of $l_p(\Gamma)$. Then:*

(i) *for the cases that $p \geq 1$ and $p \neq 2$, we have*

$$\|f+g\|^p + \|f-g\|^p = 2(\|f\|^p + \|g\|^p) \tag{2.2}$$

if and only if $|f| \wedge |g| = \theta$;

(ii) *for the cases that $0 < p < 1$, we have*

$$\|f-g\| = \|f\| + \|g\| \tag{2.3}$$

if and only if $|f| \wedge |g| = \theta$;

$$\|f-g\| + \|f+g\| = 2(\|f\| + \|g\|) \tag{2.4}$$

if and only if $|f| \wedge |g| = \theta$.

Lemma 2.2. *Let γ be an arbitrary element of Γ. Suppose that we have*

(i) $P_\gamma(T(S(E))) \subseteq \operatorname{span}(Te_\gamma)$ *when* $p \neq 1$, *and*
(ii) $T(-e_\gamma) \in \operatorname{span}(Te_\gamma)$ *when* $p = 1$.

Then, $T(-e_\gamma) = -Te_\gamma$.

Proof. We consider first the case that $p = 1$. Note that the property

$$\|u\| = \|v\| \text{ implies } v = \pm u \tag{2.5}$$

holds in every one-dimensional subspace of AL-spaces. Since T is isometric, $\|T(-e_\gamma)\| = \|Te_\gamma\| = \|e_\gamma\| = 1$. Combining (2.5), the fact that $\dim(\operatorname{span}(Te_\gamma)) = 1$ and $T(-e_\gamma) \in \operatorname{span}(Te_\gamma)$, we conclude that $T(-e_\gamma) = \pm Te_\gamma$. Suppose that $T(-e_\gamma) = Te_\gamma$. Then $0 = \|Te_\gamma - T(-e_\gamma)\| = \|2e_\gamma\|$, and thus $e_\gamma = \theta$, which is a contradiction. Therefore, we have

$$T(-e_\gamma) = -Te_\gamma. \tag{2.6}$$

We consider now the case that $p \neq 1$. Because of the projection property of AL_p-spaces, we may assume that $P = I - P_\gamma$, where I is the identity mapping on F. It is obvious that P is a band projection from F onto the orthogonal complement of B_{Te_γ}. Therefore, we have $T(-e_\gamma) = P_\gamma(T(-e_\gamma)) + P(T(-e_\gamma))$.

Since $P_\gamma(T(-e_\gamma)) \in \operatorname{span}(Te_\gamma)$, we conclude that there exists a real number a_γ such that $P_\gamma(T(-e_\gamma)) = a_\gamma Te_\gamma$.

For the cases that $p = \beta < 1$, since T is isometric, we have

$$\begin{aligned}\|T(-e_\gamma) - Te_\gamma\| &= \|-e_\gamma - e_\gamma\| = 2^\beta \\ &= \|P_\gamma(T(-e_\gamma)) + P(T(-e_\gamma)) - Te_\gamma\| = \|a_\gamma Te_\gamma - Te_\gamma\| + \|P(T(-e_\gamma))\| \\ &= |a_\gamma - 1|^\beta + \|T(-e_\gamma)\| - \|P_\gamma(T(-e_\gamma))\| = |a_\gamma - 1|^\beta + 1 - |a_\gamma|^\beta.\end{aligned}$$

Therefore, we have the following equality: $|a_\gamma - 1|^\beta - |a_\gamma|^\beta = 2^\beta - 1$.

We know that the function $h(t) = |1-t|^\beta - |t|^\beta - 2^\beta + 1$ has two zeros in $[-1, 1]$, $t_1 = -1$, $t_2 \in (0, \frac{1}{2})$.

Suppose that $0 \le a_\gamma = t_2 < \frac{1}{2}$. Now let $T(a_\gamma e_\gamma) = b_\gamma Te_\gamma + P(T(a_\gamma e_\gamma))$. Then $|b_\gamma| < \frac{1}{2}$ and

$$\begin{aligned}(1-a_\gamma)^\beta &= \|a_\gamma e_\gamma - e_\gamma\| = \|T(a_\gamma e_\gamma) - Te_\gamma\| \\ &= \|b_\gamma Te_\gamma + P(T(a_\gamma e_\gamma)) - Te_\gamma\| = \|b_\gamma Te_\gamma - Te_\gamma\| + \|P(T(a_\gamma e_\gamma))\| \\ &= (1-b_\gamma)^\beta + \|P_\gamma(T(-e_\gamma))\| = (1-b_\gamma)^\beta + a_\gamma^\beta - |b_\gamma|^\beta,\end{aligned}$$

and so

$$(\Delta) \qquad (1-a_\gamma)^\beta - a_\gamma^\beta = (1-b_\gamma)^\beta - |b_\gamma|^\beta.$$

Notice that the function $u(t) = (1-t)^\beta - t^\beta$ is strictly decreasing on $[0, \frac{1}{2}]$ but the function $v(t) = (1-t)^\beta - (-t)^\beta$ is strictly increasing on $[-\frac{1}{2}, 0]$. If $b_\gamma < 0$, then

$$(1-a_\gamma)^\beta - a_\gamma^\beta = u(a_\gamma) > u(0) = 1 = v(0) > v(b_\gamma) = (1-b_\gamma)^\beta - (-b_\gamma)^\beta,$$

contradicting $u(a_\gamma) = v(b_\gamma)$ by (Δ), and so $b_\gamma \ge 0$. Therefore, $b_\gamma = a_\gamma$, and so $P(T(a_\gamma e_\gamma)) = \theta$. We now have $T(a_\gamma e_\gamma) = a_\gamma Te_\gamma$.

On the other hand, we have

$$\begin{aligned}(1+a_\gamma)^\beta &= \|-e_\gamma - a_\gamma e_\gamma\| = \|T(-e_\gamma) - T(a_\gamma e_\gamma)\| \\ &= \|a_\gamma Te\gamma + P(T(-e_\gamma)) - T(a_\gamma e_\gamma)\| = \|P(T(-e_\gamma))\| = 1 - a_\gamma^\beta,\end{aligned}$$

that is, $(1+a_\gamma)^\beta + a_\gamma^\beta = 1$, and so $a_\gamma = 0$, contradicting the assumption $a_\gamma > 0$. It follows that $a_\gamma = t_1 = -1$, implying that $\|P_\gamma(T(-e_\gamma))\| = |a_\gamma|^\beta = 1$.

Since

$$1 = \|T(-e_\gamma)\| = \|P_\gamma(T(-e_\gamma))\| + \|P(T(-e_\gamma))\| = 1 + \|P(T(-e_\gamma))\|,$$

we derive that $P(T(-e_\gamma)) = \theta$, and thus prove (2.6) for this case.

Now suppose that $p > 1$. Since we have

$$\begin{aligned}\|T(-e_\gamma) - Te_\gamma\|^p &= \|-e_\gamma - e_\gamma\|^p = 2^p \\ &= \|P_\gamma(T(-e_\gamma)) + P(T(-e_\gamma)) - Te_\gamma\|^p = \|a_\gamma Te_\gamma - Te_\gamma\|^p + \|P(T(-e_\gamma))\|^p \\ &= |a_\gamma - 1|^p + \|T(-e_\gamma)\|^p - \|P_\gamma(T(-e_\gamma))\|^p = |a_\gamma - 1|^p + 1 - |a_\gamma|^p,\end{aligned}$$

the equality (2.6) is easily obtained using arguments similar to those of the proof for the cases that $p < 1$. □

Lemma 2.3. *Let $p = \beta$ be a real number that is less than or equal to one, n a positive integer, and $\{e_1, e_2, \cdots, e_n\}$ a subset of $S(E)$ satisfying the condition that the equality $|e_i| \wedge |e_j| = \theta$ holds for each distinct pair of indices i and j. Suppose that $x = \sum_{i=1}^n \lambda_i e_i$, where λ_i are nonzero real numbers that satisfy the equation $\sum_{i=1}^n |\lambda_i|^\beta = 1$.*

Suppose that $T\colon S(E) \to S(F)$ *is an isometric embedding satisfying*

(i) $P_i(T(S(E))) \subseteq \operatorname{span}(Te_i)$ *when* $0 < p < 1$,

(ii) $P_i(T(S(E))) \subseteq \operatorname{span}(Te_i)$ *and* $T(-e_i) \in \operatorname{span}(Te_i)$ *when* $p = 1$

for all $1 \le i \le n$. *Then, we have* $T(x) = \sum_{i=1}^{n} \lambda_i Te_i$.

Proof. By the assumptions of the lemma, and Lemma 2.2, it is easy to derive that

$$T(-e_i) = -Te_i \quad (1 \le i \le n). \tag{2.7}$$

Since T is isometric, the elements of $\{e_1, e_2, \cdots, e_n\}$ are pairwise disjoint, and the equality (2.7) holds, we conclude by (2.2) and (2.4) of Corollary 2.1 that

$$\begin{aligned}\|Te_i + Te_j\| + \|Te_i - Te_j\| &= \|e_i + e_j\| + \|e_i - e_j\| \\ &= 2(\|e_i\| + \|e_j\|) = 2(\|Te_i\| + \|Te_j\|) \quad (\forall\ 1 \le i, j \le n,\ i \ne j).\end{aligned}$$

As a result, we derive, again by applying (2.2) and (2.4) of Corollary 2.1, that the equality $|Te_i| \wedge |Te_j| = \theta$ holds for every pair of distinct indices i and j. It follows that the elements of the set $\{Te_i\}_{i=1}^n$ are mutually disjoint, and thus $\{B_{Te_i}\}_{i=1}^n$ is mutually disjoint.

Since F is order complete, it possesses the projection property. Therefore, we may assume that $P = I - \sum_{i=1}^{n} P_i$.

Since $|Te_i| \wedge |Te_j| = \theta$, it follows that $\sum_{i=1}^n P_i$ is the band projection of F onto the band generated by $\{Te_i\}_{i=1}^n$. Thus P is the band projection of F onto the orthogonal complement of $\{Te_i\}_{i=1}^n$.

Since $T\colon S(E) \to S(F)$ is isometric, by defining $\varepsilon_i = \frac{\lambda_i}{|\lambda_i|}$ for every i, $1 \le i \le n$, we have

$$\begin{aligned}\|Tx - T(\varepsilon_i e_i)\| &= \|x - \varepsilon_i e_i\| = \sum_{j=1, j\ne i}^{n} |\lambda_j|^\beta + |\lambda_i - \varepsilon_i|^\beta \\ &= 1 - |\lambda_i|^\beta + |\lambda_i - \varepsilon_i|^\beta = 1 - |\lambda_i|^\beta + (1 - |\lambda_i|)^\beta.\end{aligned} \tag{2.8}$$

On the other hand, we may assume, by the fact that $P_i(T(S(E))) \subseteq \operatorname{span}(Te_i)$ and the equality (2.7), that $P_i(Tx) = \lambda_i' T(\varepsilon_i e_i)$ for every $1 \le i \le n$. Since we have $|\lambda_i'|^\beta = \|P_i(Tx)\| \le \|Tx\| = \|x\| = 1$, and $|\lambda_i'| \le 1$ for every $1 \le i \le n$, we conclude that

$$\begin{aligned}\|Tx - T(\varepsilon_i e_i)\| &= \left\| \sum_{j=1, j\ne i}^{n} P_j(Tx) + P(Tx) + P_i(Tx) - T(\varepsilon_i e_i) \right\| \\ &= \|Tx\| - \|P_i(Tx)\| + \|P_i(Tx) - T(\varepsilon_i e_i)\| \\ &= 1 - |\lambda_i'|^\beta + \|\lambda_i' T(\varepsilon_i e_i) - T(\varepsilon_i e_i)\| = 1 - |\lambda_i'|^\beta + |\lambda_i' - 1|^\beta \\ &\ge 1 - |\lambda_i'|^\beta + (1 - |\lambda_i'|)^\beta.\end{aligned} \tag{2.9}$$

It follows by (2.8) and (2.9) that

$$|\lambda_i|^\beta - (1 - |\lambda_i|)^\beta \le |\lambda_i'|^\beta - (1 - |\lambda_i'|)^\beta.$$

For the case that $\beta = 1$, we have $2|\lambda_i| - 1 \le 2|\lambda_i'| - 1$, and thus $|\lambda_i| \le |\lambda_i'|$. For the cases that $\beta < 1$, it is well known that the function $f(t) = t^\beta - (1-t)^\beta$ is increasing on the interval $[0, 1]$. Therefore, the equality $|\lambda_i| \le |\lambda_i'|$ holds for every $1 \le i \le n$.

However, we have
$\sum_{i=1}^{n} |\lambda_i|^\beta = \|x\| = \|Tx\| = \sum_{i=1}^{n} \|P_i(Tx)\| + \|P(Tx)\| = \sum_{i=1}^{n} |\lambda_i'|^\beta + \|P(Tx)\|.$

Therefore, we conclude that $\|P_i(Tx)\| = |\lambda_i'|^\beta = |\lambda_i|^\beta$ and $P(Tx) = \theta$. Noting that (2.5) holds in every one-dimensional subspace of a β-homogeneous F^*-space, it follows from $P_i(T(S(E))) \subseteq \mathrm{span}(Te_i)$ and (2.7) that

$$P_i(Tx) = \pm|\lambda_i|T(\varepsilon_i e_i) \quad (1 \le i \le n).$$

Suppose that, in the last equation, there exists at least one equality whose right-hand side bears a negative sign. Then, we may assume that $P_j(Tx) = -|\lambda_j|T(\varepsilon_j e_j)$ for some $\lambda_j \neq 0$ $(1 \le j \le n)$. By this assumption, we derive that

$$Tx + T(\varepsilon_j e_j) = \sum_{i=1, i\neq j}^{n} P_i(Tx) + T(\varepsilon_j e_j) - |\lambda_j|T(\varepsilon_j e_j),$$

and thus

$$\begin{aligned}\|Tx + T(\varepsilon_j e_j)\| &= \left\| \sum_{i=1, i\neq j}^{n} P_i(Tx) \right\| + (1 - |\lambda_j|)^\beta \|T(\varepsilon_j e_j)\| \\ &= 1 - |\lambda_j|^\beta + (1 - |\lambda_j|)^\beta.\end{aligned}$$

However, since T is isometric and the equality (2.7) holds, we obtain that

$$\begin{aligned}\|Tx + T(\varepsilon_j e_j)\| &= \|Tx - T(-\varepsilon_j e_j)\| = \|x + \varepsilon_j e_j\| \\ &= \sum_{i=1, i\neq j}^{n} \|\lambda_i e_i\| + \|(\lambda_j + \varepsilon_j)e_j\| = 1 - |\lambda_j|^\beta + |\lambda_j + \varepsilon_j|^\beta \\ &= 1 - |\lambda_j|^\beta + (1 + |\lambda_j|)^\beta.\end{aligned}$$

As a result, we derive that $\lambda_j = 0$ holds for the last two equalities, a contradiction. Therefore, we conclude that $P_i(Tx) = |\lambda_i|T(\varepsilon_i e_i)$, $1 \le i \le n$. We obtain by (2.7) that $P_i(Tx) = \lambda_i Te_i,\ 1 \le i \le n$, thus proving the lemma. □

Example 1. In (ii) of Lemma 2.3, the condition $P_\gamma(T(S(E))) \subseteq \mathrm{span}(Te_\gamma)$ need not imply $T(-e_\gamma) \in \mathrm{span}(Te_\gamma)$. For instance, we define a mapping $T : S(l_1^2) \to S(l_1^3)$ for any $x = a_1e_1 + a_2e_2 \in S(l_1^2)$ by

$$Tx = \begin{cases} a_1e_1' + a_2e_3' & \text{if } a_1 \ge 0, \\ a_1e_2' + a_2e_3' & \text{if } a_1 < 0, \end{cases}$$

where $\{e_1 = (1,0),\ e_2 = (0,1)\}$ is a standard basis of l_1^2, and $\{e_1' = (1,0,0),\ e_2' = (0,1,0),\ e_3' = (0,0,1)\}$ is a standard basis of l_1^3. It is easy to see that T is isometric with the property that $P_n(T(S(l_1^2))) \subseteq \mathrm{span}(Te_n)$ for $n = 1, 2$. But $T(-e_1) = -e_2' \notin \mathrm{span}(Te_1) = \mathrm{span}(e_1')$.

Lemma 2.4. *Let p be a real number satisfying the conditions $p > 1$ and $p \neq 2$, and let the element x and the e_i's be as in Lemma 2.3. Suppose that the isometric embedding $T\colon S(E) \to S(F)$ satisfies $P_i(T(S(E))) \subseteq \mathrm{span}(Te_i)$ for all $1 \le i \le n$ $(n \in \mathbb{N})$. Then $T(x) = \sum_{i=1}^{n} \lambda_i Te_i$.*

Proof. We need only to note that AL_p-norms satisfy the p-additivity and that the equality (2.2) of Corollary 2.1 holds. The method used in the proof of Lemma 2.3 can be applied here for the AL_p-spaces satisfying $p > 1,\ p \neq 2$. Indeed, as in the proof of the last lemma, we conclude that

$$\begin{aligned}\|Tx - T(\varepsilon_i e_i)\|^p = \|x - \varepsilon_i e_i\|^p &= \sum_{j=1, j\neq i}^{n} |\lambda_j|^p + |\lambda_i - \varepsilon_i|^p \\ &= 1 - |\lambda_i|^p + |\lambda_i - \varepsilon_i|^p = 1 - |\lambda_i|^p + (1 - |\lambda_i|)^p,\end{aligned}$$

and that

$$\begin{aligned}\|Tx - T(\varepsilon_i e_i)\|^p &= \left\| \sum_{j=1, j\neq i}^{n} P_j(Tx) + P(Tx) + P_i(Tx) - T(\varepsilon_i e_i) \right\|^p \\ &= \|Tx\|^p - \|P_i(Tx)\|^p + \|P_i(Tx) - T(\varepsilon_i e_i)\|^p \\ &= 1 - |\lambda_i'|^p + \|\lambda_i' T(\varepsilon_i e_i) - T(\varepsilon_i e_i)\|^p = 1 - |\lambda_i'|^p + |\lambda_i' - 1|^p \\ &\geq 1 - |\lambda_i'|^p + (1 - |\lambda_i'|)^p.\end{aligned}$$

It follows that

$$|\lambda_i|^p - (1 - |\lambda_i|)^p \leq |\lambda_i'|^p - (1 - |\lambda_i'|)^p,$$

and thus $|\lambda_i| \leq |\lambda_i'|$. However, since

$$\sum_{i=1}^{n} \|\lambda_i e_i\|^p = \|x\|^p = \|Tx\|^p = \sum_{i=1}^{n} \|P_i(Tx)\|^p + \|P(Tx)\|^p,$$

we conclude that $\|P_i(Tx)\| = |\lambda_i'| = |\lambda_i|$ and $P(Tx) = \theta$.

The remainder of the proof is similar to the corresponding part of Lemma 2.3 and shall be omitted here. □

3. The main results

Theorem 3.1. *Let $0 < p < \infty$ with $p \neq 2$. An isometry $T\colon S(E) \to S(F)$ can be linearly isometrically extended if and only if, for every $\gamma \in \Gamma$,*

(i) $P_\gamma(T(S(E))) \subseteq \operatorname{span}(Te_\gamma)$ *when $p \neq 1$, and*
(ii) $P_\gamma(T(S(E))) \subseteq \operatorname{span}(Te_\gamma)$ *and $T(-e_\gamma) \in \operatorname{span}(Te_\gamma)$ when $p = 1$.*

Proof. First we show the sufficiency. Suppose that $x = \sum a_\gamma e_\gamma \in S(E)$. It is well known that E is linearly isometric to $l_p(\Gamma)$. By the characterization of elements in $l_p(\Gamma)$, the set $\{\gamma \in \Gamma :\ a_\gamma \neq 0\} := I_x$ is countable. By the continuity of isometries, Lemma 2.3, and Lemma 2.4, we conclude that

$$Tx = \sum_{i \in I_x} a_i T e_i = \sum a_\gamma T e_\gamma \quad (\forall x = \sum a_\gamma e_\gamma \in S(E)). \tag{3.10}$$

Define the operator $\widetilde{T}$ on E by

$$\widetilde{T}(x) = \begin{cases} \|x\| T\left(\frac{x}{\|x\|}\right) & \text{if } x \neq \theta, \\ \theta & \text{if } x = \theta \quad (\forall x \in E). \end{cases} \tag{3.11}$$

Suppose that $x = \sum a_\gamma e_\gamma$, $y = \sum b_\gamma e_\gamma \in E$ and $s, t \in \mathbb{R}$. By virtue of (3.10) and (3.11), we obtain that

$$\widetilde{T}(x) = \|x\| T\left(\frac{x}{\|x\|}\right) = \|x\| \sum \frac{a_\gamma}{\|x\|} T e_\gamma = \sum a_\gamma T e_\gamma.$$

Therefore, we have

$$\widetilde{T}(sx+ty)=\sum(sa_\gamma+tb_\gamma)Te_\gamma=s\sum a_\gamma Te_\gamma+t\sum b_\gamma Te_\gamma=s\widetilde{T}(x)+t\widetilde{T}(y).$$

The last equation shows that $\widetilde{T}$ is a linear isometry from E into F, and that $\widetilde{T}$ is a linearly isometric extension of T.

Now we prove the necessity. Suppose that T can be extended to a linear isometry from E to F. Following the method used in the proof of Lemma 2.3, it is not hard to show that $\{B_{Te_\gamma}\}_{\gamma\in\Gamma}$ is mutually disjoint.

The remaining part of the proof is trivial and thus is omitted. □

Throughout the remainder of this paper we use E and F to denote complex atomic AL_p-spaces.

Theorem 3.2. *Let $0<p<\infty$ with $p\neq 2$. Let E and F be complex atomic AL_p-spaces. Suppose that, for every $\gamma\in\Gamma$, we have*

(i) $P_\gamma(T(S(E)))\subseteq \operatorname{span}(Te_\gamma)$ *when $p\neq 1$, and*
(ii) $P_\gamma(T(S(E)))\subseteq \operatorname{span}(Te_\gamma)$ *and $T(-e_\gamma)\in\operatorname{span}(Te_\gamma)$ when $p=1$.*

Then, there exist an additive isometry $\widetilde{T}$ from E into F and two mutually complementary subspaces X and Y of E such that

(1) *the restriction of $\widetilde{T}$ to $S(E)$ is T,*
(2) *that to X is linear, and*
(3) *that to Y is conjugate linear.*

To show Theorem 3.2, we need the following lemma.

Lemma 3.1. *Let $D=\{\lambda\in\mathbb{C}\colon\ |\lambda|=1\}$. Suppose that T is as in Theorem 3.2. Then, for every $\gamma\in\Gamma$,*

$$\text{either } T(\lambda e_\gamma)=\lambda Te_\gamma \text{ or } T(\lambda e_\gamma)=\overline{\lambda}Te_\gamma$$

for any $\lambda\in D$.

Proof. The method Xiao Yuanhui used in [21, Lemma 7] is valid for this assertion. □

Proof of Theorem 3.2. By combining Lemma 2.3, Lemma 2.4, Lemma 3.1 and Theorem 3.1, the proof that Xiao Yuanhui introduced in [21, Theorem 2] can be easily modified to prove the assertion. □

Remark 3.1. For the case of $p=2$, Theorem 3.1 and Theorem 3.2 are also true (cf. [4]).

Corollary 3.1. *If the mapping T carries some atom $e\in S(E)$ to an atom $Te\in S(F)$, then the assertions of Theorems 3.1 and 3.2 remain true.*

Proof. It is sufficient to consider the real atomic AL_p-spaces. Since an AL_p-space is an Archimedean F-lattice and Te_γ is an atom for each $\gamma\in\Gamma$, it follows that $B_{Te_\gamma}=\operatorname{span}(Te_\gamma)$ for any $\gamma\in\Gamma$, and that $P_\gamma(T(S(E)))\subseteq B_{Te_\gamma}=\operatorname{span}(Te_\gamma)$ for any $\gamma\in\Gamma$. The last equality is the condition required in Theorems 3.1 and 3.2, thus proving the assertion. □

Corollary 3.2. *If $T\colon S(E)\to S(F)$ is a surjective isometry, then the assertions of Theorems 3.1 and 3.2 remain true.*

Proof. Again it is sufficient to consider the real case. The complex case can be obtained by applying Xiao's method of proof [21, Theorem 2].

To show that

$$T(-e_\gamma) = -Te_\gamma \quad (\forall \gamma \in \Gamma), \tag{3.12}$$

we consider three cases: $0 < p < 1$, $p = 1$ and $p > 1$.

First we consider the case that $0 < p < 1$. Since T is isometric and the elements of $\{e_\gamma\}_{\gamma\in\Gamma}$ are mutually disjoint, it follows by (2.3) of Corollary 2.1 that, for any $\gamma, \tau \in \Gamma$, $\gamma \neq \tau$,

$$\|T(\pm e_\gamma) - Te_\tau\| = \| \pm e_\gamma - e_\tau\| = \| \pm e_\gamma\| + \|e_\tau\| = \|T(\pm e_\gamma)\| + \|Te_\tau\|. \tag{3.13}$$

Consequently, we have $|T(\pm e_\gamma)| \wedge |Te_\tau| = \theta$ $(\gamma, \tau \in \Gamma,\ \gamma \neq \tau)$, and the elements of $\{Te_\gamma\}_{\gamma\in\Gamma}$ are mutually disjoint. Therefore, $\{B_{Te_\gamma}\}_{\gamma\in\Gamma}$ are mutually disjoint.

Suppose that $\gamma \in \Gamma$. We show first that

$$Tw \in B_{Te_\gamma} \Rightarrow w = \pm e_\gamma \quad \text{for any } w \in S(E). \tag{*}$$

Indeed, by the preceding proof, we have $|Tw| \wedge |Te_\tau| = \theta$ $(\tau \in \Gamma,\ \tau \neq \gamma)$, and thus $|w| \wedge |e_\tau| = \theta$ $(\tau \in \Gamma,\ \tau \neq \gamma)$. Therefore, we conclude that $w \in \operatorname{span}(e_\gamma)$. The conclusion that $w = \pm e_\gamma$ follows immediately by the fact that $w \in S(E)$.

In order to show that $T(-e_\gamma) \in B_{Te_\gamma}$, we assume, without loss of generality, that P' is the band projection of F onto the band generated by $\{Te_\tau\}_{\tau\in\Gamma,\tau\neq\gamma}$, and that $P = I - P_\gamma - P'$. Since F is order complete, it follows that P is a band projection from F onto the orthogonal complement of $\{Te_\tau\}_{\tau\in\Gamma}$. By (3.13) and (2.3) of Corollary 2.1, we derive that $|T(-e_\gamma)| \wedge |Te_\tau| = \theta$ $(\tau \in \Gamma,\ \tau \neq \gamma)$.

Therefore, we obtain that $P'(T(-e_\gamma)) = \theta$, and that $T(-e_\gamma) = P_\gamma(T(-e_\gamma)) + P(T(-e_\gamma))$.

If $P(T(-e_\gamma)) \neq \theta$, since T is onto, there exists $y \in S(E)$ such that $Ty = \frac{P(T(-e_\gamma))}{\|P(T(-e_\gamma))\|^{1/p}}$. Note that $|Ty| \wedge |Te_\tau| = \theta$ for all $\tau \in \Gamma$. It follows that $|y| \wedge |e_\tau| = \theta$ for all $\tau \in \Gamma$; thus $y = \theta$, and $P(T(-e_\gamma)) = \theta$. Therefore, $T(-e_\gamma) = P_\gamma(T(-e_\gamma)) \in B_{Te_\gamma}$.

Next we show that $T(-e_\gamma) = -Te_\gamma$. Obviously, we have $T(-e_\gamma) + Te_\gamma \in B_{Te_\gamma}$. Suppose that we have the contrary; that is, suppose that $T(-e_\gamma) + Te_\gamma \neq \theta$.

Let $\alpha = \frac{1}{\|T(-e_\gamma)+Te_\gamma\|^{1/p}}$. Then, it follows that $\alpha(T(-e_\gamma) + Te_\gamma) \in S(F) \cap B_{Te_\gamma}$. By the fact that the mapping T is onto, there exists $y \in S(E)$ with $Ty = \alpha(T(-e_\gamma) + Te_\gamma)$. Similar to the preceding proof, we conclude that $y = \pm e_\gamma$. Therefore, we have $(1 - \alpha)T(\pm e_\gamma) = T(\mp e_\gamma)$. By the fact that T is isometric, we deduce that $|1 - \alpha| = 1$. This last result implies that $\alpha = 2$, since α is not equal to zero. Therefore, we conclude that $-T(\pm e_\gamma) = T(\mp e_\gamma)$. That is, $T(-e_\gamma)+Te_\gamma = \theta$. As a result, we obtain a contradiction. Therefore, we conclude that $T(-e_\gamma)+Te_\gamma = \theta$. That is, $T(-e_\gamma) = -Te_\gamma$.

For the case $p = 1$, Ma Yumei proved in [10] that the condition that $T\colon S(E) \to S(F)$ is a surjective isometry implies that the equality $T(-x) = -Tx$ holds for every $x \in S(E)$.

For $p > 1$, for all $x \in S(E)$ we have

$$\|Tx + (-T(-x))\| = \|Tx - T(-x)\| = \|2x\| = \|Tx\| + \| - T(-x)\|.$$

Since the space F in this case is strictly convex, there exists a constant $c > 0$ for which $Tx = c(-T(-x))$. The conclusion that $c = 1$ follows immediately, because T is isometric. Therefore, $T(-x) = -Tx$ for all $x \in S(E)$.

Summing up the above discussion, we conclude that the equality (3.12) holds for every positive number $p \neq 2$.

Similarly to the proof of Lemma 2.3, it follows by (2.2) and (2.3) of Corollary 2.1 and (3.12) that $\{B_{Te_\gamma}\}_{\gamma\in\Gamma}$ are mutually disjoint.

For a number p satisfying $0 < p < \infty$, $p \neq 2$, and an arbitrary $\gamma \in \Gamma$, we now show that $B_{Te_\gamma} = \operatorname{span}(Te_\gamma)$.

For an element $u \in B_{Te_\gamma}$, if $u = \theta$, then $u \in \operatorname{span}(Te_\gamma)$. Therefore, we may assume that $\theta \neq u$. Let

$$u_0 = \begin{cases} \frac{u}{\|u\|^{1/p}} & \text{if } 0 < p < 1, \\ \frac{u}{\|u\|} & \text{if } p \geq 1. \end{cases}$$

Then, we have $u_0 \in S(F) \cap B_{Te_\gamma}$. Note that the equality $(*)$ also holds for the cases that $p \geq 1$. Since T is a surjective isometry, it follows by $(*)$ that there exists $x_0 \in S(E)$ such that $Tx_0 = u_0$; thus $x_0 = \pm e_\gamma$. Therefore, we conclude that $u = \|u\|^{1/p}u_0 = \|u\|^{1/p}Tx_0 = \|u\|^{1/p}T(\pm e_\gamma) = \pm\|u\|^{1/p}Te_\gamma \in \operatorname{span}(Te_\gamma)$ for $0 < p < 1$, and that $u = \pm\|u\|Te_\gamma \in \operatorname{span}(Te_\gamma)$ for $p \geq 1$.

It follows easily by Theorem 3.1 that T can be linearly isometrically extended to E, and this proves the theorem. □

Remark 3.2. For the cases that $p > 1$, Corollary 3.2 is just a result of Xiao [21, Theorem 1]. In the proof of Corollary 3.2, we have actually shown that if T is surjective, then it maps atoms of $S(E)$ into atoms of $S(F)$ in a way that is different from what Professor Ding Guanggui had described in [5]. The converse is not true, however. For example, let $T\colon S(l_1) \to S(l_1)$ be defined by $Tx = \sum \lambda_i e_{2i-1}$ for all $x = \sum \lambda_i e_i \in S(l_1)$, where $\{e_i\}_{i\in\mathbb{N}}$ is a standard basis. Obviously, T maps atoms of $S(l_1)$ into atoms of $S(l_1)$, but T is not surjective.

Remark 3.3. We can see in the proofs of Lemma 2.3 and Lemma 2.4 that an isometry $T\colon S(E) \to S(F)$ can be linearly isometrically extended if and only if, for any $\gamma \in \Gamma$, there exists $\{a_{\gamma,\gamma'}\}_{\gamma'\in\Gamma'_\gamma} \subset \mathbb{R}$ with $\sum_{\gamma'\in\Gamma'_\gamma} |a_{\gamma,\gamma'}|^p = 1$ such that

$$Tx = \sum_{\gamma\in\Gamma} \sum_{\gamma'\in\Gamma'_\gamma} \lambda_\gamma a_{\gamma,\gamma'} e'_{\gamma'} \quad (\forall x = \sum_{\gamma\in\Gamma} \lambda_\gamma e_\gamma \in S(E)),$$

where $\{e'_{\gamma'}\}_{\gamma'\in\Gamma'}$ is a norm-one complete disjoint system of F, *and* $\Gamma'_\gamma = \{\gamma' \in \Gamma' : e'_{\gamma'} \in B_{Te_\gamma}\}$.

Acknowledgment

I would like to express my deep gratitude to my Ph.D. advisor, Professor Ding Guanggui for his advice. I also would like to express my sincere gratitude to referees for their comments. Finally, I would like to thank Dr. Zhaoyang Wu for his helpful suggestions in English writing.

References

[1] C. D. Aliprantis and O. Burkinshaw, *Locally solid Riesz spaces*, Pure and Applied Math., Vol. 76, Academic Press, New York-San Francisco-London, 1978. MR 58:12271

[2] C. D. Aliprantis and O. Burkinshaw, *Positive operators*, Pure and Applied Mathematics, Vol. 119, Academic Press, Orlando, FL, 1985. MR 87h:47086

[3] Ding Guanggui, *On the extension of isometries between unit spheres of E and $C(\Omega)$*, Acta Math. Sinica (to appear).

[4] Ding Guanggui, *The* 1*-Lipschitz mapping between the unit spheres of two Hilbert spaces can be extended to a real linear isometry of the whole space*, Science in China, Ser. A, **45**(4) (2002), 479—483. (English, English summary) MR 2003d:46029

[5] Ding Guanggui, *The isometric extension problem in the unit spheres of* $l^p(\Gamma), (p > 1)$ *type spaces*, Science in China, Ser. A, **46**(3) (2003), 333—338. (English, English summary)

[6] Ding Guanggui, *On perturbations and extensions of isometric operators*, Taiwanese J. Math., **5** (2001), 109—115. (English, English summary) MR 2002f:47031

[7] H. E. Lacey, *The isometric theory of classical Banach spaces*, Springer-Verlag, New York, 1974. MR 58:12308

[8] J. Lamperti, *On the isometries of certain function-spaces*, Pacific J. Math., **8** (1958), 459—466. MR 21:3764

[9] P. Mankiewicz, *Fat equicontinuous groups of homeomorphisms of linear topological spaces and their application to the problem of isometries in linear metric spaces*, Studia Math., **64** (1979), 13—23. MR 80g:46004

[10] Y.-M. Ma, *Isometries of the unit spheres*, Acta. Math. Sci., **12** (1992), 366—373. (English, English summary) MR 94c:46021

[11] H. L. Royden, *Real Analysis*, Macmillan, New York, 1968. MR 27:1540

[12] H. H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, Berlin, Heidelberg, New York, 1974. MR 54:11023

[13] D. Tingley, *Isometries of the unit spheres*, Geometriae Dedicata, **22** (1987), 371—378. MR 88f:46047

[14] R.-S. Wang, *Isometries between the unit spheres of* $C_0(\Omega)$ *type spaces*, Acta Math. Sci., **14**(1) (1994), 82—89. (English, English summary) MR 95b:46041

[15] R.-S. Wang, *Isometries on the* $C_0(\Omega, E)$ *type spaces*, J. Math. Sci. Univ. Tokyo, **2** (1995), 117—130. (English, English summary) MR 96h:46047

[16] R.-S. Wang, *Isometries of* $C_0(\Omega, \sigma)$ *type spaces*, Kobe J. Math., **12** (1995), 31—43. (English, English summary) MR 97e:46024

[17] R.-S. Wang, *Isometries of* $C_0^{(n)}(X)$, Hokkaido Math. J., **25**(3) (1996), 465—519. (English, English summary) MR 97m:46033

[18] R.-S. Wang and A. Orihara, *Isometries on the* l^1*-sum of* $C_0(\Omega, E)$ *type spaces*, J. Math. Sci. Univ. Tokyo, **2** (1995), 131—154. (English, English summary) MR 96h:46048

[19] R.-S. Wang, *Isometries on the* l^p*-sum of* $C_0(\Omega, E)$ *type spaces*, J. Math. Sci. Univ. Tokyo, **3** (1996), 471—493. (English, English summary) MR 97i:46071

[20] R.-S. Wang and A. Orihara, *Isometries between the unit spheres on the* l^1*-sum of strictly convex spaces*, (to appear).

[21] Y.-H. Xiao, *On extension of isometries between unit spheres of abstract* L_p*-spaces*, Acta. Sci. Nat. Univ. Nankai, **28** (1995), 11—14. (English, English summary)

[22] Y.-H. Xiao and R.-S. Wang, *On extension of isometries between unit spheres of abstract* AM*-spaces*, Chinese Ann. Math. (A), **22**(1) (2001), 97—104. (Chinese, English summary); English transl., Chinese J. Contemp. Math. **22** (2001), 79—88. MR 2002a:46002

[23] D.-P. Zhan, *Extensions of isometries between unit spheres*, Acta. Mathematica Sinica, **41**(2) (1998), 275—280. (Chinese, English summary) MR 99m:46046

[24] D.-P. Zhan, *On extension of isometries between unit spheres on the* $L^p(\Omega, H)$ *type spaces*, Acta. Sci. Nat. Univ. Nankai, **30**(4) (1997), 31—35. (Chinese, English summary)

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