

## VECTOR MEASURE BANACH SPACES CONTAINING A COMPLEMENTED COPY OF $c_0$

A. PICÓN AND C. PIÑEIRO

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ABSTRACT. Let  $X$  a Banach space and  $\Sigma$  a  $\sigma$ -algebra of subsets of a set  $\Omega$ . We say that a vector measure Banach space  $(\mathcal{M}(\Sigma, X), \|\cdot\|_{\mathcal{M}})$  has the bounded Vitaly-Hahn-Sacks Property if it satisfies the following condition: Every vector measure  $m : \Sigma \rightarrow X$ , for which there exists a bounded sequence  $(m_n)$  in  $\mathcal{M}(\Sigma, X)$  verifying  $\lim_{n \rightarrow \infty} m_n(A) = m(A)$  for all  $A \in \Sigma$ , must belong to  $\mathcal{M}(\Sigma, X)$ . Among other results, we prove that, if  $\mathcal{M}(\Sigma, X)$  is a vector measure Banach space with the bounded V-H-S Property and containing a complemented copy of  $c_0$ , then  $X$  contains a copy of  $c_0$ .

### 1. INTRODUCTION

Throughout the paper  $X$  will be a Banach space and  $\Sigma$  a  $\sigma$ -algebra of subsets of a set  $\Omega$ .  $\mathcal{M}(\Sigma, X)$  will denote an arbitrary vector space formed for bounded vector measures  $m : \Sigma \rightarrow X$  (finitely additive and with bounded semivariation). We only consider vector measure Banach spaces  $(\mathcal{M}(\Sigma, X), \|\cdot\|_{\mathcal{M}})$  with the following property:

**(P):** “There is some constant  $c > 0$  so that  $\sup_{A \in \Sigma} \|m(A)\| \leq c \|m\|_{\mathcal{M}}$  for every  $m \in \mathcal{M}(\Sigma, X)$ .”

**Definition 1.** We say that a vector measure Banach space  $(\mathcal{M}(\Sigma, X), \|\cdot\|_{\mathcal{M}})$  has the bounded Vitaly-Hahn-Sacks property (briefly, bounded V-H-S property) if it satisfies the following condition:

“If  $m : \Sigma \rightarrow X$  is a vector measure such that there is a bounded sequence  $(m_n)_n$  in  $(\mathcal{M}(\Sigma, X), \|\cdot\|_{\mathcal{M}})$  satisfying  $\lim_{n \rightarrow \infty} m_n(A) = m(A)$  for all  $A \in \Sigma$ , then  $m$  belongs to  $\mathcal{M}(\Sigma, X)$ .”

Our main result establishes that  $X$  contains a copy of  $c_0$  whenever  $\mathcal{M}(\Sigma, X)$  has the bounded V-H-S property and contains a complemented copy of  $c_0$ .

Since the classical vector measure Banach spaces  $ca(\Sigma, X)$ ,  $ca(\Sigma, \mu, X)$ ,  $bvca(\Sigma, X)$ ,  $bvca(\Sigma, \mu, X)$ ,  $rbvca(\Sigma, X)$  and  $rbvca(\Sigma, \mu, X)$  have the bounded V-H-S property, our result subsumes some particular cases considered in [D], [F] and [FE].

In the final section we only consider vector measure Banach spaces on a dual space and study a stronger property.

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**Definition 2.** We say that a vector measure Banach space  $\mathcal{M}(\Sigma, X^*)$  is  $w^*$ -boundedly closed if the following holds:

“If  $m : \Sigma \rightarrow X$  is a vector measure such that there is a bounded net  $(m_i)_{i \in I}$  in  $\mathcal{M}(\Sigma, X^*)$  satisfying  $w^* - \lim_i m_i(A) = m(A)$  for all  $A \in \Sigma$ , then  $m$  belongs to  $\mathcal{M}(\Sigma, X^*)$ .”

We prove that  $\mathcal{M}(\Sigma, X^*)$  is complemented in its bidual whenever it has that property.

## 2. MAIN RESULTS

To start, we establish the equivalence of property (P) and other conditions we will need. We omit its proof because it is a straightforward verification, using the closed graph theorem.

**Proposition 3.** *Let  $(\mathcal{M}(\Sigma, X), \|\cdot\|_{\mathcal{M}})$  be a vector measure Banach space. The following statements are equivalent:*

- (1) *There exists a constant  $c > 0$  such that  $\sup_{A \in \Sigma} \|m(A)\| \leq c \|m\|_{\mathcal{M}}$  for every  $m \in \mathcal{M}(\Sigma, X)$ .*
- (2) *The linear map  $m \in \mathcal{M}(\Sigma, X) \mapsto m(A) \in X$  is continuous, for all  $A \in \Sigma$ .*
- (3) *The linear form  $\chi_A \otimes x^* : m \in \mathcal{M}(\Sigma, X) \mapsto \langle m(A), x^* \rangle \in \mathbf{R}$  is continuous, for all  $A \in \Sigma$  and  $x^* \in X^*$ .*

If  $(\mathcal{M}(\Sigma, X), \|\cdot\|_{\mathcal{M}})$  is an arbitrary vector measure Banach space, we define  $\mathcal{M}(\Sigma, X)_s$  as the vector space of all vector measures  $m : \Sigma \rightarrow X$  such that there exists a bounded sequence  $(m_n)$  in  $\mathcal{M}(\Sigma, X)$  satisfying

$$\lim_{n \rightarrow \infty} m_n(A) = m(A) \text{ for all } A \in \Sigma.$$

For every  $m \in \mathcal{M}(\Sigma, X)_s$ , we define

$$\|m\|_s = \inf_n \sup \|m_n\|_{\mathcal{M}},$$

where the infimum runs over all admissible sequences  $(m_n)$  in  $\mathcal{M}(\Sigma, X)$ . It is a standard argument to prove the next result.

**Proposition 4.**  *$(\mathcal{M}(\Sigma, X)_s, \|\cdot\|_s)$  is a vector measure Banach space containing  $\mathcal{M}(\Sigma, X)$  and, for every  $m \in \mathcal{M}(\Sigma, X)$ , we have  $\|\cdot\|_s \leq \|\cdot\|_{\mathcal{M}}$ .*

Now we are ready to establish our main result.

**Theorem 5.** *Let  $(\mathcal{M}(\Sigma, X), \|\cdot\|_{\mathcal{M}})$  be an arbitrary vector measure Banach space. Then we have:*

- (a) *If  $X$  does not contain a copy of  $c_0$ , every operator  $\mathbf{U} : c_0 \rightarrow \mathcal{M}(\Sigma, X)$  admits a continuous linear extension  $\tilde{\mathbf{U}} : \ell_\infty \rightarrow \mathcal{M}(\Sigma, X)_s$ .*
- (b) *If  $\mathcal{M}(\Sigma, X)$  has the bounded V-H-S property and contains a complemented copy of  $c_0$ , then  $X$  contains  $c_0$  isomorphically.*
- (c) *If  $X$  does not contain a copy of  $c_0$  and  $\mathcal{M}(\Sigma, X)$  has the bounded V-H-S property, then  $\mathcal{M}(\Sigma, X)$  contains  $c_0$  if and only if it contains  $\ell_\infty$ .*

*Proof.* (a) Put  $m_n = \mathbf{U}e_n$ , for all  $n \in \mathbf{N}$ . From Proposition 3 it follows that, for all  $A \in \Sigma$ , the series  $\sum m_n(A)$  is w.u.c. in  $X$  and, therefore, unconditionally

convergent. Then, for every  $\psi = (\psi_n)_n \in \ell_\infty$ , we can define a finitely additive measure  $\tilde{U}(\psi) : \Sigma \rightarrow X$  by

$$\tilde{U}(\psi)(A) = \sum_{n=1}^{\infty} \psi_n m(A_n), \quad \forall A \in \Sigma.$$

Since  $\sum_{k=1}^n \psi_k m_k = \mathbf{U}(\psi_1, \psi_2, \dots, \psi_n, 0, \dots)$ , the sequence  $(\sum_{k=1}^n \psi_k m_k)_n$  is bounded in  $\mathcal{M}(\Sigma, X)$ . Then the vector measure  $\tilde{U}(\psi)$  belongs to  $\mathcal{M}(\Sigma, X)_s$ . So, we can consider the linear map

$$\tilde{U} : \psi \in \ell_\infty \mapsto \tilde{U}(\psi) \in \mathcal{M}(\Sigma, X)_s,$$

which is continuous because of the closed graph theorem.

(b) Now assume  $\mathbf{U} : c_0 \rightarrow \mathcal{M}(\Sigma, X)$  is an isomorphism onto a complemented subspace  $H$  of  $\mathcal{M}(\Sigma, X)$ . By contradiction, suppose  $X$  does not contain a copy of  $c_0$ . Statement (a) tells us that there is a continuous linear extension  $\tilde{U} : \ell_\infty \rightarrow \mathcal{M}(\Sigma, X)$ . If  $P$  is a continuous projection from  $\mathcal{M}(\Sigma, X)$  onto  $H$ , then  $\mathbf{U}^{-1}|_H \circ P \circ \tilde{U}$  is a projection from  $\ell_\infty$  onto  $c_0$  and this is a contradiction.

(c) We only need to prove that  $\mathcal{M}(\Sigma, X)$  contains  $\ell_\infty$  whenever it contains  $c_0$ . Since  $X$  does not contain a copy of  $c_0$ , statement (a) and [CM, Theorem 1.3.1] complete the proof. □

Using the Vitaly-Hahn-Sacks theorem, it is easy to prove that the classical vector measure Banach spaces, above-mentioned, have the bounded V-H-S property. So, we omit its proofs. Anyway, in the next section we are going to prove that  $rbvca(\Sigma, X^*)$  is  $w^*$ -boundedly closed.

*Remark.* (i)  $L_1(\mu, X)$  has not the bounded V-H-S property, in general. For an example, take  $X = c_0$  and consider the vector measure:

$$m : A \in \Sigma \rightarrow \left( \int_A \sin(2^n \pi t) d\mu(t) \right)_{n=1}^\infty \in c_0,$$

where  $\Sigma$  is the  $\sigma$ -field of Lebesgue measurable subsets of  $[0,1]$  and  $\mu$  the Lebesgue measure on  $\Sigma$ . In [DU, p. 60] it is proved that  $m$  has no Radon-Nikodým derivative with respect to  $\mu$ . Nevertheless,  $\lim_{n \rightarrow \infty} m_n(A) = m(A)$ , for all  $A \in \Sigma$ ,  $m_n$  being the vector measure defined by

$$m_n(A) = \sum_{k=1}^n \left( \int_A \sin(2^k \pi t) d\mu(t) \right) \cdot e_k, \quad \text{for all } A \in \Sigma \text{ and } n \in \mathbf{N}.$$

(ii) Obviously, if  $X$  has the Radon-Nikodým property, then  $L_1(\mu, X) = bvca(\Sigma, \mu, X)$ . So, in this case  $L_1(\mu, X)$  has the bounded V-H-S property.

(iii) When  $X$  has a Schauder basis, it is easy to show that

$$L_1(\mu, X)_s = bvca(\Sigma, \mu, X).$$

But there exist Banach spaces  $X$  for which the last equality is not true. To see this, note that every vector measure  $m$  belonging to  $L_1(\mu, X)_s$  has separable range. Then it suffices to find a vector measure  $m \in bvca(\Sigma, \mu, X)$  whose range is nonseparable. Choose a nonseparable space  $X = L_1(\mu)$  (see [L, Theorem 12, Chapter 5]), and consider the vector measure  $m : \Sigma \rightarrow X$  defined by  $m(A) = \chi_A$ , for all  $A \in \Sigma$ . Since  $rg(m)$  is nonseparable, it follows that  $m$  does not belong to  $L_1(\mu, X)_s$ .

The converse of the above theorem is not true, in general. For an example, consider the vector measure Banach space  $rbvca(\Sigma, X)$  of all regular countably additive  $X$ -valued measures with bounded variation defined on the Borel subsets of a compact space  $\Omega$  (endowed with the variation norm). It is well known that this Banach space is isomorphically isometric to  $\Pi_1(\mathcal{C}(\Omega), X)$ , the Banach space of all 1-summing operators from  $\mathcal{C}(\Omega)$  into  $X$  (see [DU]). Take a Banach space  $X$  such that  $X^{**}$  contains a copy of  $c_0$  and nevertheless  $rbvca(\Sigma, X^{**})$  does not contain a complemented copy of  $c_0$ , because it is isomorphically isometric to the dual of  $\mathcal{N}_\infty(X, \mathcal{C}(\Omega))$  (the Banach space of all  $\infty$ -nuclear operators from  $X$  into  $\mathcal{C}(\Omega)$ ). Then  $rbvca(\Sigma, X^{**})$  is the desired example.

Anyway, for certain vector measure Banach spaces  $\mathcal{M}(\Sigma, X)$  it is true that  $\mathcal{M}(\Sigma, X)$  contains a complemented copy of  $c_0$  whenever  $X$  contains a copy of  $c_0$ . We say that the measurable space  $(\Omega, \Sigma)$  is nontrivial if there exists an infinite sequence  $(A_n)_n$  of pairwise disjoint, nonempty, members of  $\Sigma$ .

**Proposition 6.** *Let  $(\Omega, \Sigma)$  be a nontrivial measurable space, and let  $\mathcal{M}(\Sigma, X)$  be a vector measure Banach space of strongly additive measures, with the following property:*

(\*): “For all w.u.c. sequences  $(x_n)$  in  $X$  and all null sequences  $(\mu_n)$  in  $ca(\Sigma)$ , the vector measure  $m = \sum_{n=1}^\infty \mu_n \otimes x_n$  belongs to  $\mathcal{M}(\Sigma, X)$ .”

If  $X$  contains a copy of  $c_0$ , then  $\mathcal{M}(\Sigma, X)$  contains a complemented copy of  $c_0$ .

*Proof.* Let  $(x_n)$  be a copy in  $X$  of the unit vector basis of  $c_0$ , and let  $(x_n^*)$  be a sequence of biorthogonal coefficients. Take a sequence  $(A_n)$  of pairwise disjoint, nonempty, members of  $\Sigma$  and choose  $w_n \in A_n$ , for all  $n \in \mathcal{N}$ . The sequence  $(\chi_{A_n} \otimes x_n^*)$  is  $*$ -weak null in  $(\mathcal{M}(\Sigma, X), \|\cdot\|_{\mathcal{M}})$ .

So

$$\mathbf{V} : m \in \mathcal{M}(\Sigma, X) \mapsto (\langle m, \chi_{A_n} \otimes x_n^* \rangle)_n \in c_0$$

defines an operator. We can consider a second operator

$$\mathbf{U} : e_n \in c_0 \mapsto \delta_{w_n} \otimes x_n \in \mathcal{M}(\Sigma, X).$$

$\mathbf{U}$  is well-defined by (\*) and continuous by the closed graph theorem. Note that  $\mathbf{V}\mathbf{U}$  is the identity on  $c_0$ . Therefore  $\mathbf{U}\mathbf{V}$  is a projection from  $\mathcal{M}(\Sigma, X)$  onto a subspace  $H$ . It is easy to prove that  $\mathbf{U}\mathbf{V}\mathbf{U}$  is an isomorphism from  $c_0$  onto  $H$ .  $\square$

### 3. VECTOR MEASURE BANACH SPACES COMPLEMENTED IN THEIR BIDUAL

In this section we only consider vector measure Banach spaces on a dual  $X^*$ . We will denote by  $ba(\Sigma, X^*)$  the vector space of all bounded additive measures  $m : \Sigma \rightarrow X^*$ . We endow it with the locally convex topology  $\tau$  defined by the family of seminorms

$$\rho_{A, X}(m) = |\langle x, m(A) \rangle|, \text{ for all } m \in ba(\Sigma, X^*),$$

where  $A$  and  $X$  range over  $\Sigma$  and  $X$ , respectively.

If  $\mathcal{M}(\Sigma, X^*)$  is an arbitrary vector measure Banach space and  $B_0$  is its closed unit ball, put  $B = \overline{B_0}^\tau$ .  $B$  is absolutely convex,  $\tau$ -closed and bounded in  $(ba(\Sigma, X^*), \tau)$ . For the sake of simplicity, put  $Y = ba(\Sigma, X^*)$ . It is well known that  $Y_B = \bigcup_n nB$  is a normed space if we endow it with the norm

$$\rho_B(m) = \inf\{\lambda > 0 : m \in \lambda B\}$$

and  $B$  becomes the closed unit ball of  $Y_B$ . It is easy to prove that  $Y_B$  satisfies property (P) (with the same constant as  $\mathcal{M}(\Sigma, X^*)$ ) and the equality

$$\rho_B(m) = \inf \sup_i \|m\|_{\mathcal{M}}$$

holds, where the infimum is taken over all bounded nets  $(m_i)_{i \in I}$  in  $\mathcal{M}(\Sigma, X^*)$  for which  $\lim_i m_i(A) = m(A)$ , for every  $A \in \Sigma$ .

**Lemma 7.**  $(Y_B, \rho_B)$  is a Banach space.

*Proof.* It suffices to prove that every absolutely convergent series  $\sum_n m_n$  in  $Y_B$  is convergent. So, suppose  $\sum_{n=1}^\infty \rho_B(m_n) \leq 1$ . By property (P), there is a constant  $c > 0$  such that

$$\sup_{A \in \Sigma} \|m(A)\| \leq c \rho_B(m), \quad \forall m \in Y_B.$$

Then  $\sum_{n=1}^\infty \|m(A)\| < \infty$  for all  $A \in \Sigma$ , and we can define  $m : \Sigma \rightarrow X^*$  by  $m(A) = \sum_{n=1}^\infty m_n(A)$ . Since

$$\rho_B\left(\sum_{n=1}^N m_n\right) \leq \sum_{n=1}^N \rho_B(m_n) \leq 1,$$

it follows that  $\sum_{n=1}^N m_n \in B$  for all  $n \in \mathcal{N}$ .

Then  $m \in B$  because  $B$  is  $\tau$ -closed.

We must prove that  $m = \sum_{n=1}^\infty m_n$  in  $Y_B$ . For this, given  $\varepsilon > 0$ , choose  $n_0 \in \mathcal{N}$  such that  $\sum_{n > n_0} \rho_B(m_n) < \varepsilon$ . This yields  $\rho_B(\sum_{n \in H} m_n) < \varepsilon$ , for all finite sets  $H \subset \{n_0 + 1, n_0 + 2, \dots\}$ . Then  $\sum_{n \in H} m_n \in \varepsilon B$ , for all such sets  $H$ . Now, in the same way, it is easy to show that  $\sum_{k=n}^\infty m_k \in \varepsilon B$  for all  $n > n_0$ , and this implies that

$$\rho_B\left(m - \sum_{k=1}^n m_k\right) = \rho_B\left(\sum_{k=n+1}^\infty m_k\right) \leq \varepsilon, \quad \text{for all } n > n_0.$$

□

From now on, we denote  $Y_B$  by  $\mathcal{M}(\Sigma, X^*)_{w^*}$  and  $\rho_B(m)$  by  $\|m\|_{w^*}$ .

Since  $B$  is  $\tau$ -closed, the following equality follows obviously:

$$(\mathcal{M}(\Sigma, X^*)_{w^*})_{w^*} = \mathcal{M}(\Sigma, X^*)_{w^*}.$$

The next result tells us that  $\mathcal{M}(\Sigma, X^*)$  is complemented in its bidual whenever it is  $w^*$ -boundedly closed.

**Theorem 8.** Let  $\mathcal{M}(\Sigma, X^*)$  be a vector measure Banach space  $w^*$ -boundedly closed. Then there is a continuous projection from  $\mathcal{M}(\Sigma, X^*)^{**}$  onto  $\mathcal{M}(\Sigma, X^*)$ .

*Proof.* We define  $P : \phi \in \mathcal{M}(\Sigma, X^*)^{**} \mapsto P\phi \in \mathcal{M}(\Sigma, X^*)$  by

$$\langle x, (P\phi)(A) \rangle = \phi(x \otimes \chi_A) \quad \text{for all } A \in \Sigma \text{ and } x \in X.$$

So,  $P\phi$  is a finitely additive  $X^*$ -valued measure satisfying

$$\|P\phi(A)\| \leq c\|\phi\| \quad \text{for all } A \in \Sigma.$$

On the other hand, there exists a net  $(m_i)_{i \in I}$  in  $\mathcal{M}(\Sigma, X^*)$  for which we have

$$w^* \text{-} \lim_i m_i = \phi \quad \text{and} \quad \|m_i\| \leq \|\phi\|, \quad \forall i \in I.$$

This yields that  $w^*\text{-}\lim_i m_i(A) = (P\phi)(A)$ . So,  $P\phi \in \mathcal{M}(\Sigma, X^*)$ . Since  $\mathcal{M}(\Sigma, X^*)$  is  $w^*$ -boundedly closed, the open mapping theorem tells us that there is a positive constant  $c' > 0$  such that  $\|m\|_{\mathcal{M}} \leq c'\|m\|_{w^*}$  for all  $m \in \mathcal{M}(\Sigma, X^*)$ . Then we have

$$\|P\phi\|_{\mathcal{M}} \leq c'\|P\phi\|_{w^*} \leq c' \sup_i \|m_i\| \leq c'\|\phi\|;$$

so  $P$  is continuous.

We are going to prove that  $rbvca(\Omega, X^*)$  has this property. Suppose  $m : \Sigma \rightarrow X^*$  is a vector measure such that there exists a bounded net  $(m_i)_{i \in I}$  in  $rbvca(\Omega, X^*)$  satisfying

$$w^*\text{-}\lim_i m_i(A) = m(A) \text{ for all } A \in \Sigma.$$

For every  $i \in I$ , we define

$$T_i : \mathcal{C}(\Omega) \rightarrow X^*$$

by  $T_i\varphi = \int_{\Omega} \varphi dm_i$ . It is well known that  $T_i$  is 1-summing and  $\pi_1(T_i) = |m_i|(\Omega)$  (see [DU]). So,  $(T_i)_{i \in I}$  is a bounded net in  $\Pi_1(\mathcal{C}(\Omega, X^*))$ . On the other hand, it is easy to prove that  $m$  is bounded. So, we can consider the Bartle integral with respect to  $m$ . Then the map

$$T : \varphi \in \mathcal{C}(\Omega) \rightarrow \int_{\omega} \varphi dm \in X^*$$

is linear and continuous. Since  $m(A) = w^*\text{-}\lim_i m_i(A)$  for every  $A \in \Sigma$ , it follows that  $w^*\text{-}\lim_i T_i = T$  pointwisely.

Now, it is an standard argument to show that  $T$  is 1-summing. Hence, its representing measure  $m$  belongs to  $rbvca(\Omega, X^*)$ .  $\square$

Other  $w^*$ -boundedly closed vector measure Banach spaces are  $ba(\Sigma, X^*)$  and  $bvfa(\Sigma, X^*)$ .

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DEPARTAMENTO DE MATEMÁTICAS, FACULTAD DE CIENCIAS EXPERIMENTALES, CAMPUS UNIVERSITARIO DE EL CARMEN, UNIVERSIDAD DE HUELVA, 21071, HUELVA, SPAIN

DEPARTAMENTO DE MATEMÁTICAS, FACULTAD DE CIENCIAS EXPERIMENTALES, CAMPUS UNIVERSITARIO DE EL CARMEN, UNIVERSIDAD DE HUELVA, 21071, HUELVA, SPAIN

*E-mail address:* candido@uhu.es