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FINITE s-ARC TRANSITIVE CAYLEY GRAPHS AND FLAG-TRANSITIVE PROJECTIVE PLANES

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ABSTRACT. In this paper, a characterisation is given of finite s-arc transitive Cayley graphs with $s \geq 2$. In particular, it is shown that, for any given integer k with $k \geq 3$ and $k \neq 7$, there exists a finite set (maybe empty) of s-transitive Cayley graphs with $s \in \{3, 4, 5, 7\}$ such that all s-transitive Cayley graphs of valency k are their normal covers. This indicates that s-arc transitive Cayley graphs with $s \geq 3$ are very rare. However, it is proved that there exist 4-arc transitive Cayley graphs for each admissible valency (a prime power plus one). It is then shown that the existence of a flag-transitive non-Desarguesian projective plane is equivalent to the existence of a very special arc transitive normal Cayley graph of a dihedral group.

1. INTRODUCTION

A graph Γ is a *Cayley graph* if there exist a group G and a subset $S \subset G$ with $S = S^{-1} := \{s^{-1} \mid s \in S\}$ such that the vertices of Γ may be identified with the elements of G in such a way that x is connected to y if and only if $yx^{-1} \in S$. The Cayley graph Γ is denoted by $\mathsf{Cay}(G, S)$. Cayley graphs stem from a type of diagram now called a Cayley color diagram, introduced by Cayley in 1878. In this paper, we investigate the symmetric Cayley graphs and a relation between them and finite flag-transitive projective planes.

A graph Γ is said to be (X, s)-*arc transitive* if $X \leq \mathrm{Aut}\Gamma$ is transitive on vertices and s-arcs of Γ where s is a positive integer. (A sequence $v_0, v_1, \ldots, v_s$ of vertices of Γ is called an s-*arc* if v_i is adjacent to v_{i+1} for $0 \leq i \leq s-1$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$.) An (X, s)-arc transitive graph is called (X, s)-*transitive* if it is not $(X, s+1)$-arc transitive. In particular, if $X = \mathrm{Aut}\Gamma$, then an (X, s)-arc transitive graph and an (X, s)-transitive graph are simply called s-*arc transitive* and s-*transitive*, respectively.

Interest in s-arc transitive graphs stems from a seminal result of Tutte in 1947, who proved that there exist no finite s-transitive cubic graphs for $s \geq 6$. Tutte's Theorem was generalized by Weiss [24] who proved that there exist no finite s-transitive graphs of valency at least 3 for $s = 6$ and $s \geq 8$. Since then, characterizing

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s-arc transitive graphs has received considerable attention in the literature (see, for example, [11, 14, 19, 24]).

Let G be a group, and let $\Gamma = \mathsf{Cay}(G, S)$. Then Γ has an automorphism group:

$$\hat{G} = \{\hat{g} : \ x \to xg \text{ for all } x \in G \mid g \in G\},$$

consisting of right multiplications of elements $g \in G$. The subgroup $\hat{G}$ acts regularly on the vertex set of Γ. If $\hat{G}$ is normal in AutΓ, then Γ is called a *normal Cayley graph*; if the core of $\hat{G}$ in AutΓ has index 2 in $\hat{G}$, then Γ is called a *bi-normal Cayley graph*, while if $\hat{G}$ is core free in AutΓ, then Γ is called a *core free Cayley graph*. (Recall that the *core* of a subgroup H of a group X is the largest normal subgroup of X contained in H.) These three classes of Cayley graphs are important for studying s-arc transitive Cayley graphs. Normal Cayley graphs have some very nice properties; refer to [21, 25], and also to Lemma 2.2 and Proposition 2.3. Several nice properties of bi-normal Cayley graphs are given in Lemma 2.4 and Corollary 2.7. Some properties for core-free Cayley graphs are given in Section 3. An interesting result of Section 3 is Proposition 3.2, which says that

"almost all" vertex primitive Cayley graphs are normal Cayley graphs.

Let $N \lhd X$, and let $\mathcal{B}$ be the set of N-orbits in V. Then the *normal quotient graph* Γ_N of Γ induced by N is the graph with vertex set $\mathcal{B}$ such that $B, B' \in \mathcal{B}$ are adjacent if and only if some vertex $u \in B$ is adjacent in Γ to some vertex $v \in B'$. If Γ and Γ_N have the same valency, then Γ is called a *normal cover* of Γ_N.

Some special classes of 2-arc transitive Cayley graphs have been studied; see [1, 2, 11, 15, 18]. One of the main results of this paper is stated in Theorem 1.1, which tells us that s-arc transitive Cayley graphs with $s \geq 3$ are rare. Throughout this paper, denote by $\mathcal{G}(s,k)$ the set of core-free s-transitive Cayley graphs of valency k.

Theorem 1.1. *For any integers $s \in \{2,3,4,5,7\}$ and $k \geq 3$, the set $\mathcal{G}(s,k)$ is finite, and for each s-transitive Cayley graph Γ, one of the following statements holds:*

(i) *Γ is a normal or a bi-normal Cayley graph, and either $s = 2$, or $(s,k) = (3,7)$ and Γ is bi-normal (so bipartite);*
(ii) *Γ is a normal cover of a member of $\mathcal{G}(s,k)$.*

Moreover, $\mathcal{G}(2,k)$ and $\mathcal{G}(3,k)$ are not empty for all $k \geq 3$, and $\mathcal{G}(4, q+1)$ is not empty for all prime powers q; in particular, there exist 4-transitive Cayley graphs of all admissible valencies.

Remarks. (a) It is easily shown that complete graphs are members of $\mathcal{G}(2,k)$, and complete bipartite graphs are members of $\mathcal{G}(3,k)$. Some other examples of members in $\mathcal{G}(2,k)$ and $\mathcal{G}(3,k)$ will be constructed in Section 4.

(b) The examples in $\mathcal{G}(4, q+1)$ with q being a power of a prime are the incidence graphs of Desarguesian projective planes. It is known that the valency of a 4-transitive graph is $r+1$ such that r is a power of a prime; see [24]. Thus, Theorem 1.1 tells us that for all admissible valencies, there exist 4-transitive Cayley graphs. However, we do not know any other examples of 4-transitive Cayley graphs, and we do not have any examples of 5- or 7-transitive Cayley graphs at all.

(c) There have been some results regarding certain special classes of 2-arc transitive normal and bi-normal Cayley graphs; see [2, 11, 15, 20]. However, the general case is still not quite well-understood. In particular, we do not know any examples of 3-transitive bi-normal Cayley graphs.

We would like to propose the following question and problem.

Question 1.2. *(a) Do there exist* 3*-transitive bi-normal Cayley graphs?*
(b) Do there exist s-transitive Cayley graphs for $s = 5$ and $s = 7$?

Problem 1.3. *(a) Determine members of $\mathcal{G}(s,k)$ for $k \geq 3$ and $s \geq 2$.*
(b) Give a satisfactory description of bi-normal 2*-transitive Cayley graphs.*

Let $\mathit{\Pi}$ be a *projective plane of order n* with point set $\mathcal{P}$ and line set $\mathcal{L}$. Then $\mathit{\Pi}$ has n^2+n+1 points and n^2+n+1 lines such that (a) any two points lie on exactly one line and any two lines intersect at exactly one point, (b) each line contains exactly $n+1$ points and each point lies on exactly $n+1$ lines. A *flag* of $\mathit{\Pi}$ is a pair of point p and line l such that p lies on l. A permutation of flags preserving the incidence of $\mathit{\Pi}$ is an *automorphism* of $\mathit{\Pi}$, and all automorphisms of $\mathit{\Pi}$ form a group $\mathrm{Aut}\mathit{\Pi}$; that is, the automorphism group of $\mathit{\Pi}$. A plane $\mathit{\Pi}$ is called *flag-transitive* if $\mathrm{Aut}\mathit{\Pi}$ is transitive on its flags. Let Γ be the *incidence graph* of $\mathit{\Pi}$; that is, the vertex set of Γ is $\mathcal{P} \cup \mathcal{L}$, and the edge set of Γ is the set of flags of $\mathit{\Pi}$. Then $\mathrm{Aut}\Gamma = \mathrm{Aut}\mathit{\Pi}$ or $(\mathrm{Aut}\mathit{\Pi}).\mathbb{Z}_2$; so if $\mathit{\Pi}$ is flag-transitive, then Γ is edge-transitive.

The classical (Desarguesian) projective planes have been well studied, and their incidence graphs give the first family of 4-arc transitive Cayley graphs, presented in Example 4.5. The existence problem of non-Desarguesian flag-transitive projective planes is a long-standing open problem in finite geometry; see, for example, books [6, 17], and articles [7, 12, 22]. Here we prove that the existence of a non-Desarguesian projective plane is equivalent to the existence of a type of Cayley graph of a dihedral group satisfying very restricted properties.

Let n be a 2-power such that $p := n^2 + n + 1$ and $q := n + 1$ are primes. Let $G = \langle a \rangle \rtimes \langle z \rangle \cong \mathsf{D}_{2p}$, where $a^z = a^{-1}$. Let $\sigma \in \mathrm{Aut}(\langle a \rangle)$ be such that $\sigma z = z\sigma$ and $o(\sigma z) = 2q$. Let

$$S = (az)^{\langle \sigma \rangle} = \{az, a^r z, a^{r^2} z, \ldots, a^{r^{q-1}} z\},$$

and set $\Gamma(q) = \mathsf{Cay}(G, S)$. It will be shown in Lemma 5.1 that $\Gamma(q)$ has full automorphism group $\mathrm{Aut}\Gamma(q) = G \rtimes \langle \sigma z \rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{2q}$, a Frobenius group of order $2pq$, and thus $\Gamma(q)$ is a normal Cayley graph of the dihedral group $G \cong \mathsf{D}_{2p}$. The following theorem shows that the girth of $\Gamma(q)$ determines the existence of flag-transitive non-Desarguesian projective planes of order $q-1$.

Theorem 1.4. *Using the notation defined above, there exists a flag-transitive non-Desarguesian projective plane of order n if and only if the girth of the Cayley graph $\Gamma(q)$ is equal to* 6*.*

As widely believed, all flag-transitive projective planes are Desarguesian, and hence we make the following conjecture; see Lemma 5.1.

Conjecture 1.5. For each prime power q, the graph $\Gamma(q)$ defined above has diameter 4 and girth 4.

Theorem 1.4 tells us that if this conjecture is true, then there exist no finite non-Desarguesian flag-transitive projective planes.

2. Automorphism groups of Cayley graphs: Normality

It is known that a graph Γ is a Cayley graph of a group G if and only if its automorphism group contains a subgroup that is isomorphic to G and acts regularly on vertices; see, for example, [3, Proposition 16.3]. It is hence natural to use the regular subgroup to describe properties of Γ and $\mathrm{Aut}\Gamma$.

2.1. **Normal Cayley graphs.** Let G be a finite group, and let $\Gamma = \mathsf{Cay}(G, S)$. Let

$$\mathrm{Aut}(G, S) = \{\sigma \in \mathrm{Aut}(G) \mid S^\sigma = S\}.$$

Let $\mathbb{1}$ be the vertex of Γ corresponding to the identity of G. It follows from the definition that each element of $\mathrm{Aut}(G, S)$ induces an automorphism of Γ fixing $\mathbb{1}$, and so $\mathrm{Aut}(G, S) \leq (\mathrm{Aut}\Gamma)_{\mathbb{1}}$. Moreover, we have the following statement.

Lemma 2.1 ([8]). *For a Cayley graph $\Gamma = \mathsf{Cay}(G, S)$, the following holds:*

$$\mathsf{N}_{\mathrm{Aut}\Gamma}(\hat{G}) = \hat{G} \rtimes \mathrm{Aut}(G, S).$$

The subgroup $\mathrm{Aut}(G, S)$ plays an important role in the study of Cayley graphs; refer to [8, 21, 25]. Some structural information may be read out from $\mathrm{Aut}(G, S)$; for instance, if $\mathrm{Aut}(G, S)$ is transitive on S, then the Cayley graph Γ is $\mathsf{N}_{\mathrm{Aut}\Gamma}(\hat{G})$-arc transitive; if $\mathrm{Aut}(G, S)$ is 2-transitive on S, then Γ is $(\mathsf{N}_{\mathrm{Aut}\Gamma}(\hat{G}), 2)$-arc transitive.

For a group X with $\hat{G} \leq X \leq \mathrm{Aut}\Gamma$, if $\hat{G}$ is normal in X, then $X_{\mathbb{1}} \leq \mathrm{Aut}(G, S)$, and Γ is called an *X-normal Cayley graph* of G. In particular, if $X = \mathrm{Aut}\Gamma$, then Γ is a normal Cayley graph, and $\mathrm{Aut}\Gamma = \hat{G} \rtimes \mathrm{Aut}(G, S)$. The action of X on the vertices of Γ behaves in a very nice way:

Lemma 2.2. *Let $\Gamma = \mathsf{Cay}(G, S)$, and let $X \leq \mathrm{Aut}\Gamma$ be such that $\hat{G}$ is a normal subgroup of X. Then $X = \hat{G} \rtimes X_{\mathbb{1}}$, and for an element $x = \hat{g}y \in X$, where $\hat{g} \in \hat{G}$ and $y \in X_{\mathbb{1}}$, and for any vertex $h \in G$, we have*

$$h^x = h^{\hat{g}y} = (hg)^y = y^{-1}hgy.$$

This property implies the following result about s-arc transitive Cayley graphs, which was obtained in [13].

Proposition 2.3. *Let $\Gamma = \mathsf{Cay}(G, S)$, and let $X \leq \mathrm{Aut}\Gamma$ be such that $\hat{G}$ is a normal subgroup of X. Then Γ is not $(X, 3)$-arc transitive.*

2.2. **Bi-normal Cayley graphs.** Let $\Gamma = \mathsf{Cay}(G, S)$, and let $\hat{G} \leq X \leq \mathrm{Aut}\Gamma$. In this section we study the important case where $\hat{G}$ is not normal in X but $\mathsf{core}_X(\hat{G})$ is "big". A Cayley graph $\mathsf{Cay}(G, S)$ is called an *X-bi-normal Cayley graph* of G if the core $\mathsf{core}_X(\hat{G})$ has index 2 in $\hat{G}$. Hence if $X = \mathrm{Aut}\Gamma$, then an X-bi-normal Cayley graph is a bi-normal Cayley graph.

For an X-normal Cayley graph $\Gamma = \mathsf{Cay}(G, S)$, $X_{\mathbb{1}}$ acts on the vertex set G by conjugation, and so if Γ is connected, then $X_{\mathbb{1}}$ is faithful on S. For an X-bi-normal Cayley graph $\mathsf{Cay}(G, S)$, the $X_{\mathbb{1}}$-action only on half of its vertices is by conjugation; however, $X_{\mathbb{1}}$ is still faithful on S under certain conditions, as shown below.

Lemma 2.4. *Let Γ be a connected bipartite graph with biparts U and U'. Assume that $X \leq \mathrm{Aut}\Gamma$ is transitive on the vertex set $V\Gamma$, and assume furthermore that X has a normal subgroup $N \leq X^+$ such that N is intransitive on $V\Gamma$, N_v is transitive on $\Gamma(v)$ for some vertex $v \in U$, and N has a normal subgroup that is regular on both U and U'. Then N_v acts faithfully on $\Gamma(v)$.*

Proof. Choose $v \in U$, and let $R = MN_v^{[1]}$. Since U is an orbit of N and M acts transitively on U, we have $N = MN_v$. Since M is normal in N and $N_v^{[1]}$ is normal in N_v, it follows that R is normal in N and that $R = MN_w^{[1]}$ (and hence $R_w \leq N_w^{[1]}$) for all $w \in U$. Now let $\{u, v\}$ be an arbitrary edge. We may assume that $u \in U$ and $v \in U'$. Let $x \in R_u$. Since $R = MN_v^{[1]}$, we have $x = yz$ where $y \in M$ and $z \in N_v^{[1]}$. Since $N_v^{[1]} \leq R_u$, we have $z \in R_u$. It then follows that $y \in M \cap R_u$. By the assumption that M is regular on U, we have $M \cap R_u = 1$. Thus $y = 1$. We conclude that $x = z \in N_v^{[1]}$, and so $R_u \leq N_v^{[1]} \leq R_v$. Since $\{u, v\}$ is arbitrary, it follows that $R_u \leq N_u^{[1]}$ for every $u \in U'$. Thus $R_u \leq N_u^{[1]}$ for every $u \in V\Gamma$. Since Γ is connected, it follows that $R_{uv} = 1$ for every edge $\{u, v\}$. Therefore, $N_u^{[1]} = 1$ for every vertex $u \in V\Gamma$. □

By inspecting the classification of finite 2-transitive permutation groups (see [4]), the statement of the next lemma is easily obtained.

Lemma 2.5. *Let T be a 2-transitive permutation group on a set Ω of degree n. Then for two points $\omega, \omega' \in \Omega$, the point stabilizer $T_{\omega\omega'}$ has a transitive permutation representation of degree $n-1$ if and only if $\mathsf{soc}(T) = \mathrm{A}_7$ or S_7 and $n = 7$.*

The next result tells us that for a (G, s)-arc transitive graph, if the vertex stabilizer is faithful on the neighborhood, then s is small.

Proposition 2.6. *Let Γ be an (X, s)-transitive graph of valency k, where $G \leq \mathrm{Aut}\Gamma$ and $s \geq 1$. Assume that X_v is faithful on $\Gamma(v)$. Then either $s \leq 2$, or $(k, s) = (7, 3)$ and $X_v \cong \mathrm{A}_7$ or S_7.*

Proof. By Lemma 2.4, the vertex stabilizer X_v acts faithfully on $\Gamma(v)$; that is, $X_v \cong X_v^{\Gamma(v)}$, and thus X_v is a 2-transitive permutation group on $\Gamma(v)$. Suppose furthermore that Γ is $(X, 3)$-arc transitive. Then for distinct vertices $u, w \in \Gamma(v)$, the stabilizer X_{uvw} of the 2-arc (u, v, w) is transitive on $\Gamma(w) \setminus \{v\}$. In particular, X_{uvw} has a transitive permutation representation of degree $k-1$. Note that X_{uvw} is the stabilizer of u, w of the 2-transitive permutation group $X_v \cong X_v^{\Gamma(v)}$. Thus by Lemma 2.5, we conclude that $X_v \cong \mathrm{A}_7$ or S_7, $X_{vw} \cong \mathrm{A}_6$ of S_6, and $X_{uvw} \cong \mathrm{A}_5$ or S_5, respectively. In particular, $k = 7$, completing the proof of the proposition. □

Combining Lemma 2.4 and Proposition 2.6, we have the following statement for bi-normal Cayley graphs.

Corollary 2.7. *Let $\Gamma = \mathsf{Cay}(G, S)$ be a connected graph of valency k. Assume that $\hat{G} < X \leq \mathrm{Aut}\Gamma$ is such that Γ is X-bi-normal and $(X, 2)$-arc transitive. Then $X_{\mathbb{1}}$ is faithful on $\Gamma(\mathbb{1})$, and furthermore, either*

(i) *Γ is not $(X, 3)$-arc transitive, or*
(ii) *$k = 7$, $X_{\mathbb{1}} \cong \mathrm{A}_7$ or S_7, and Γ is $(X, 3)$-transitive.*

3. Core-free Cayley graphs and their group duals

Let $\Gamma = \mathsf{Cay}(G, S)$, and let $X \leq \mathrm{Aut}\Gamma$ contain $\hat{G}$. Assume furthermore that $\hat{G}$ is not normal in X. If the core of $\hat{G}$ in X is trivial, then Γ is a core-free Cayley graph for if $\hat{G}$ is core-free in X, then $\hat{G}$ is core-free in $\mathrm{Aut}\Gamma$.

Suppose now that $\hat{G}$ is core-free in X. Then X has an exact factorization with core-free factors:

$$X = \hat{G}H, \text{ with } H, \hat{G} \text{ core-free, and } \hat{G} \cap H = 1,$$

where $H = X_{\mathbb{1}}$. The action of X on $[X : \hat{G}]$ gives rise to Cayley graphs of the group H, called *dual Cayley graphs* of Γ. In this case, we observe that the order of X and hence the order of G are up-bounded in terms of the order of $X_{\mathbb{1}}$; that is,

$$\hat{G} < X \leq \mathrm{Sym}(|X_{\mathbb{1}}|).$$

It is known that for many important classes of graphs, the order of $X_{\mathbb{1}}$ is further up-bounded in terms of the valency: for example, cubic edge-transitive graphs, 2-arc transitive graphs, and vertex-primitive graphs. This therefore gives an up-bound of $|G|$ and $|X|$ in terms of the valency. This implies that X has two faithful transitive permutation representations, that is, on $[X : \hat{G}]$ and on $[X : H]$, and the statements in the following lemma are obviously true.

Lemma 3.1. *Let Γ be a Cayley graph of G, and let $X \leq \mathrm{Aut}\Gamma$ be such that $\hat{G} < X$ and $\mathsf{core}_X(\hat{G}) = 1$. Then X has a faithful transitive permutation representation on the set $[X : \hat{G}]$, and $X_{\mathbb{1}}$ is a regular subgroup of X acting on $[X : \hat{G}]$. In particular, the order $|X|$ is up-bounded in terms of $|X_{\mathbb{1}}|$.*

Each generalized orbital graph of X on the set $[X : \hat{G}]$ is therefore a Cayley graph of the group $H = X_{\mathbb{1}}$, which is a dual Cayley graph of $\mathsf{Cay}(G, S)$. Considering the dual permutation representation of a vertex-primitive automorphism group of a Cayley graph, we have the following result.

Proposition 3.2. *For every given integer k, there are only finitely many vertex-primitive Cayley graphs of valency k that are not normal Cayley graphs.*

Proof. Let $\Gamma = \mathsf{Cay}(G, S)$ be such that $X = \mathrm{Aut}\Gamma$ is primitive on $V\Gamma$. It follows that $\hat{G}$ in X is either normal or core-free. By [5], the order of $X_{\mathbb{1}}$ is up-bounded in terms of the valency of an orbital graph of X on $V\Gamma$, where $\mathbb{1}$ is the identity of G. Thus $|X_{\mathbb{1}}|$ is up-bounded in terms of the valency of Γ. If Γ is core-free, then X has a faithful representation on $[X : \hat{G}]$, which is of degree $|X_{\mathbb{1}}|$. Hence $|X| \leq |X_{\mathbb{1}}|!$, and so the order of X is up-bounded in terms of the valency of Γ, and so the number of vertex-primitive Cayley graphs of valency k that are not normal Cayley graphs is up-bounded in terms of k. □

This shows that, although they are very special, normal Cayley graphs are not very rare. Similarly, we have the next proposition regarding 2-arc transitive Cayley graphs.

Proposition 3.3. *For any given positive integer k, there are at most finitely many core-free 2-arc transitive Cayley graphs of valency k.*

Proof. Let $\Gamma = \mathsf{Cay}(G, S)$ be an $(X, 2)$-arc transitive graph of valency $k = |S|$ such that $\hat{G} < X \leq \mathrm{Aut}\Gamma$ and $\hat{G}$ is core-free in X. By [23], the order of $X_{\mathbb{1}}$ is up-bounded in terms of the valency k, where $\mathbb{1}$ is the identity of G. Since $\hat{G}$ is core-free in X, the group X has a faithful transitive representation on $[X : \hat{G}]$, which is of degree $|X_{\mathbb{1}}| = |X : \hat{G}|$. Hence $|X| \leq |X_{\mathbb{1}}|!$, and so the order of X is up-bounded in terms of k. In particular, the number of core-free 2-arc transitive Cayley graphs of valency k is up-bounded in terms of k. □

4. Examples

In this section, we construct examples of s-arc transitive core-free Cayley graphs, where $s \geq 2$.

It is obvious that a complete graph K_n of n vertices with $n \geq 4$ has automorphism group isomorphic to S_n and is a 2-transitive Cayley graph of an arbitrary group of order n. Also, it is easy to see that a complete bipartite graph $\mathsf{K}_{n,n}$ with $n \geq 3$ has automorphism group $\mathsf{S}_n \wr \mathsf{S}_2$, and is a 3-transitive Cayley graph of valency n. This shows that $\mathcal{G}(2,k)$ and $\mathcal{G}(3,k)$ are nonempty for all $k \geq 3$. Except for these "trivial" examples, various other s-arc transitive core-free Cayley graphs will be constructed in this section.

Let X be a group, and let H be a core-free subgroup of X. Let $[X:H] = \{Hx \mid x \in X\}$. For an element $g \in X$ such that $g^2 \in H$, a *coset graph*

$$\Gamma := \mathsf{Cos}(X, H, HgH)$$

is defined as the graph with vertex set $V = [X:H]$ such that Hx is adjacent to Hy if and only if $yx^{-1} \in HgH$. We have the following well-known properties.

Lemma 4.1. *Let $\Gamma = \mathsf{Cos}(X, H, HgH)$. Then*
- (i) Γ *is connected if and only if $\langle H, g\rangle = X$;*
- (ii) Γ *is $(X,2)$-arc transitive if and only if H is 2-transitive on $[H : H \cap H^g]$.*

The first example gives a family of 2-arc transitive core-free Cayley graphs of prime-power valency.

Example 4.2. Let $X = \mathsf{S}_{p^r}$ where p is a prime, acting on $\Omega = \{1, 2, \ldots, p^r\}$. Then X has two subgroups G, H such that $G = X_{1,2} \cong \mathsf{S}_{p^r-2}$ and $H \cong \mathrm{AGL}_1(p^r) \cong \mathbb{Z}_p^r \rtimes \mathbb{Z}_{p^r-1}$ is 2-transitive on Ω. It follows that $G \cap H = 1$ and $X = GH$. Write $H = N \rtimes \langle z\rangle$, where $N \cong \mathbb{Z}_p^r$ is regular on Ω, and $z = (1, 2, 3, \ldots, p^r - 1)$. It is easily shown that there exists an involution $g \in X$ such that $z^g = z^{-1}$ and $\langle H, g\rangle = X$. Let $\Gamma = \mathsf{Cos}(X, H, HgH)$. Then Γ is a connected 2-arc transitive graph of valency p^r and is a Cayley graph of G. □

The next two examples were constructed in [16] as examples of quasiprimitive 2-arc transitive graphs of a certain type. Here we prove they are Cayley graphs.

Example 4.3. Let Γ be a graph constructed in Example 4.2 with $r = 1$ and $p \geq 5$. Then Γ is a 2-arc transitive Cayley graph of S_{p-2}. Let l be a divisor of $(p-1)/2$, and let

$$Y = (T_1 \times T_2 \times \cdots \times T_l) \rtimes (O \times L),$$

where $T_i \cong \mathrm{A}_p$, $O \cong \mathbb{Z}_2$ normalizes but does not centralize T_i for each i, and $L = \langle (1, 2, 3, \ldots, l)\rangle \cong \mathbb{Z}_l$ permutes cyclically the direct factors $T_1, T_2, \ldots, T_l$.

It is shown in [16] that Y has a subgroup $K \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}) \times \mathbb{Z}_l$ and an element $g \in Y$ such that the coset graph $\Gamma := \mathsf{Cos}(Y, K, KgK)$ is connected and $(Y,2)$-arc transitive of valency p. It is now easily shown that the subgroup $R \times T_2 \times \cdots \times T_l$, where $\mathsf{S}_{p-2} \cong R < T_1$, acts regularly on $V\Gamma$. Hence Γ is a Cayley graph. □

The following example presents a family of 3-arc transitive Cayley graphs.

Example 4.4. Let $T = \mathrm{PSL}(2,q)$, where $q = 2^e$ with $e \geq 2$, and let $H = \mathbb{Z}_2^e{:}D$, where $D \cong \mathbb{Z}_{2^e-1}$. Let $f \in T$ be such that $\langle D, f\rangle \cong \mathsf{D}_{2(p^e-1)}$. Then $\Sigma = \mathsf{Cos}(T, H, HfH) \cong \mathsf{K}_{q+1}$ is $(T,2)$-arc transitive. Let

$$Y = (T_1 \times T_2 \times \cdots \times T_{q-1}) \rtimes L,$$

where $T_i \cong \mathrm{A}_p$, and $L = \langle(1, 2, 3, \ldots, q-1)\rangle \cong \mathbb{Z}_{q-1}$ permutes cyclically the direct factors $T_1, T_2, \ldots, T_{q-1}$.

It is shown in [16] that Y has a subgroup $K \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_{q-1}) \times \mathbb{Z}_{q-1}$ and an element $g \in Y$ such that the coset graph $\Gamma := \mathsf{Cos}(Y, K, KgK)$ is connected and $(Y, 3)$-arc transitive of valency p. It is now easily shown that the subgroup $R \times T_2 \times \cdots \times T_l$, where $\mathbb{Z}_{q+1} \cong R < T_1$, acts regularly on $V\Gamma$. Hence Γ is a Cayley graph. □

Finally, we present a family of 4-arc transitive Cayley graphs.

Example 4.5. Let Γ be the incidence graph of a Desarguesian projective plane $\mathit{\Pi}$ of order n. Then n is a prime power, and $\mathrm{Aut}\mathit{\Pi} = \mathrm{PGL}(3, n)$; see [6]. It is known that the incidence graph Γ is 4-arc transitive of valency $n+1$, and $\mathrm{Aut}\Gamma = \mathrm{Aut}\mathit{\Pi}.\mathbb{Z}_2 = \mathrm{Aut}(\mathrm{PSL}(3, n))$. Now the Singer subgroup $C \cong \mathbb{Z}_{n^2+n+1}$ is regular on both the point set $\mathcal{P}$ and the line set $\mathcal{L}$. There exists an involution $\tau \in \mathrm{Aut}\Gamma \setminus \mathrm{Aut}\mathit{\Pi}$ such that τ normalizes C (refer to [10, Lemma 3.2]). Hence $\mathrm{Aut}\Gamma$ has a subgroup $G := \langle C, \tau\rangle \cong \mathbb{Z}_{n^2+n+1} \rtimes \mathbb{Z}_2$ that is regular on the vertices of the graph Γ. So Γ is a Cayley graph of a group isomorphic to G. □

5. Proofs of Theorems 1.1 and 1.4

In this section, we prove Theorems 1.1 and 1.4.

Proof of Theorem 1.1. Let $\Gamma = \mathsf{Cay}(G, S)$ be an (X, s)-arc transitive graph of valency k with vertex set V, where $s \geq 2$. Assume furthermore that $\hat{G} < X$. Let $C = \mathsf{core}_X(\hat{G})$.

Assume first that $C = \hat{G}$. Then $\hat{G}$ is normal in X, and hence by Proposition 2.3, we have that $s = 2$, as in part (i) of Theorem 1.1.

Assume that C has exactly two orbits in V. Then $\hat{G}$ is bi-normal in X, and Γ is bipartite. Hence by Corollary 2.7, we have that either $s = 2$, or $(s, k) = (3, 7)$, as in part (i) of Theorem 1.1.

Finally, assume that C has at least three orbits in V. Then $\hat{G}/C$ is core-free in X/C and regular on the vertex set of Γ_C, $X/C \leq \mathrm{Aut}\Gamma_C$, and Γ_C is an $(X/C, s)$-arc transitive graph of valency k. Thus Γ_C is a member of $\mathcal{G}(s, k)$, and Γ is a normal cover of Γ_C, as in part (ii) of Theorem 1.1.

The existence of graphs in $\mathcal{G}(s, k)$ with $s \leq 4$ is justified by the examples given in Section 4. This completes the proof of Theorem 1.1. □

To prove Theorem 1.4, we first investigate the graphs $\Gamma(q)$. Let n be a prime power such that $p := n^2 + n + 1$ and $q := n + 1$ are primes. Let

$$G = \mathsf{D}_{2p} = \langle a\rangle \rtimes \langle z\rangle,$$

where $a^z = a^{-1}$. Let $\sigma \in \mathrm{Aut}(\langle a\rangle)$ be such that $\sigma z = z\sigma$ and $o(\sigma z) = 2q$. Let

$$S = (az)^{\langle\sigma\rangle} = \{az, a^r z, a^{r^2} z, \ldots, a^{r^{q-1}} z\},$$

and set $\Gamma(q) = \mathsf{Cay}(G, S)$.

Lemma 5.1. *Let $\Gamma = \Gamma(q)$, and let $d(\Gamma)$ and $g(\Gamma)$ be the diameter and the girth of Γ, respectively. Then*

(i) Γ *is of valency q, and* $\mathrm{Aut}\Gamma = \langle a\rangle \rtimes \langle z\sigma\rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{2q}$;
(ii) *either $(d(\Gamma), g(\Gamma)) = (3, 6)$, or $g(\Gamma) = 4$.*

Proof. The graph Γ is a bipartite graph of valency q with biparts Δ and Δ' say. Write $A = \mathrm{Aut}\Gamma$ and $A^+ = A+\Delta = A_{\Delta'}$. Then $A \geq G \rtimes \langle\sigma\rangle$, and $A^+ \geq \langle a\rangle \rtimes \langle\sigma\rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$. In particular, A^+ is a primitive permutation group of degree p. Thus by Burnside's theorem, either A^+ is affine or an almost simple group. It then follows from the result of [9] that either $\langle a\rangle \lhd A^+$, or $\mathsf{soc}(A^+) = \mathrm{PSL}(d,r)$ with $p = \frac{r^d-1}{r-1}$, or A_p. Suppose that $\mathsf{soc}(A^+) = \mathrm{PSL}(d,r)$. Then the cyclic group $\langle a\rangle$ is a Singer cycle of $\mathrm{PSL}(d,r)$, and so $\mathsf{N}_{\mathrm{PSL}(d,r)}(\langle a\rangle) \leq \langle a\rangle \rtimes \mathbb{Z}_d$. Thus the order $o(\sigma)$ divides d, which is not possible. Suppose that $\mathsf{soc}(A^+) = \mathrm{A}_p$. Then $A_{\mathbb{1}} = \mathrm{A}_{p-1}$. Since the smallest transitive permutation representation of A_{p-1} is $p-1$, it follows that Γ has valency at least $p-1$, which is a contradiction. Thus $\langle a\rangle$ is normal in A^+. It then follows that $\mathrm{Aut}\Gamma = G \rtimes \langle z\sigma\rangle$, as in part (i).

Suppose that the girth $g(\Gamma)$ of Γ is at least 6. Since $|\Gamma(\mathbb{1})| = n+1$, we have that $|\Gamma_2(\mathbb{1})| = n^2+n$. Furthermore, since $|V\Gamma| = 2(n^2+n+1)$, we conclude that $|\Gamma_3(\mathbb{1})| \leq |V\Gamma| - (1 + (n+1) + (n^2+n)) = n^2$. It then follows that $g(\Gamma) = 6$, $|\Gamma_3(\mathbb{1})| = n^2$, and the diameter $d(\Gamma) = 3$. Suppose finally that the girth $g(\Gamma)$ of Γ is less than 6. Then since Γ is bipartite, $g(\Gamma) = 4$. $\square$

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let $\mathit{\Pi}$ be a projective plane of order n, where n is a natural integer. Let Γ be the incidence graph of $\mathit{\Pi}$; that is, the vertex set of Γ is $\mathcal{P}\cup\mathcal{L}$, and the edge set of Γ is the set of flags of $\mathit{\Pi}$. Then Γ is a bipartite graph with biparts $\mathcal{P}$ and $\mathcal{L}$, and either $\mathrm{Aut}\Gamma = \mathrm{Aut}\mathit{\Pi}$, or $\mathrm{Aut}\Gamma = (\mathrm{Aut}\mathit{\Pi}).\mathbb{Z}_2$.

Assume that $\mathit{\Pi}$ is not Desarguesian, and assume that $\mathit{\Pi}$ is flag-transitive. By [12] (or refer to [7]), n is a 2-power, both $q := n+1$ and $p := n^2+n+1$ are primes, and $\mathrm{Aut}\mathit{\Pi} = \langle a\rangle \rtimes \langle\sigma\rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$ is a Frobenius group, where $o(a) = p$ and $o(\sigma) = q$.

The subgroup $\mathrm{Aut}\mathit{\Pi}$ of $\mathrm{Aut}\Gamma$ acts transitively on the edge set of Γ, and furthermore, the cyclic subgroup $\langle a\rangle$ of $\mathrm{Aut}\mathit{\Pi}$ is regular on each of the biparts $\mathcal{P}$ and $\mathcal{L}$. Thus by [10, Lemma 3.2], there exists an element $z \in \mathrm{Aut}\Gamma$ interchanging points and lines of $\mathit{\Pi}$. It follows that $\mathrm{Aut}\Gamma = (\mathrm{Aut}\mathit{\Pi}).\mathbb{Z}_2 = \langle a,\sigma,z\rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{2q}$, and the subgroup $G := \langle a,z\rangle$ is regular on the vertex set $\mathcal{P}\cap\mathcal{L}$. Thus Γ may be viewed as a Cayley graph of G, say $\Gamma = \mathsf{Cay}(G,S)$ for some subset $S \subset G$.

Since Γ is the incidence graph of the projective plane $\mathit{\Pi}$, Γ has girth 6 and diameter 3. It follows that G is not abelian and that $\mathrm{Aut}\Gamma$ is a Frobenius group. In particular, G is a dihedral group of order $2p$.

Now G is normal in $\mathrm{Aut}\Gamma$, and hence Γ is a normal Cayley graph of G. By Lemma 2.1, we have that $\mathrm{Aut}\Gamma = G \rtimes \mathrm{Aut}(G,S)$. Hence $(\mathrm{Aut}\Gamma)_{\mathbb{1}} = \mathrm{Aut}(G,S) \cong \mathbb{Z}_q$. Since all subgroups of $\mathrm{Aut}\Gamma$ of order q are conjugate, we may assume that $\mathrm{Aut}(G,S) = \langle\sigma\rangle$. Since Γ is bipartite, we conclude that $S \cap \langle a\rangle = \emptyset$, and so all elements of S are involutions. Thus $a^iz \in S$ for some i with $1 \leq i \leq p-1$. Let $a^\sigma = a^r$ for some integer r. Then $S = (a^iz)^{\langle\sigma\rangle} = \{a^iz, a^{ri}z, \ldots, a^{r^{q-1}i}z\}$, and so $\Gamma \cong \Gamma(q)$, as defined before.

Conversely, assume that $\Gamma(q)$ has girth 6 for some q. By Lemma 5.1, the diameter of $\Gamma(q)$ equals 3. Let $\mathcal{P}$ and $\mathcal{L}$ be the biparts of $\Gamma(q)$, and call the vertices in $\mathcal{P}$ *points* and the vertices in $\mathcal{L}$ *lines*. Then it is easily shown that the incidence structure $\mathit{\Pi} = (\mathcal{P},\mathcal{L})$ is a projective plane of order $q-1$. By Lemma 5.1, we have $\mathrm{Aut}\Gamma(q) \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{2q}$. It is known that a Desarguesian projective plane of order $q-1$ has automorphism group $\mathsf{PGL}(3,q-1)$; see, for example, [6]. So $\mathit{\Pi}$ is not Desarguesian. $\square$

Added in proof

A remark on Question 1.2(b). A 5-arc transitive Cayley graph is constructed by Xu, Fang, Wang and Xu in *On cubic s-arc transitive Cayley graphs of finite simple groups*, to appear in European Journal of Combinatorics. Also M. Conder announced that he has constructed some infinite families of s-arc transitive Cayley graphs for $s = 5$ or 7.

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