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F_σ–ADDITIVE FAMILIES AND THE INVARIANCE OF BOREL CLASSES

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Abstract. We prove that any F_σ–additive family $\mathcal{A}$ of sets in an absolutely Souslin metric space has a σ–discrete refinement provided every partial selector set for $\mathcal{A}$ is σ–discrete. As a corollary we obtain that every mapping of a metric space onto an absolutely Souslin metric space, which maps F_σ–sets to F_σ–sets and has complete fibers, admits a section of the first class. The invariance of Borel and Souslin sets under mappings with complete fibers, which preserves F_σ-sets, is shown as an application of the previous result.

1. Introduction

One of the main results of our contribution is an extension of results of R.W. Hansell and G. Koumoullis ([4, Theorem 2.1] and [10, Theorem 2]). G. Koumoullis proved that a disjoint cover $\mathcal{A}$ of an absolutely Souslin metric space Y consisting of F_σ–sets is either σ–discretely decomposable or there exists a compact set $K \subset Y$ that meets uncountably many sets of $\mathcal{A}$.

Later on, R.W. Hansell in [4, Theorem 2.1] improved this result for point–countable families of F_σ–sets, proving the following assertion. Given a point–countable cover $\mathcal{A}$ of an absolutely Souslin metric space Y consisting of F_σ-sets, then either $\mathcal{A}$ has a σ–discrete refinement or there exists a compact set $K \subset Y$ that is not covered by any countable subfamily of $\mathcal{A}$.

He used this result to obtain a corollary, which states that any F_σ–additive point–countable family has a σ–discrete refinement ([4, Theorem 3.3]). This enabled him to obtain a first-class selector theorem for F_σ–weakly measurable multivalued mappings $\Phi : Y \to X$ with complete separable values. He showed that the assumption of separability of values of Φ cannot be omitted.

Nevertheless, we are able to obtain a first-class selector without this assumption for a particular case of an inverse mapping; namely, we prove that for a mapping $f : X \to Y$ from a metric space X onto an absolutely Souslin metric space Y, which preserves F_σ–sets and has complete fibers, there exists a function $g : Y \to X$ of the first class such that $f(g(y)) = y$, $y \in Y$.

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The crucial step in the proof is the following result on decomposability of F_σ–additive families:

Let $\mathcal{A}$ be an F_σ–additive family of sets in an absolutely Souslin metric space Y such that each partial selector set for $\mathcal{A}$ is σ–discrete. Then $\mathcal{A}$ has a σ–discrete refinement.

The assumption imposed on the family $\mathcal{A}$ is quite natural, because it can be easily verified that an image of a discrete family of F_σ–sets under a mapping preserving F_σ–sets possesses this property (see Proposition 3.3). The proof of the emphasized assertion is the main content of Section 3.

In the next Section 4 we apply this result to the aforementioned selection theorem for the inverse mapping $f^{-1} : Y \to X$, where $f : X \to Y$ has complete fibers and preserves F_σ–sets. We combine well–known techniques (see e.g. [8, Theorem 1] or [2, Theorem 3.5]) to get a selection function $g : Y \to X$ of the mapping $f^{-1} : y \mapsto f^{-1}(y)$, $y \in Y$, so that g satisfies some auxiliary conditions. This function is then substantially used in the proof of a result concerning the preservation of Borel and Souslin sets.

2. Preliminaries

In this section we recall definitions and facts needed throughout our contribution. We point out that every space in our paper is considered to be metrizable. We write (X, ρ) when a metrizable space X is considered with a specified compatible metric ρ.

Let $\mathcal{B}$ be a family of subsets of a set X. We denote by $\mathcal{A}_1(\mathcal{B})$, or by $\mathcal{M}_1(\mathcal{B})$, the family of all countable unions, or countable intersections, of elements of $\mathcal{B}$. Having defined $\mathcal{A}_\beta(\mathcal{B})$ and $\mathcal{M}_\beta(\mathcal{B})$ for $1 \le \beta < \alpha < \omega_1$, let $\mathcal{A}_\alpha(\mathcal{B})$, or $\mathcal{M}_\alpha(\mathcal{B})$, be the family of all unions, or intersections, of countable families of sets, each being contained in $\bigcup\{\mathcal{A}_\beta(\mathcal{B}) \cup \mathcal{M}_\beta(\mathcal{B}) : 1 \le \beta < \alpha\}$.

If X is a metric space and $\mathcal{B} = \{F \cap G : F \text{ closed}, \ G \text{ open}\}$, we obtain the usual hierarchy of Borel sets in X. In this case we write $\mathcal{A}_\alpha(X)$, or $\mathcal{M}_\alpha(X)$, for the family of all sets of additive, or multiplicative, class α. For the sake of completeness we recall that the class $\mathcal{A}_0(X)$, or $\mathcal{M}_0(X)$, is just the family of all open, or closed, sets in X. Elements of the first additive class are called F_σ–sets, and their complements are called G_δ–sets.

We denote by $\mathbb{N}^{<\mathbb{N}}$ the space of finite sequences of positive integers. Let $l(s)$ be the length of s and $\|s\| = \sum_{i=1}^{l(s)} s_i$. As usual, for $s, t \in \mathbb{N}^{<\mathbb{N}}$ we write $s \prec t$ if t is an extension of s, i.e., $l(s) \le l(t)$ and $s_i = t_i$, $1 \le i \le l(s)$. We denote by $\emptyset$ the empty sequence, and adopt the convention that the length of the empty sequence is 0. For $s \in \mathbb{N}^{<\mathbb{N}}$ and $n \in \mathbb{N}$ we write $s^\wedge n$ for the sequence $(s_1, \dots, s_{l(s)}, n)$.

The space $\mathbb{N}^{\mathbb{N}}$ with the usual product topology will be denoted by $\mathbb{I}$. For $\sigma \in \mathbb{I}$, $\sigma = \{\sigma_k\}_{k=1}^\infty$, and $n \in \mathbb{N}$ we write $\sigma \restriction n = (\sigma_1, \dots, \sigma_n)$. We adopt the convention that $\sigma \restriction 0 = \emptyset$.

We recall that a set A in a metric space X is said to be *Souslin* if there exists a family of closed sets $\{F_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$ in X so that

$$A = \bigcup_{\sigma \in \mathbb{I}} \bigcap_{n=1}^{\infty} F_{\sigma \restriction n} \ .$$

Let $\mathcal{A} = \{A_i : i \in I\}$ be a family of sets in a metric space X. Then $\mathcal{A}$ is said to be *discrete* in X if each point $x \in X$ has a neighbourhood that meets at most

one set from $\mathcal{A}$. If I can be written as a countable union of sets $I(n)$ so that each family $\{A_i : i \in I(n)\}$ is discrete, then $\mathcal{A}$ is said to be *σ–discrete*. It readily follows that $A \subset X$ is a discrete (respectively σ–discrete) set in X if and only if the family $\{\{x\} : x \in A\}$ is discrete (respectively σ–discrete) in X.

If $\mathcal{A}$ is a σ–discrete family in a metric space X and X is homeomorphically embedded in a metric space Y, then $\mathcal{A}$ is σ–discrete in Y likewise. This easily follows from [3, Lemma 2]. Thus the property of σ–discreteness is topologically invariant.

We say that $\mathcal{A}$ is *σ–discretely decomposable* if every set $A_i \in \mathcal{A}$ can be written as $A_i = \bigcup_{n\in\mathbb{N}} A_i(n)$ such that $\{A_i(n) : i \in I\}$ is a discrete family for every $n \in \mathbb{N}$.

A family $\{R_j : j \in J\}$ is called a *refinement* of $\mathcal{A}$ if $\bigcup_{j\in J} R_j = \bigcup_{i\in I} A_i$ and for each $j \in J$ there exists $i \in I$ with $R_j \subset A_i$. We say that $\mathcal{A}$ has a *σ–discrete refinement* if there exists a refinement $\mathcal{R}$ of $\mathcal{A}$ that is a σ–discrete family. Clearly, any σ–discretely decomposable family has a σ–discrete refinement, but the converse need not hold in general.

We say that a family $\{\{x_i\} : i \in \widehat{I}\}$ is a *partial selector* for a family $\{A_i : i \in I\}$ if $\widehat{I} \subset I$ and $x_i \in A_i$ for every $i \in \widehat{I}$. The set $\bigcup\{\{x_i\} : i \in \widehat{I}\}$ is called the *set of the partial selector*, and such sets are said to be *partial selector sets*.

We say that a family $\mathcal{A} = \{A_i : i \in I\}$ of sets in a metric space is *F_σ–additive* if $\bigcup_{i\in\widehat{I}} A_i$ is an F_σ–set for every $\widehat{I} \subset I$.

We recall that a metric space X is called a *Baire space* if the intersection of every sequence of dense open sets in X is dense. Any complete metric space is a well–known example of a Baire space. If $A \subset X$ can be covered by a countable union of nowhere dense sets, then A is said to be *meager*.

A metric space X is called an *absolutely Souslin metric space* if X is homeomorphic to a Souslin subset of some complete metric space. It follows from [5, Theorem 1.1] and [6, Theorem 4.1] that X is an absolutely Souslin metric space if and only if X is a Souslin subset of any metric space Y containing X, and this is the case if and only if there exists a complete metric space T and a continuous mapping f of T onto X such that f maps any discrete family of sets in T onto a σ–discretely decomposable family in X.

We shall frequently use the fact that an open covering of a metric space has a σ–discrete refinement consisting of open sets (see [11, §21, Corollary 1a]). Moreover, any metric space has a σ–discrete base of open sets.

If A is a subset of a metric space X, we write $\overline{A}$ for the closure of A. If ρ is a metric on X and $A, B \subset X$, then $\operatorname{dist}(A, B)$ stands for the distance of sets A and B and $\operatorname{diam} A$ for the diameter of A.

If $f : X \to Y$ is a mapping between two sets and $\mathcal{A} = \{A_i : i \in I\}$ is a family of subsets of X, then $f(\mathcal{A}) = \{f(A_i) : i \in I\}$. Similarly we use $f^{-1}(\mathcal{B})$ for a family $\mathcal{B}$ in Y. If $\mathcal{A} = \{A_i : i \in I\}$ is a family of sets in X and $B \subset X$, we write $\mathcal{A} \wedge B$ for the family $\{A_i \cap B : i \in I\}$. Similarly, if $\mathcal{B} = \{B_j : j \in J\}$ is another family of sets in X, then $\mathcal{A} \wedge \mathcal{B}$ stands for the family $\{A_i \cap B_j : i \in I, j \in J\}$.

For a family $\mathcal{A} = \{A_i : i \in I\}$ of subsets of a set X and a point $x \in X$ we put $I(x) = \{i \in I : x \in A_i\}$.

3. F_σ–ADDITIVE FAMILIES

The main part of this section is Theorem 3.2. In its proof we refine techniques used in [10, Theorem 2 and Remarks] and [4, Theorem 2.1]. More precisely, as in

the mentioned theorems we suppose that an F_σ–additive cover $\mathcal{A}$ of an absolutely Souslin space without a σ–discrete refinement is given and construct a suitable compact set that meets "a lot of" members of $\mathcal{A}$.

Before the proof of Theorem 3.2 we need to mention a preliminary lemma. Its assertions are modifications of Lemma 3 and Corollary 1 in [3]. We omit its easy proof since it is quite analogous to the proofs of the cited results.

Lemma 3.1. *Let X be a metric space.*

(i) *Let $\mathcal{B}$ be a family of sets in X admitting a σ–discrete refinement. Let $\mathcal{A}$ be a family such that, for every $B \in \mathcal{B}$, the family $\mathcal{A} \wedge B$ has a σ–discrete refinement. Then $\mathcal{A}$ has a σ–discrete refinement as well.*

(ii) *Let $\mathcal{A}$ be a family of sets in X. Assume that for every $x \in X$ there exists an open set U containing x such that the family $\mathcal{A} \wedge U$ has a σ-discrete refinement. Then $\mathcal{A}$ has a σ-discrete refinement.*

Theorem 3.2. *Let $\mathcal{A}$ be an F_σ–additive family of sets in an absolutely Souslin metric space Y such that each partial selector set for $\mathcal{A}$ is σ–discrete. Then $\mathcal{A}$ has a σ–discrete refinement.*

Proof. Without loss of generality, we may assume that $\mathcal{A} = \{A_i : i \in I\}$ is an F_σ–additive cover of Y such that every partial selector set of $\mathcal{A}$ is σ–discrete.

Let $g : \widehat{Y} \to Y$ be a continuous mapping of a complete metric space $\widehat{Y}$ onto Y such that g maps any discrete family of sets in $\widehat{Y}$ onto a σ–discretely decomposable family of sets in Y. Put

$$B_i = g^{-1}(A_i)\ ,\ i \in I\ , \qquad \mathcal{B} = \{B_i : i \in I\}\ ,$$
$$D = \{y \in Y : I(y) \text{ is uncountable}\}\ , \qquad \widehat{D} = g^{-1}(D)\ ,$$
$$E = \{y \in Y : I(y) \text{ is countable}\}\ , \qquad \widehat{E} = g^{-1}(E)\ .$$

Then $\mathcal{B}$ is an F_σ–additive cover of $\widehat{Y}$. Let us suppose that $\mathcal{A}$ has no σ–discrete refinement.

Claim 1. *If $F \subset D$ is a Souslin subset of Y, then F is a σ–discrete set.*

Suppose that $F \subset D$ is a Souslin set in Y that is not σ–discrete. Since F, as a Souslin subset of an absolutely Souslin space, is also an absolutely Souslin space, according to El'kin's theorem in [1] (cf. Corollary 3 in [3]) there exists an uncountable compact set $K \subset F$. It is easy to verify that every set $S \subset K$ of cardinality at most $\aleph_1$ is a partial selector set for $\mathcal{A}$.

Indeed, given a set $S \subset K$ of cardinality at most $\aleph_1$ we can enumerate it as a transfinite sequence $\{s_\alpha : \alpha < \omega_1\}$. Then, for every $\alpha < \omega_1$, we find by transfinite induction an index $i_\alpha \in I$ so that $s_\alpha \in A_{i_\alpha}$ and $i_\alpha \notin \{i_\beta : \beta < \alpha\}$. Then $\widehat{I} = \{i_\alpha : \alpha < \omega_1\}$ is an index set witnessing that S is a partial selector set for $\mathcal{A}$.

It follows from our assumption that every set $S \subset K$ of cardinality at most $\aleph_1$ is σ–discrete, which yields that the compact set K contains an uncountable discrete set. This contradiction finishes the proof of Claim 1.

The next assertion immediately follows from our assumption and the properties of the mapping g.

Claim 2. *The family $\mathcal{B}$ has no σ–discrete refinement.*

We define the following subsets of $\widehat{Y}$:

$$G = \{\widehat{y} \in \widehat{Y} : \mathcal{B} \wedge U \text{ has a } \sigma\text{–discrete refinement for some open } U \text{ containing } \widehat{y}\} ,$$

$$H = \widehat{Y} \setminus G .$$

Then G is an open set, H is closed and it follows from Lemma 3.1 (ii) and Claim 2 that H is nonempty. Even the following stronger assertion holds true.

Claim 3. *For any $\widehat{y} \in H$ and any open set U containing $\widehat{y}$, the family $\mathcal{B} \wedge (U \cap H)$ has no σ–discrete refinement.*

Indeed, suppose that U is an open subset of $\widehat{Y}$ intersecting H so that $\mathcal{B} \wedge (U \cap H)$ has a σ–discrete refinement. Since $\mathcal{B} \wedge (U \cap G)$ has a σ–discrete refinement due to Lemma 3.1 (ii), we get that $\mathcal{B} \wedge U$ has a σ–discrete refinement. Hence $U \subset G$, which contradicts the assumption that U meets H.

In the sequel we fix a compatible complete metric ρ on $\widehat{Y}$.

Claim 4. *Let $\varepsilon > 0$, $\widehat{y} \in H$ and a countable set $\widehat{I} \subset I$ be given. Then there exists $\widehat{z} \in H \cap \widehat{E}$ so that $\rho(\widehat{y}, \widehat{z}) < \varepsilon$ and $\widehat{z} \notin \bigcup\{B_i : i \in \widehat{I}\}$.*

Suppose the contrary, i.e., assume there exists an open set U in $\widehat{Y}$ intersecting H and a countable set $\widehat{I} \subset I$ so that

$$U \cap H \cap \widehat{E} \subset U \cap H \cap \bigcup_{i \in \widehat{I}} B_i . \tag{1}$$

Put

$$V = (U \cap H) \setminus \bigcup_{i \in \widehat{I}} B_i . \tag{2}$$

Then V is a G_δ–set in $\widehat{Y}$ that is contained in $\widehat{D}$ by (1) and the definition of $\widehat{E}$.

Then $g(V)$ is a Souslin subset of Y according to the characterization of absolutely Souslin metric space mentioned in Section 2. Moreover, $g(V) \subset D$. Claim 1 yields that $g(V)$ is σ–discrete and thus $\mathcal{A} \wedge g(V)$ has a σ–discrete refinement. Hence $\mathcal{B} \wedge g^{-1}(g(V))$ and consequently $\mathcal{B} \wedge V$ has a σ–discrete refinement.

Combining this assertion with (2) we get that $\mathcal{B} \wedge (U \cap H)$ has a σ–discrete refinement, which contradicts Claim 3. Hence the proof of Claim 4 is finished.

Claim 5. *There exists a set $P = \{\widehat{y}_s : s \in \mathbb{N}^{<\mathbb{N}}\}$ with $P \subset H \cap \widehat{E}$ so that, for every $s \in \mathbb{N}^{<\mathbb{N}}$ and $n \in \mathbb{N}$,*

(i) *$\widehat{y}_{s^\wedge n} \notin \bigcup_{i \in I(\widehat{y}_s)} B_i$ and*
(ii) *$\rho(\widehat{y}_s, \widehat{y}_{s^\wedge n}) < 2^{-\|s^\wedge n\|}$.*

Pick $y_\emptyset \in H \cap \widehat{E}$ arbitrary. Suppose that $\{\widehat{y}_s : l(s) \le k-1\}$ are chosen for $k \in \mathbb{N}$. Let a sequence $t = s^\wedge n$ of length k be given. If we put $\varepsilon = 2^{-\|t\|}$ and $\widehat{I} = I(\widehat{y}_s)$ in Claim 4, we obtain the required point $\widehat{y}_t$. Properties (i) and (ii) are clearly satisfied, and the proof of Claim 5 is finished.

By condition (ii) of Claim 5 we know that P is totally bounded. Hence $L = \overline{P}$ is a compact subset of $\widehat{Y}$. Put

$$I(P) = \bigcup_{s \in \mathbb{N}^{<\mathbb{N}}} I(\widehat{y}_s) , \quad K = g(L) \quad \text{and} \quad Q = g(P) .$$

Claim 6. *Let $\widehat{I} \subset I$ be countable. Then $K \cap \bigcup_{i \in \widehat{I}} A_i$ is meager in K.*

Let $\widehat{I} \subset I$ be countable such that $\bigcup\{A_i \cap K : i \in \widehat{I}\}$ is not meager in K. Since every set $A_i \cap K$ is an F_σ–set in K, there is an index $i_0 \in \widehat{I}$ so that $A_{i_0} \cap K$ has a nonempty interior in K. Then

$$B_{i_0} \cap L = g^{-1}(A_{i_0}) \cap L$$

has a nonempty interior in L. Find $s \in \mathbb{N}^{<\mathbb{N}}$ and $n \in \mathbb{N}$ so that $\widehat{y}_s$ and $\widehat{y}_{s\wedge n}$ are contained in B_{i_0}. This is possible thanks to the density of P in L and condition (ii) in Claim 5. But this contradicts (i) in Claim 5, which concludes the proof of Claim 6.

We are now ready to finish the proof of the theorem. Put

$$M = K \setminus \bigcup_{i \in I \setminus I(P)} A_i \ .$$

Then M is a G_δ–set in K that contains Q.

Indeed, if $y = g(\widehat{y}_s)$ for some $s \in \mathbb{N}^{<\mathbb{N}}$, then

$$\widehat{y}_s \notin \bigcup_{i \in I \setminus I(\widehat{y}_s)} B_i$$

and hence also

$$y \in K \setminus \bigcup_{i \in I \setminus I(\widehat{y}_s)} A_i \subset K \setminus \bigcup_{i \in I \setminus I(P)} A_i \ .$$

Thus M is a dense G_δ–set in K. Since K is a Baire space, M is not meager in K. On the other hand, if we put

$$N = K \cap \bigcup_{i \in I(P)} A_i \ ,$$

we obtain an F_σ–set in K so that $M \subset N$.

Indeed, for $y \in K \setminus \bigcup_{i \in I \setminus I(P)} A_i$ given, there exists $i \in I$ with $y \in A_i$. Here we use the fact that $\mathcal{A}$ is a cover. Then $i \notin I \setminus I(P)$, and thus $y \in \bigcup_{i \in I(P)} A_i$ as needed.

An appeal to Claim 6 gives that N is a meager set in K. Hence we have arrived at the conclusion that M is both meager and non–meager in K, which is impossible. This contradiction finishes the proof. □

Proposition 3.3. *Let $f : X \to Y$ be a mapping of a metric space X into an absolutely Souslin metric space Y such that f maps closed sets in X to F_σ–sets in Y.*

If $\mathcal{A}$ is a discrete family of closed sets in X, then $f(\mathcal{A})$ has a σ–discrete refinement.

Proof. Let a discrete family $\mathcal{A} = \{A_i : i \in I\}$ of closed sets in X be given. Then $\mathcal{B} = f(\mathcal{A})$ is an F_σ–additive family in an absolutely Souslin metric space Y. If we verify that every partial selector set for $\mathcal{B}$ is σ–discrete, the proof will be finished by an application of the previous Theorem 3.2.

Let $S = \bigcup\{\{y_i\} : i \in \widehat{I}\}$, $\widehat{I} \subset I$, be a partial selector set for $\mathcal{B}$. For every $i \in \widehat{I}$ find a point $x_i \in A_i$ with $f(x_i) = y_i$. Since $T = \{x_i : i \in \widehat{I}\}$ is a discrete set in X and f preserves F_σ–sets, we get that every subset of S is an F_σ–set. Since S is an absolutely Souslin space, it follows from Corollary 4 in [3] that S is σ–discrete. Hence the assertion follows from Theorem 3.2. □

Lemma 3.4. *Let $\alpha \geq 1$ (respectively $\alpha \geq 0$) and $\mathcal{A}$ be a family of sets in a metric space X consisting of sets of additive (respectively multiplicative) class α.*

If $\mathcal{A}$ has a σ–discrete refinement, then there exists a σ–discrete refinement of $\mathcal{A}$ consisting of sets of additive (respectively multiplicative) class α.

Proof. Let $\mathcal{B}$ be a σ–discrete refinement of $\mathcal{A}$. For any $B \in \mathcal{B}$ find $A_B \in \mathcal{A}$ with $B \subset A_B$. Put $C_B = \overline{B} \cap A_B$ and $\mathcal{C} = \{C_B : B \in \mathcal{B}\}$. It follows from the definition of $\mathcal{C}$ and the equality

$$\bigcup \mathcal{A} = \bigcup \mathcal{B} \subset \bigcup \mathcal{C} \subset \bigcup \mathcal{A}$$

that $\mathcal{C}$ refines $\mathcal{A}$. Plainly, $\mathcal{C}$ consists of sets of additive (respectively multiplicative) class α, and it is quite obvious that $\mathcal{C}$ is σ–discrete. □

Theorem 3.5. *Let $f : X \to Y$ be a mapping of a metric space X into an absolutely Souslin metric space Y such that f maps F_σ–sets in X to F_σ–sets in Y.*

If $\mathcal{F}$ is a σ–discrete family of F_σ–sets in X, then $f(\mathcal{F})$ has a σ–discrete refinement consisting of F_σ–sets.

Proof. Let $\mathcal{F} = \{F_i : i \in \bigcup_n I(n)\}$ be a family of F_σ–sets in X such that $\mathcal{F}(n) = \{F_i : i \in I(n)\}$ is discrete. Write $F_i = \bigcup_k F_i(k)$, where each $F_i(k)$ is a closed set in X, and apply Proposition 3.3 along with Lemma 3.4 to families $\{f(F_i(k)) : i \in I(n)\}$, $k, n \in \mathbb{N}$. By putting the obtained refinements together we get the desired refinement of the family $f(\mathcal{F})$. □

4. The invariance of Borel classes and first-class sections

Since the existence of a σ–discrete refinement is a sufficient tool in the proofs of a number of selection theorems, the validity of Theorem 3.5 enables us to show that a mapping f from a metric space X onto an absolutely Souslin metric space Y admits a section of the first class provided f preserves F_σ–sets and the fibers $f^{-1}(y)$, $y \in Y$, are complete. We recall that a *section of f* is a mapping $g : Y \to X$ satisfying $f(g(y)) = y$ for all $y \in Y$, i.e., g is a selection function of the multivalued mapping $y \mapsto f^{-1}(y)$, $y \in Y$.

By combining methods of Theorem 1 in [8] and Lemma 8 in [7] we get a section that satisfies some useful auxiliary conditions. This section is the main tool in the proof of Theorem 4.4 on the invariance of Borel and Souslin sets.

Before the proofs we briefly recall a few definitions concerning multivalued maps.

We say that $\Phi : Y \to X$ is a *multivalued map* if Φ assigns to each $y \in Y$ a set in X. If F is a subset of X, the inverse image $\Phi^{-1}(F)$ is defined as

$$\Phi^{-1}(F) = \{y \in Y : \Phi(y) \cap F \neq \emptyset\} .$$

If $\mathcal{A}$ is a family of sets in X, we write $\Phi^{-1}(\mathcal{A})$ for the family $\{\Phi^{-1}(A) : A \in \mathcal{A}\}$.

Lemma 4.1. *Let $\Phi : Y \to X$ be a multivalued mapping with nonempty closed values from a metric space Y to a metric space (X, ρ) such that*

(i) *$\Phi^{-1}(H)$ is an F_σ–set in Y for any closed $H \subset X$, and*
(ii) *$\Phi^{-1}(\mathcal{A})$ has a σ–discrete refinement for any discrete family $\mathcal{A}$ of closed sets in X.*

Let $\mathcal{F}$ be a discrete family of closed sets in X and $\varepsilon > 0$.

Then there exists a multivalued mapping $\Psi : Y \to X$ with nonempty closed values such that, for every $y \in Y$,

(iii) $\operatorname{diam} \Psi(y) \leq \varepsilon$,

(iv) *for any* $F \in \mathcal{F}$*, either* $\Psi(y) \subset F$ *or* $\Psi(y) \cap F = \emptyset$*,*
(v) $\Psi(y) \subset \Phi(y)$*,*
(vi) $\Psi^{-1}(H)$ *is an* F_σ*–set in* Y *for any closed set* H *in* X*, and*
(vii) $\Psi^{-1}(\mathcal{A})$ *has a* σ*–discrete refinement for any discrete family* $\mathcal{A}$ *of closed sets in* X*.*

Proof. Since $\bigcup \mathcal{F}$ is a closed set, we can find a sequence $\{F_k\}$ of closed sets in X such that $X \setminus \bigcup \mathcal{F} = \bigcup_{k=1}^{\infty} F_k$. Put $\widehat{\mathcal{F}} = \mathcal{F} \cup \{F_k : k \in \mathbb{N}\}$.

Let $\widehat{\mathcal{H}}$ be a σ–discrete covering of Y consisting of closed sets with diameter smaller than ε. Put $\mathcal{H} = \widehat{\mathcal{H}} \wedge \widehat{\mathcal{F}}$. It readily follows that the family $\mathcal{H}$ is σ–discrete and consists of closed sets with diameter smaller than ε. Furthermore, for every set $H \in \mathcal{H}$ and $F \in \mathcal{F}$,

$$\text{either} \quad H \subset F \quad \text{or} \quad H \cap F = \emptyset \,. \tag{3}$$

Thanks to the assumptions and Lemma 3.4, the family $\Phi^{-1}(\mathcal{H})$ is a cover of Y that admits a σ–discrete refinement $\mathcal{U}$ consisting of F_σ–sets. According to [9, Lemma 4], there exists a disjoint σ–discrete cover $\mathcal{D} = \{D_U : U \in \mathcal{U}\}$ of Y such that $D_U \subset U$, $U \in \mathcal{U}$, and $\mathcal{D}$ is F_σ–additive.

For every $U \in \mathcal{U}$ find a set $H_U \in \mathcal{H}$ so that $U \subset \Phi^{-1}(H_U)$. Define the mapping $\Psi : Y \to X$ as

$$\Psi(y) = \Phi(y) \cap H_U, \quad y \in D_U,\ U \in \mathcal{U} \,.$$

Then, for any $y \in Y$, $\Psi(y)$ is a nonempty closed set in X with $\operatorname{diam} \Psi(y) \le \varepsilon$. Obviously, Φ satisfies condition (v). Thanks to (3), for any $F \in \mathcal{F}$ the set $\Phi(y)$ is either contained in F or does not intersect F. Thus we have verified condition (iv).

To verify (vi), pick a closed set K in X. Then

$$\begin{aligned} \Psi^{-1}(K) &= \bigcup_{U \in \mathcal{U}} \Psi^{-1}(K) \cap D_U \\ &= \bigcup_{U \in \mathcal{U}} \Phi^{-1}(K \cap H_U) \cap D_U \end{aligned}$$

is an F_σ–set, for the latter set is a σ–discrete union of F_σ–sets.

It remains to check the last condition (vii). Let $\mathcal{A}$ be a discrete family of closed sets in X. Then, for $U \in \mathcal{U}$, we have

$$\Psi^{-1}(\mathcal{A}) \wedge D_U = \Phi^{-1}(\mathcal{A} \wedge H_U) \wedge D_U \,.$$

The latter set has a σ–discrete refinement thanks to the assumptions on Φ. Hence the family $\Psi^{-1}(\mathcal{A}) \wedge D_U$ has a σ–discrete refinement for every $U \in \mathcal{U}$. Since the family $\{D_U : U \in \mathcal{U}\}$ is a σ–discrete cover of Y, the family $\Psi^{-1}(\mathcal{A})$ has a σ–discrete refinement due to Lemma 3.1 (i). This concludes the proof. □

Theorem 4.2. *Let* $\Phi : Y \to X$ *be a multivalued mapping with nonempty closed values from a metric space* Y *to a complete metric space* X *such that* $\Phi^{-1}(H)$ *is an* F_σ*–set in* Y *for any closed set* $H \subset X$ *and* $\Phi^{-1}(\mathcal{H})$ *has a* σ*–discrete refinement for any discrete family* $\mathcal{H}$ *of closed sets in* X*. Let* $\mathcal{F}$ *be a* σ*–discrete family of* F_σ*–sets in* X*.*

Then there exists a selection function $f : Y \to X$ *of* Φ *so that* $f^{-1}(F)$ *is an* F_σ*–set in* Y *for every* $F \in \mathcal{F}$ *and* $f^{-1}(G)$ *is an* F_σ*–set in* Y *for any open* $G \subset X$*.*

Proof. Let $\mathcal{F} = \bigcup_n \mathcal{H}_n$, where each $\mathcal{H}_n$ is a discrete family. For $H \in \mathcal{H}_n$ find closed sets $F_H(n,k)$ so that $H = \bigcup_k F_H(n,k)$. Then $\mathcal{F}_{n,k} = \{F_H(n,k) : H \in \mathcal{H}_n\}$ is a discrete family of closed sets for every $n, k \in \mathbb{N}$. Relabel these families into a single sequence $\{\mathcal{F}_n\}$.

Fix a compatible complete metric on X, and denote $\Phi_0 = \Phi$. Inductive use of Lemma 4.1 provides a sequence of mappings $\Phi_n : Y \to X$ such that, for every $y \in Y$ and $n \in \mathbb{N}$,

(i) $\Phi_n(y)$ is a nonempty closed set in X,
(ii) $\Phi_n^{-1}(H)$ is an F_σ–set in Y for every closed set H in X,
(iii) $\Phi_n^{-1}(\mathcal{A})$ has a σ–discrete refinement for every discrete family $\mathcal{A}$ of closed sets in X,
(iv) $\Phi_n(y) \subset \Phi_{n-1}(y)$,
(v) $\operatorname{diam} \Phi_n(y) \le \frac{1}{n}$, and
(vi) for every $F \in \mathcal{F}_n$, either $\Phi_n(y) \subset F$ or $\Phi_n(y) \cap F = \emptyset$.

Define $f : Y \to X$ as

$$f(y) = \bigcap_{n=1}^{\infty} \Phi_n(y) \ , \quad y \in Y \ .$$

Since $\{\Phi_n(y)\}$ is a sequence of closed sets with diameters converging to zero and X is complete, f is well defined. For $n \in \mathbb{N}$, condition (vi) implies

$$f^{-1}(F) = \Phi_n^{-1}(F) \ , \quad F \in \mathcal{F}_n \ .$$

Since the set $\Phi_n^{-1}(F)$ is an F_σ–set in Y according to condition (ii), we get that $f^{-1}(F)$ is an F_σ–set for every $F \in \mathcal{F}_n$ as required.

Let G be an open subset of X. If we put

$$H_k = \{z \in X : \operatorname{dist}(z, X \setminus G) \ge \frac{1}{k}\} \ ,$$

using condition (v) it can be easily verified that

$$f^{-1}(G) = \{y \in Y : \operatorname{dist}(f(y), X \setminus G) > 0\} = \bigcup_{k=1}^{\infty} \bigcup_{n=k+1}^{\infty} \Phi_n^{-1}(H_k) \ .$$

Thus $f^{-1}(G)$ is an F_σ–set in Y according to condition (ii). □

Theorem 4.3. *Let Y be an absolutely Souslin metric space, X be a metric space and $f : X \to Y$ be a surjective mapping such that f maps F_σ–sets in X to F_σ–sets in Y and the fiber $f^{-1}(y)$ is complete for each $y \in Y$. Let $\mathcal{F}$ be a σ–discrete family of F_σ–sets in X.*

Then there exists a section $g : Y \to X$ of f such that $g^{-1}(F)$ is an F_σ–set in Y for every $F \in \mathcal{F}$ and $g^{-1}(G)$ is an F_σ–set for every open set G in X.

Proof. Let $\widehat{X}$ be the completion of the metric space X. Then each fiber $f^{-1}(y)$, $y \in Y$ is a closed set in $\widehat{X}$. We define a multivalued map $\Phi : Y \to \widehat{X}$ as $\Phi(y) = f^{-1}(y)$, $y \in Y$. If F is a closed set in $\widehat{X}$, then

$$\Phi^{-1}(F) = \{y \in Y : \ \Phi(y) \cap F \neq \emptyset \ \} = f(F \cap X)$$

is an F_σ–set in Y.

Furthermore, let $\mathcal{A}$ be a discrete family of closed sets in $\widehat{X}$. Then $\mathcal{A} \wedge X$ is a discrete family of closed sets in X. According to Theorem 3.5, the family

$$\Phi^{-1}(\mathcal{A}) = \Phi^{-1}(\mathcal{A} \wedge X) = f(\mathcal{A} \wedge X)$$

has a σ–discrete refinement. Thus the conclusion follows by the previous Theorem 4.2. □

Theorem 4.4. *Let X be a metric space, Y be an absolutely Souslin metric space and $f : X \to Y$ be a surjective mapping such that f maps F_σ–sets in X to F_σ–sets in Y and the fiber $f^{-1}(y)$ is complete for each $y \in Y$.*

Let $\alpha \geq 1$ be a countable ordinal. If B is a subset of Y such that $f^{-1}(B)$ is a set of additive, or multiplicative, class α in X, then B is of the same class in Y.

Similarly, if $f^{-1}(B)$ is a Souslin set in X, then B is a Souslin set in Y.

Proof. Let $\alpha \geq 1$ and B be a subset of Y such that $A = f^{-1}(B)$ is of multiplicative class α in X. Since the case $\alpha = 1$ is obvious, we will assume that $\alpha \geq 2$. It easily follows by transfinite induction that there exists a countable family $\mathcal{H}$ of F_σ–sets in X such that $A \in \mathcal{M}_\alpha(\mathcal{H})$.

Use Theorem 4.3 to find a section g of f, $g : Y \to X$, such that $g^{-1}(H)$ is an F_σ–set in Y for every $H \in \mathcal{H}$. It is easy to check by transfinite induction that $g^{-1}(A) \in \mathcal{M}_\alpha(g^{-1}(\mathcal{H})) \subset \mathcal{M}_\alpha(Y)$. Since $g^{-1}(A) = g^{-1}(f^{-1}(B)) = B$, we obtain the required conclusion.

The case when $f^{-1}(B)$ is of additive class α, $\alpha \geq 2$, can be deduced from the previous argument by taking the complement of B.

The case when $f^{-1}(B)$ is a Souslin set in X can be treated similarly. Indeed, find a family $\mathcal{H} = \{F_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$ of closed sets in X such that $f^{-1}(B) = \bigcup_{\sigma \in \mathbb{I}} \bigcap_n F_{\sigma \restriction n}$. As above choose a selection function $g : Y \to X$ so that $g^{-1}(F_s)$ is an F_σ–set in Y for every $s \in \mathbb{N}^{<\mathbb{N}}$. Then $B = \bigcup_{\sigma \in \mathbb{I}} \bigcap_n g^{-1}(F_{\sigma \restriction n})$, which proves that B is a Souslin subset of Y. □

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