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INTEGRABLE FACTORS IN COMPACT SCHUR MULTIPLIERS

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ABSTRACT. It is shown that a Schur multiplier is compact if and only if it is the Schur product of two multipliers, one of which is a Hankel–Schur multiplier generated by an integrable function. This is illuminated by factoring exotic, singular measures and is brought into relation with Paley set-based multipliers.

As a model of general, separable Hilbert spaces the classical space $\ell^2 = \ell^2(\mathbb{N})$ is universal and operators on ℓ^2 are conveniently identified with infinite matrices as determined by the action on the standard coordinate basis $\{e_n\}_{n=0}^{\infty}$. Matrices will be indexed by non-negative integers in this essay. A matrix $A = \{a_{jk}\}$ is a Schur multiplier if the Schur (or entrywise) product $B \mapsto A * B = \{a_{jk}b_{jk}\}$ maps the full space of bounded operators $B(\ell^2)$ into itself. The collection of all Schur multipliers – here denoted V_2 – becomes a commutative Banach algebra when given the operator norm as acting on $B(\ell^2)$ and supplied with the usual addition and the Schur product as multiplication. The multiplier norm will be denoted $\|\ \|_m$. Bennett [Be] is an excellent reference on Schur multipliers. Some additional considerations – see Hladnik [H] for a concise presentation – reveals that V_2 coincides with the multiplier space of itself (acting via the Schur product) and this with identical norms. Thus the functional analytic properties of $A\colon B(\ell^2) \to B(\ell^2)$ and $A\colon V_2 \to V_2$ coincide.

In order to simplify later formulations the notation Γ_ϕ for the Hankel matrix $\{\phi(j+k)\}_{j,k\geq 0}$ determined by the diagonal $\phi \in \ell^\infty(\mathbb{N})$ will be used. The subspace of V_2 consisting of Hankel matrices is clearly a closed subalgebra of V_2. The corresponding diagonals induce a Banach algebra M^H of sequences in ℓ^∞ with norm $\|c\|_{M^H} = \|\Gamma_c\|_m$. It can be demonstrated by various means that

$$\|c\|_{\ell^\infty} \leq \|c\|_{M(H^1)} \leq \|c\|_{M^H} \leq \|c\|_{B(\mathbb{N})}$$

holds. Here $M(H^1)$ is the coefficient multiplier space of the Hardy space H^1, and $B(\mathbb{N})$ is the restricted Fourier-Stieltjes algebra with norm

$$\|c\|_{B(\mathbb{N})} = \inf\ \big\{\|\rho\|_{M(\mathbb{T})}\,;\ \hat{\rho}\big|_{\mathbb{N}} = c\big|_{\mathbb{N}}\big\}.$$

The purpose of this paper is to recognize the compact Schur multipliers in surprisingly sturdy terms. It was proved in [H] that the subalgebra of compact Schur multipliers can be characterised as the tensor product $c_0 \otimes^h c_0$ supplied with the Haagerup tensor norm. A completely different characterisation will be achieved below.

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Consider for an introductory example a Paley set $P \subseteq \mathbb{N}$, i.e., a finite union of lacunary sets, or what is the same, a subset of natural numbers such that $\sup_m |P \cap [m, 2m]| < \infty$. View $\ell^\infty(P)$ as a closed subspace of $\ell^\infty = \ell^\infty(\mathbb{N})$ simply by assigning 0 to every index $n \in \mathbb{N} \setminus P$. By Bożejko's theorem [Bo] the norm of Γ_b as a Littlewood function, with $b \in \ell^\infty$, is equivalent to the quantity $\sup_m \big\{ \sum_{k=m}^{2m} |b(m)|^2 \big\}^{1/2}$. The following is then immediate.

Proposition. *The Hankel-Schur multiplier Γ_b with $b \in \ell^\infty(P)$ is compact if and only if $b \in c_0(P)$.*

Writing $b = b_1 b_2$ coordinate-wise with $b_j \in c_0(P)$, a result of Rudin [R, Thm. 2.1] supplies some $f \in L^1(\mathbb{T})$ such that $\hat{f}\big|_P = b_2\big|_P$, since P is a Sidon set. Thus Γ_b with $b \in \ell^\infty(P)$ is a compact Schur multiplier if and only if it factors as $\Gamma_b = \Gamma_{b_1} * \Gamma_{\widehat{f}}$. This is in fact a general phenomenon, as the main result displays.

Let the matrix F_N have elements $f_{jk} = \max\{0, (N+1)^{-1}(N+1-j-k)\}$. Then $F_N = \Gamma_{\widehat{k_N}}$ is a Hankel matrix and corresponds to the positive Fejér kernel k_N. It follows that $\|F_N\|_m = \|k_N\|_{L^1} = 1$, all $N \geq 0$.

Theorem. *Let the Schur multiplier A be given. Then the following statements are equivalent.*

a. *$A\colon V_2 \to V_2$ is compact.*
b. *For some subsequence $N(j) \to \infty$ the convergence $\|A - F_{N(j)} * A\|_m \to 0$ is obtained.*
c. *$\|A - F_N * A\|_m \to 0$ as $N \to \infty$.*
d. *$A = \Gamma_{\widehat{f}} * B$, where $B\colon V_2 \to V_2$ is a (compact) Schur multiplier and $\hat{f}$ is the Fourier sequence given by some integrable $f \in L^1(\mathbb{T})$.*

Proof. Keep in mind that $\{F_N\}_{N=0}^\infty$ is a bounded set in V_2. Since statement b. expresses A as a limit of finite rank multipliers, clearly a. and b. are equivalent. On the other hand, any accumulation point of any sequence $\{F_{N(j)} * A\}_1^\infty$ is necessarily represented by the same matrix as A is, so standard functional analysis equates statements b. and c. Presupposing statement d., one has

$$\|A - F_N * A\|_m = \|B * (\Gamma_{\widehat{f}} - F_N * \Gamma_{\widehat{f}})\|_m \leq \|B\|_m \, \|f - \sigma_N f\|_{M(\mathbb{T})} \to 0$$

as $N \to \infty$. Here $\sigma_N f$ is the N-th Fejér sum of $f \in L^1(\mathbb{T})$. Hence A is compact by the previous argument.

Finally, the action $L^1(\mathbb{T}) \times V_2 \to V_2, (f, A) \mapsto \Gamma_{\widehat{f}} * A$ introduces on V_2 a structure of an $L^1(\mathbb{T})$-module, fully compatible with the multiplier norm topology. Likewise the closed subalgebra of compact multipliers becomes a module under this action:

$$\|\Gamma_{\widehat{f}} * A\|_m \leq \|\Gamma_{\widehat{f}}\|_m \|A\|_m \leq \|f\|_{L^1} \|A\|_m.$$

Since the range of this action is dense in the space of compact multipliers, the Module Factorization Theorem of Cohen–Hewitt (cf. [HR]) says exactly that statement d. holds provided A indeed should be compact. The proof is complete. □

Corollary. *Any compact Hankel–Schur multiplier is approximable by finitely supported Hankel–Schur multipliers.*

Proof. Any $F_N * A$ is Hankel-shaped if A is so! □

Additionally, the same proof generalizes mutatis mutandis to handle the multiplier spaces $M(H^1)$ and $M(L^1) \simeq M(\mathbb{T})$. The first statement below is of course known to everyone.

Proposition. *A multiplier $a \in M(L^1)$ is a compact mapping $L^1 \to L^1$, as well as $M(L^1) \to M(L^1)$, if and only if $a = \hat{f}$ for some $f \in L^1(\mathbb{T})$. Correspondingly, a multiplier $a \in M(H^1)$ is a compact mapping $M(H^1) \to M(H^1)$ if and only if there are $b \in M(H^1)$ and $f \in L^1(\mathbb{T})$ such that $a(n) = b(n)\hat{f}(n)$, all $n \geq 0$.*

Observe that the above discrepancy with integrable factors for the algebras V_2 and $M(H^1)$, but identification with Fourier sequences for the algebra $M(L^1)$, is necessary in order to accommodate the compact, Paley set-based multipliers. For another source of examples – somewhat surprising – a classification of the author comes in handy:

Lemma ([A]). *The Riesz product measure $\mu = \prod_{k=1}^{\infty}\left(1 + a_k e^{in_k\theta} + \overline{a_k} e^{-in_k\theta}\right)$, where $\{n_k\}$ is lacunary, defines a multiplier $\hat{\mu}\big|_{\mathbb{N}} \in M(H^1, H^2)$ if and only if*

$$\sup_k |a_k|^2 \prod_{j=1}^{k-1}\left(1 + 2|a_k|^2\right) < \infty.$$

Lemma ([A, Examples 8 and 9]). *There exist singular Riesz products that generate multipliers in $M(H^1, H^2)$.*

By Bożejko's theorem, membership in $M(H^1, H^2)$ above can freely be replaced by being a Littlewood–Hankel function in T_2^H, a certain subspace of Hankel–Schur multipliers. It is fairly straightforward to verify that every multiplier in $M(H^1, H^2)$ is weakly compact when seen as acting $H^1 \to H^1$; in particular, every Paley set makes the members of $\ell^\infty(P) \subset M(H^1)$ weakly compact and those in $c_0(P) \subset M(H^1)$ compact multipliers. In contrast, $\ell^\infty(P) \setminus c_0(P) \subset M^H$ generates non-weakly compact Hankel-Schur multipliers. This can be verified by studying permutation operators on ℓ^2.

Now let μ satisfy the last lemma; by singularity $\sum |a_k|^2 = \infty$. Trivially there exists a decreasing null-sequence $\{b_k\} \subset \,]0,1]$ such that $\sum |a_k b_k|^2 = \infty$ still holds. Let the Riesz product ν correspond to $\{a_k b_k\}_1^\infty$. By construction ν is singular and

$$|a_k b_k|^2 \prod_{j=1}^{k-1}\left(1 + 2|a_k b_k|^2\right) \leq b_k^2 \cdot \mathcal{O}(1) = o(1).$$

Therefore the technique of [A, Proposition 4] shows that $\hat{\nu}|_{\mathbb{N}}$ produces a compact Hankel–Schur multiplier $\Gamma_{\hat{\nu}|\mathbb{N}}$, and in fact the truncation $\hat{\nu}_M = \hat{\nu}\chi_{[0,M]}$ has the property

$$\|\Gamma_{\hat{\nu}} - \Gamma_{\hat{\nu}_M}\|_m \leq C \sup_{m \geq M+1} \left[\sum_{k=m}^{2m} |\hat{\nu}(k)|^2\right]^{1/2} \to 0 \quad \text{as} \quad M \to \infty.$$

Proposition. *There exist compact Hankel–Schur multipliers that are generated by Fourier–Stieltjes sequences of singular measures.*

Observe that in the restricted Fourier–Stieltjes algebra $B(\mathbb{N})$ the norm inequality $\|\hat{\nu} - \hat{\nu}_M\|_{B(\mathbb{N})} \geq 1$ holds for all $M \geq 0$, since ν is singular and of total variation 1. Therefore the compactness of $\Gamma_{\hat{\nu}}$ is a phenomenon in the multiplier algebra V_2 and is not due to any conditions inside $B(\mathbb{N})$ or $M(\mathbb{T})$.

Applying the theorem above, the stated compactness as a multiplier produces some compact $b \in M^H$ and $f \in L^1(\mathbb{T})$ such that $\hat{\nu}(n) = b(n)\hat{f}(n)$, all $n \geq 0$. However, the singularity of ν as a measure prevents the factor b from being a member of $B(\mathbb{N})$, so b cannot possibly be the restriction to $\mathbb{N}$ of a Fourier–Stieltjes transform. One should note that the support of b is not contained in any Paley set, since this is true of $\hat{\nu}\big|_{\mathbb{N}}$.

Corollary. *There exist singular measures $\nu \in M(\mathbb{T})$ that are divisible by integrable functions in the sense that $\hat{\nu}(n) = b(n)\hat{f}(n)$, all $n \geq 0$, for a suitable function $f \in L^1(\mathbb{T})$ and a sequence b generating a (compact) Schur multiplier Γ_b.*

The following material is intended to display the implications of the Paley spectrum upon Hankel–Schur multipliers. Recall first an important result of Klemes:

Theorem ([K])**.** *Any idempotent multiplier of H^1 is the characteristic function of a set arising as a finite boolean combination of* i) *arithmetic progressions,* ii) *finite sets, and* iii) *lacunary sets.*

Thus the same thing holds for the diagonals of idempotent Hankel–Schur multipliers. Invoking Bożejko's criterion, the largest sub-C^*-algebras of M^H may be identified using a straightforward argument:

Corollary. *Any set $E \subseteq \mathbb{N}$ that allows the natural embedding $c_0(E) \to M^H$ or $c_0(E) \to M(H^1)$ to be an isomorphism onto its range, is necessarily a Paley set.*

Therefore M^H contains sub-C^*-algebras $\ell^\infty(P)$ and $c_0(P)$, on which the supremum and multiplier norms are equivalent, if and only if P is a Paley set; clearly $c_0(P)$ consists entirely of compact Hankel–Schur multipliers.

To understand why the Fourier spectrum of the previous example with measure ν turned out by necessity to be thicker than Paley sets, some classical results from Fourier analysis come in handy. Recall that a set $E \subset \mathbb{Z}$ is a $\Lambda(2)$-set if the norm on $L^2_E(\mathbb{T})$, functions with spectrum in E, is equivalent to the L^1-norm.

Lemma. *Let $E \subset \mathbb{N}$ be a $\Lambda(2)$-set. Then there is a positive constant k_E such that every $a \in \ell^\infty$ with support in E satisfies*

$$k_E \, \|a\|_{\ell^2} \leq \|a\|_{B(\mathbb{N})} \leq \|a\|_{\ell^2}.$$

Proof. The upper inequality is immediate since $f \sim \sum_{n \in E} a_n e^{in\theta}$ has

$$\|a\|_{B(\mathbb{N})} \leq \|f\|_{L^1} \leq \|f\|_{L^2} = \|a\|_{\ell^2}.$$

Let on the other hand $a \in \ell^\infty$ be supported in E and such that $\|a\|_{B(\mathbb{N})}$ is finite. Invoking the weak-$*$-topology on $M(\mathbb{T})$ there is some $\nu \in M(\mathbb{T})$ with $\hat{\nu}|_{\mathbb{N}} = a$ and $\|\nu\|_{M(\mathbb{T})} = \|a\|_{B(\mathbb{N})}$. Now E is also a $\Lambda(1)$-set, since it is a $\Lambda(2)$-set, so by [R, Thm. 5.7], the Fourier spectrum in $E \cup \mathbb{Z}_-$ forces ν to be absolutely continuous.

Next, based on the $\Lambda(2)$-property, a theorem of Fournier [F] produces a positive number k_E such that any integrable function with Fourier spectrum in $E \cup \mathbb{Z}_-$ has $k_E \|\hat{f}\|_{\ell^2(E)} \leq \|f\|_{L^1}$. It follows that

$$k_E \, \|a\|_{\ell^2} = k_E \, \|\hat{\nu}\|_{\ell^2(E)} \leq \|\nu\|_{M(\mathbb{T})} = \|a\|_{B(\mathbb{N})},$$

which is the claimed lower inequality. □

Since every Paley set is a Sidon set and thus also a $\Lambda(2)$-set, the next results are immediate.

Corollary. *Let $P \subset \mathbb{N}$ be a Paley set. Then every $a \in \ell^\infty(P) \subset \ell^\infty(\mathbb{N})$ is the diagonal of a Littlewood function and generates as such (Bożejko's criterion) a Hankel–Schur multiplier with diagonal a, but alas this multiplier corresponds to a measure if and only if $a \in \ell^2$.*

Corollary. *For a Paley set P, every $a \in c_0(P) \setminus \ell^2$ is the diagonal of a compact Hankel–Schur multiplier that is not generated by a measure: $\|\Gamma_a\|_m \leq C_P \|a\|_\infty$, but $\|a\|_{B(\mathbb{N})} = \infty$. In contrast, every $c \in \ell^2(\mathbb{N})$ corresponds via the diagonal to a compact Hankel–Schur multiplier generated by an absolutely continuous measure.*

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