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A SIMPLE CLOSURE CONDITION FOR THE NORMAL CONE INTERSECTION FORMULA

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ABSTRACT. In this paper it is shown that if C and D are two closed convex subsets of a Banach space X and $x \in C \cap D$, then $N_{C\cap D}(x) = N_C(x) + N_D(x)$ whenever the convex cone, $(\operatorname{Epi}\sigma_C + \operatorname{Epi}\sigma_D)$, is weak* closed, where σ_C and N_C are the support function and the normal cone of the set C respectively. This closure condition is shown to be weaker than the standard interior-point-like conditions and the bounded linear regularity condition.

1. Introduction

Never mind the regularity condition, in a normal cone intersection formula it's the interior-point-like condition [11, 15] which we need to avoid. The formula for closed convex sets is not always true without a regularity condition. The purpose of this paper is to show that a normal cone intersection formula for closed convex sets holds under a simple closure condition that is weaker than the interior-point-like conditions and bounded linear regularity condition [3, 4, 5].

A normal cone intersection formula states [7, 11, 4] that the normal cone of the intersection of sets equals the sum of the normal cones of the sets. A fundamental problem in convex analysis is to determine conditions under which the intersection formula holds at every point of the intersection of the sets. Such an intersection formula plays a key role in characterizing solutions of optimization problems and constrained best approximation problems. For instance, consider the optimization model problem

$$\inf\{f(x) \mid x \in C \cap D\}, \tag{1}$$

where C and D are closed and convex subsets of a Banach X and $f : X \to \mathbb{R} \cup \{\infty\}$ is a proper convex function. The model problem arises, for example, in convex

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programming problems [6, 14, 15], where $D = \{x \in X \mid g_i(x) \leq 0, i = 1, 2, \ldots, m\}$ and the g_i's are convex functions, and in constrained best approximation problems [9, 17], where $f(x) = ||y - x||$ and $y \in X$. It is known that if f is continuous at $x^* \in C \cap D$, then x^* is an optimal solution of (1) if and only if

$$0 \in \partial f(x^*) + N_{C\cap D}(x^*), \tag{2}$$

where ∂f is the subdifferential of f and $N_{C\cap D}$ is the normal cone of the set $C \cap D$. The significance of this characterization relies on the description of $N_{C\cap D}(x^*)$ in terms of $N_C(x^*)$ and $N_D(x^*)$. If the interior-point condition that $(\operatorname{int} D) \cap C \neq \emptyset$ (or its recent generalization that the cone generated by $(C - D)$, $\operatorname{cone}(C - D)$, is a closed subspace [13, 15, 10]) is satisfied or the bounded linear regularity condition [4, 5], when X is a Euclidean space, is satisfied, then the normal cone intersection formula

$$N_{C\cap D}(x) = N_C(x) + N_D(x), \ \forall x \in C \cap D,$$

holds. The interior-point-like conditions are often restrictive in applications as for instance the set D may not have interior points or the set C may not have a point in the interior of D. For a simple example, let $C := [0, 1]$ and $D := (-\infty, 0]$. Then $N_{C\cap D}(0) = \mathbb{R} = (-\infty, 0] + [0, \infty) = N_C(0) + N_D(0)$, and so the normal cone intersection formula holds at $x \in C \cap D = \{0\}$, whereas $(\operatorname{int} D) \cap C = \emptyset$ and $\operatorname{cone}(C - D) = [0, \infty)$, which is not a subspace. For other examples, see [4, 5].

In this paper we show that if C and D are two closed convex subsets of a Banach space and $x \in C \cap D$, then $N_{C\cap D}(x) = N_C(x) + N_D(x)$ whenever the convex cone, $(\operatorname{Epi} \sigma_C + \operatorname{Epi} \sigma_D)$, is weak* closed, where σ_C is the support function of the set C. We give a proof using a separation theorem [8, 12]. Our closure condition is shown to be weaker than the popularly known generalized interior point conditions and the bounded linear regularity condition [4, 5].

2. Preliminaries

We begin by fixing some definitions and notation. Let X be a Banach space. The continuous dual space of X will be denoted by X' and will be endowed with the weak* topology. For the set $D \subset$ X, the **closure** of D and the **interior** of D will be denoted $\operatorname{cl} D$ and $\operatorname{int} D$ respectively. If a set $A \subset X'$, $\operatorname{cl} A$ will stand for the weak* closure. Let **cone**$(D) = \{\lambda x \mid \lambda \in \mathbb{R}, \lambda \geq 0, \ x \in D\}$. The **indicator function** δ_D is defined as $\delta_D(x) = 0$ if $x \in D$ and $\delta_D(x) = +\infty$ if $x \notin D$. The **support function** σ_D is defined by $\sigma_D(u) = \sup_{x\in D} u(x)$. The **dual cone** of D is given by $D^+ = \{\theta \in X' : \theta(k) \geq 0, \ \forall k \in D\}$ and the **normal cone** of D is given by $N_D(x) := \{v \in X' \ : \ \sigma_D(v) = v(x)\} = \{v \in X' : \ v(y - x) \leq 0, \forall y \in D\}$ when $x \in D$, and $N_D(x) = \emptyset$ when $x \notin D$.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. Then, the **conjugate** function of f, $f^* : X' \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$f^*(v) = \sup\{v(x) - f(x) \mid x \in \ \operatorname{dom} f\}$$

where the domain of f, dom f, is given by

$$\operatorname{dom} f = \{x \in X \mid f(x) < +\infty\}.$$

The epigraph of f, Epi f, is defined by

$$\operatorname{Epi} f = \{(x, r) \in X \times \mathbb{R} \mid x \in \ \operatorname{dom} f, \ f(x) \leq r\}.$$

Note that for a set $C \subset X$, $\delta_C^* = \sigma_C$.

For the proper lower semi-continuous functions $f, g : X \to \mathbb{R} \cup \{+\infty\}$, the *infimal convolution* of f with g is denoted by $f \oplus g : X \to \mathbb{R} \cup \{+\infty\}$ and is defined by

$$f \oplus g(x) := \inf_{x_1+x_2=x} \{f(x_1) + g(x_2)\}.$$

The infimal convolution of f with g is said to be *exact* provided the infimum above is achieved for every $x \in X$. It is known (see, e.g., [19, Theorem 2.2(c)]) that if the infimal convolution is exact, then

$$\text{Epi } (f \oplus g) = \text{Epi } f + \text{Epi}\, g. \tag{3}$$

Moreover, if cone(dom $f -$dom g) is a closed subspace, then the infimal convolution of f^* and g^* is exact, and $f^* \oplus g^* = (f+g)^*$. For details, see [2, 19].

The conclusion of the following lemma, which plays a useful role in our development of a new closure condition, follows from a separation theorem.

Lemma 2.1. *Let C and D be closed convex subsets of X. Then*

$$C \cap D \neq \emptyset \iff (0, -1) \notin \text{cl}\,(\text{Epi}\,\sigma_C + \text{Epi}\,\sigma_D)\,.$$

Proof. (i) Let $(u, \alpha) \in (\text{Epi } \sigma_C + \text{Epi } \sigma_D) = \text{Epi } \delta_C^* + \text{Epi } \delta_D^*$. Then there exist $v, w \in X'$, and $\beta, \delta \in \mathbb{R}$ such that $(v, \beta) \in$ Epi δ_D^* , $(w, \delta) \in$ Epi δ_C^*, and $u = v+w$, $\alpha = \beta + \delta$. So, for each $x \in D, v(x) \leq \beta$, and for each $x \in C$, $w(x) \leq \delta$. Now, if $x \in A := C \cap D$, then

$$u(x) = (v + w)(x) \leq \beta + \delta = \alpha,$$

which proves that $(u, \alpha) \in$ Epi σ_{A}. This, together with the fact that Epi σ_A is weak* closed, gives us

$$\text{cl}\,(\text{Epi } \delta_D^* + \text{Epi } \delta_C^*) \subset \text{ epi } \sigma_A. \tag{4}$$

If $C \cap D \neq \emptyset$, then clearly $(0, -1) \notin$ epi σ_A, and so from the above inclusion $(0, -1) \notin \text{cl}\,(\text{Epi } \delta_D^* + \text{Epi } \delta_C^*)$. Conversely, if $(0, -1) \notin \text{cl}\,(\text{Epi } \delta_D^* + \text{Epi } \delta_C^*)$, then by the separation theorem [[8], Theorem 4.3.5] there is $(x, \alpha) \in X \times \mathbb{R}$, $(x, \alpha) \neq (0, 0)$, such that $-\alpha < 0$ and

$$u(x) + \gamma\alpha \geq 0, \ \ \forall (u, \gamma) \in \text{cl}\,(\text{Epi } \delta_D^* + \text{Epi } \delta_C^*)\,.$$

Let $\bar{x} = \frac{x}{\alpha}$. Then we have

$$u(-\bar{x}) - \gamma \leq 0, \ \forall (u, \gamma) \in \text{cl}\,(\text{Epi } \delta_D^* + \text{Epi } \delta_C^*)\,.$$

So, for each $v \in$ domδ_{D}^* and for each $w \in$ domδ_{C}^*, we have $(v+w, \delta_D^*(v)+\delta_C^*(w)) \in \text{cl}\,(\text{Epi } \delta_D^* + \text{Epi } \delta_C^*)$ and hence,

$$(v + w)(-\bar{x}) - \delta_D^*(v) - \delta_C^*(w) \leq 0.$$

By letting $w = 0$, we get $v(-\bar{x}) - \delta_D^*(v) \leq 0$, for each $v \in$ dom δ_D^*. Since δ_D is lower semi-continuous,

$$\delta_D(-\bar{x}) = \delta_D^{**}(-\bar{x}) = \sup_v [v(-\bar{x}) - \delta_D^*(v)] \leq 0.$$

This implies that $-\bar{x} \in D$. Similarly, we can show that $-\bar{x} \in C$. Thus $-\bar{x} \in C \cap D \neq \emptyset$. $\square$

3. The normal cone intersection formula

In this section we derive the normal cone intersection formula for closed convex sets. We first obtain a key extension of the dual cone intersection formula for closed convex cones C and D such that $(C \cap D)^+ = \mathrm{cl}(C^+ + D^+)$ to closed convex sets C and D which are not necessarily cones. The extension, which is expressed in terms of the epigraphs of the support functions of C and D, then leads to a closure condition, ensuring the normal cone intersection formula.

Lemma 3.1. *Let C and D be closed convex subsets of X. If $C \cap D \neq \emptyset$, then*

$$\mathrm{Epi}\,\sigma_{C\cap D} = \mathrm{cl}\,(\mathrm{Epi}\,\sigma_C + \mathrm{Epi}\,\sigma_D)\,.$$

Proof. Let $A := C \cap D \neq \emptyset$. Then, as we saw in the proof of Lemma 2.1 (see (4)), we have the inclusion

$$\mathrm{cl}\,(\mathrm{Epi}\,\sigma_D + \mathrm{Epi}\,\sigma_C) \subset \mathrm{Epi}\ \sigma_A.$$

To show the reverse inclusion, let $(u, \alpha) \notin \mathrm{cl}\,(\mathrm{Epi}\,\sigma_D + \mathrm{Epi}\,\sigma_C)$. Since $A \neq \emptyset$, it follows from Lemma 2.1 that $(0, -1) \notin \mathrm{cl}\,(\mathrm{Epi}\,\sigma_D + \mathrm{Epi}\,\sigma_C)$. So,

$$B \cap (\mathrm{cl}\,(\mathrm{Epi}\,\sigma_D + \mathrm{Epi}\,\sigma_C)) = \emptyset,$$

where $B := \{\delta(u, \alpha) + (1-\delta)(0, -1) \in X' \times \mathbb{R} \mid \delta \in [0, 1]\}$ is the segment connecting the points (u, α) and $(0, -1)$. Otherwise, there is $\delta_0 \in (0, 1)$ such that

$$\delta_0(u, \alpha) + (1 - \delta_0)(0, -1) \in \mathrm{cl}\,(\mathrm{Epi}\,\sigma_D + \mathrm{Epi}\,\sigma_C)\,;$$

thus,

$$(\delta_0 u, \delta_0\alpha - (1 - \delta_0)) \in \mathrm{cl}\,(\mathrm{Epi}\,\sigma_D + \mathrm{Epi}\,\sigma_C)\,.$$

Also $\{0\} \times \mathbb{R}_+ \subset \mathrm{cl}\,(\mathrm{Epi}\,\sigma_D + \mathrm{Epi}\,\sigma_C)$. Then,

$$(\delta_0 u, \delta_0\alpha) = (\delta_0 u, \delta_0\alpha - (1 - \delta_0)) + (0, 1 - \delta_0) \in \mathrm{cl}\,(\mathrm{Epi}\,\sigma_D + \mathrm{Epi}\,\sigma_C)\,.$$

This gives us that

$$(u, \alpha) = \frac{1}{\delta_0}(\delta_0 u, \delta_0\alpha) \in \mathrm{cl}\,(\mathrm{Epi}\,\sigma_D + \mathrm{Epi}\,\sigma_C)\,,$$

which is a contradiction.

Now, by applying the separation theorem, there is $(x, \beta) \in X \times \mathbb{R}$, $(x, \beta) \neq (0, 0)$ such that

$$[\delta(u, \alpha) + (1 - \delta)(0, -1)](x, \beta) < 0, \quad \forall \delta \in [0, 1],$$

and

$$v(x) + \gamma\beta \geq 0,\ \forall (v, \gamma) \in \mathrm{cl}\,(\mathrm{Epi}\,\sigma_D + \mathrm{Epi}\,\sigma_C)\,.$$

By letting $\delta = 0$ we get $\beta > 0$ and by letting $\delta = 1$ we obtain $u(x) + \alpha\beta < 0$; thus, $u(\frac{-x}{\beta}) > \alpha$. Also, the same argument as in the proof of Lemma 2.1 leads to

$$\frac{-x}{\beta} \in C \cap D.$$

This together with the fact that $u(\frac{-x}{\beta}) > \alpha$ implies that $(u, \alpha) \notin \mathrm{Epi}\,\sigma_A$. $\square$

Observe that if C and D are closed and convex cones of X, then the conclusion of Lemma 3.1 gives us that

$$-(C \cap D)^+ \times \mathbb{R}_+ = \mathrm{Epi}\,\sigma_{C\cap D} = \mathrm{cl}\,(\mathrm{Epi}\,\sigma_C + \mathrm{Epi}\,\sigma_D) = \mathrm{cl}\left[(-C^+ - D^+) \times \mathbb{R}_+\right],$$

and so, $(C \cap D)^+ = \mathrm{cl}\,(C^+ + D^+)$.

We now derive the main result as an application of Lemma 3.1. Note that the set (Epi σ_C + Epi σ_D) is a convex cone.

Theorem 3.1. *Let X be a Banach space and let C and D be two closed and convex subsets of X. If the set* (Epi σ_C + Epi σ_D) *is weak* closed, then*

$$\forall x \in C \cap D, \qquad N_{C\cap D}(x) = N_C(x) + N_D(x).$$

Proof. Let $x \in C \cap D$. Then, clearly $N_{C\cap D}(x) \supset N_C(x) + N_D(x)$. To show the reverse inclusion, let $v \in N_{C\cap D}(x)$. This means that $\sigma_{C\cap D}(v) = v(x)$, which in turn gives $(v, v(x)) \in \text{Epi } \sigma_{C\cap D}$. Since Epi σ_C + Epi σ_D is weak* closed, it follows from Lemma 3.1 that $(v, v(x)) \in \text{Epi } \sigma_C + \text{Epi } \sigma_D$. Then there exist two elements $(v_1, \alpha_1) \in \text{Epi } \ \sigma_C$ and $(v_2, \alpha_2) \in \text{Epi } \ \sigma_D$ such that $v_1 + v_2 = v$ and $\alpha_1 + \alpha_2 = v(x)$. So,

$$v(x) = \alpha_1 + \alpha_2 \geq \sigma_C(v_1) + \sigma_D(v_2) \geq v_1(x) + v_2(x) = v(x),$$

which gives us $\sigma_C(v_1) + \sigma_D(v_2) = v(x)$. Now,

$$\begin{aligned} 0 \geq v_1(x) - \sigma_C(v_1) &= (v - v_2)(x) + \sigma_D(v_2) - v(x) = \sigma_D(v_2) - v_2(x) \\ &\geq v_2(z) - v_2(x) = v_2(z - x), \end{aligned}$$

for each $z \in D$. Thus, $v_2 \in N_D(x)$. Similarly, we can show also that $v_1 \in N_C(x)$. Hence, $v \in N_C(x) + N_D(x)$. □

We now see that Theorem 3.1 leads to the sum formula for dual cones under our closure condition. Note that if $0 \in C$, it follows from the definitions that $N_C(0) = -C^+$.

Corollary 3.1. *Let C and D be two closed and convex subsets of X such that $0 \in C \cap D$ and the set* (Epi σ_C + Epi σ_D) *is weak* closed. Then*

$$(C \cap D)^+ = C^+ + D^+. \tag{5}$$

Proof. Clearly, $N_C(0) = -C^+$ and $N_D(0) = -D^+$. It follows from Theorem 3.1 that $N_{C\cap D}(0) = N_C(0) + N_D(0)$ and hence

$$-(C \cap D)^+ = -(C^+) - (D^+) = -(C^+ + D^+),$$

which readily implies (5). □

We also see that the known interior-point-like conditions yield our closure condition. Recall that $\textbf{core}(A) := \{a \in A \mid (\forall x \in X)(\exists \varepsilon > 0) \text{ such that } (\forall \lambda \in [-\varepsilon, \varepsilon])\ a + \lambda x \in A\}$.

Proposition 3.1. *Let C and D be closed and convex subsets of X. Then the set* (Epi σ_C + Epi σ_D) *is weak* closed if one of the following conditions holds:*

(i) $(\text{int } D) \cap C \neq \emptyset$,
(ii) $0 \in \text{core}(C - D)$,
(iii) $\text{cone}(C - D)$ *is a closed subspace.*

Proof. Clearly, (i) implies (ii), which in turn implies (iii). So, it suffices if we show that (iii) implies that the set (Epi σ_C + Epi σ_D) is weak*-closed. Indeed, if (iii) holds, then, using [19, Theorem 3.6], we get $\sigma_{D\cap C} = \sigma_D \oplus \sigma_C$, with exact infimal convolution. As a consequence of the exactness (see equation (3)), we have that Epi $\sigma_{C\cap D}$ = Epi σ_C + Epi σ_D. Since the set in the left-hand side is weak* closed, the conclusion holds. □

The following simple example illustrates the situation where the conditions (i)-(iii) fail; whereas our closure condition holds.

Example 3.1. Let $C := [0, 1]$ and $D := (-\infty, 0]$. Then $N_{C\cap D}(0) = \mathbb{R} = (-\infty, 0] + [0, \infty) = N_C(0) + N_D(0)$, and so the normal cone intersection formula holds at $x \in C \cap D = \{0\}$, whereas $(\text{int } D) \cap C = \emptyset$ and $\text{cone}(C - D) = [0, \infty)$, which is not a subspace. However, $\text{Epi}\, \sigma_C + \text{Epi}\, \sigma_D = \{(x, t) \in \mathbb{R} \times \mathbb{R} |\ \max\{x, 0\} \le t\} + \mathbb{R}_+ \times \mathbb{R}_+ = \mathbb{R} \times \mathbb{R}_+$ is a closed convex cone.

For related conditions guaranteeing the closure of the sum of two closed convex sets, see [1].

4. Regularity and closed convex cones

We now examine the connections between our closure condition and the bounded linear regularity condition [4, 5] in the case where C and D are closed and convex cones.

Proposition 4.1. *Let C and D be closed convex cones of X. Then the set $(\text{Epi } \sigma_C + \text{Epi } \sigma_D)$ is weak* closed if and only if $(C^+ + D^+)$ is weak* closed.*

Proof. If the set $(\text{Epi } \sigma_C + \text{Epi } \sigma_D)$ is weak* closed, then it follows from Corollary 3.1 that $(C^+ + D^+) = (C \cap D)^+$, which is weak* closed. Conversely, assume that $C^+ + D^+$ is weak* closed. Since C and D are closed convex cones,

$$-(C^+ + D^+) \times \mathbb{R}_+ = (-C^+ \times \mathbb{R}_+) + (-D^+ \times \mathbb{R}_+) = \text{Epi } \sigma_C + \text{Epi } \sigma_D.$$

Hence, if $(C^+ + D^+)$ is weak* closed, then the set $(\text{Epi } \sigma_C + \text{Epi } \sigma_D)$ is weak* closed. $\square$

In the case where X is a Euclidean space, we will now show that our closure condition is equivalent to the normal cone intersection formula for closed convex cones. Recall that the negative dual cone of a set D is given by $D^- := -D^+$.

Proposition 4.2. *Let X be a Euclidean space. Let C and D be closed convex cones of X. Then the following statements are equivalent.*

(i) *The set $(\text{Epi } \sigma_C + \text{Epi } \sigma_D)$ is closed.*
(ii) *$(C^+ + D^+)$ is closed.*
(iii) *For each $x \in C \cap D$, $N_{C\cap D}(x) = N_C(x) + N_D(x)$.*

Proof. [(i)$\Longleftrightarrow$ (ii)]. This follows from Proposition 4.1.
[(ii) $\Longleftrightarrow$ (iii)]. The set $C^+ + D^+$ is closed if and only if $C^- + D^-$ is closed. Now, the equivalence follows from Proposition 4.1 and Proposition 20 of [4]. $\square$

Recall that the pair $\{C, D\}$ is said to be **boundedly linearly regular** [3] if for every bounded set S in X, there exists $\kappa_S > 0$ such that the distances to the sets C, D and $C \cap D$ are related by

$$d(x, C \cap D) \le \kappa_S \max\{d(x, C),\, d(x, D)\},$$

for every $x \in S$, where $d(x, C) := \inf\{||x - c||\,|\, c \in C\}$ is the distance function.

Proposition 4.3. *Let X be a Euclidean space. Let C and D be closed convex cones of X. If the pair $\{C, D\}$ is boundedly linearly regular, then the set $(\text{Epi } \sigma_C + \text{Epi } \sigma_D)$ is closed.*

Proof. Theorem 3 of [4] gives us that bounded linear regularity implies that the normal cone intersection formula, called *strong* CHIP in [4], holds for closed convex sets. Hence, the conclusion follows from Proposition 4.2. □

Note that in the case where C and D are closed convex cones in a Euclidean space, the condition that the set $(\text{Epi }\sigma_C + \text{Epi }\sigma_D)$ is closed does not imply that the pair $\{C, D\}$ is boundedly linearly regular. Indeed, it has recently been shown in [5, Corollary 3.2] that the normal cone intersection formula may hold for certain closed convex cones C and D, whereas the pair $\{C, D\}$ is not boundedly linearly regular. Thus, the counterexample in $\mathbb{R}^4$ in Section 3 of [5] shows that the set $(\text{Epi }\sigma_C + \text{Epi }\sigma_D)$ is closed, whereas the pair $\{C, D\}$ is not boundedly linearly regular.

On the other hand, Proposition 5.16 of [3] shows that if C and D are closed subspaces in a Hilbert space, then $(C^{\perp} + D^{\perp})$ is closed if and only if $\{C, D\}$ is boundedly linearly regular. Note in this case that $(C^{\perp} + D^{\perp})$ is closed if and only if the set $\text{Epi}\,\sigma_C + \text{Epi}\,\sigma_D$ is closed. This follows from the fact that $\text{Epi}\,\sigma_C + \text{Epi}\,\sigma_D = (C^{\perp} \times \mathbb{R}_+) + (D^{\perp} \times \mathbb{R}_+) = (C^{\perp} + D^{\perp}) \times \mathbb{R}_+$, where the set $V^{\perp}$ is the subspace perpendicular to the subspace V.

Hence, if C and D are closed subspaces in a Hilbert space, then the following conditions are equivalent:

(i) The set $(\text{Epi }\sigma_C + \text{Epi }\sigma_D)$ is closed.
(ii) $(C^{\perp} + D^{\perp})$ is closed.
(iii) $\{C, D\}$ is boundedly linearly regular.

We end this section by pointing out that, in the particular case in which C and D are subspaces of a Euclidean space X, the set $\text{Epi}\,\sigma_C + \text{Epi}\,\sigma_D$ is always closed, since the subspace $(C^{\perp} + D^{\perp})$ is always closed in a Euclidean space.

5. Conclusion and further research

In this paper we have shown that the normal cone intersection formula for closed convex sets holds under a simple closure condition. In other words, we have established the subdifferential sum formula [2, 4, 16], that $\partial(\delta_C + \delta_D)(x) = \partial\delta_C(x) + \partial\delta_D(x)$, for the indicator functions of two closed convex sets C and D under a closure condition that is much weaker than the interior-point-like conditions. The following questions naturally arise: Does the subdifferential sum formula for two arbitrary proper lower semi-continuous convex functions hold under a similar closure condition that is weaker than the interior-point-like conditions? Is the pointwise sum of two maximal monotone operators a maximal monotone operator under an appropriate extension of our closure condition? The answers to these questions appear to be in the affirmative and will be investigated in a further study.

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