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E-ALGEBRAS WHOSE TORSION PART IS NOT CYCLIC

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ABSTRACT. We consider algebras A over a Dedekind domain R with the property $A \cong \operatorname{EndAlg}_R A$ and generalize Schultz' structure theory of the case $R = \mathbb{Z}$ to Dedekind domains. We construct examples of mixed $E(R)$-algebras, which are non-split extensions of the submodule of elements infinitely divisible by the relevant prime ideals. This is also new in the case $R = \mathbb{Z}$.

1. INTRODUCTION

1.1. **Notion of an $E(R)$-algebra.** Let R be a commutative ring and M an R-module. The endomorphism ring $\operatorname{End} M$ of M is an R-algebra acting on M on the right. We will denote the endomorphism ring by $\operatorname{Endo} M$ if we think of it as an R-module.

If M and $\operatorname{Endo} M$ are isomorphic, then M is called an *$E(R)$-module*.

If A is an R-algebra, then $\mu \colon A \to \operatorname{End} A$ mapping $a \in A$ to right multiplication by a is an algebra homomorphism. If μ is an isomorphism, then A is an *$E(R)$-algebra*.

An R-algebra A is a *generalized $E(R)$-algebra* if it is isomorphic to $\operatorname{End} A$.

Note that $E(R)$-algebras are precisely the commutative generalized $E(R)$-algebras. Every generalized $E(R)$-algebra is an $E(R)$-module, and every $E(R)$-module admits a generalized $E(R)$-algebra structure, which is unique up to isomorphism. Thus $E(R)$-modules and generalized $E(R)$-algebras are essentially the same.

The above notions are generalizations of E-rings, i.e. $E(\mathbb{Z})$-algebras.

These notions go back to Schultz; see [5]. Examples of $E(R)$-modules are given in Section 2.

1.2. **Main results.** This subsection summarizes our main results, which will be proved in later sections.

From now on let R be a Dedekind domain. Let Q denote the quotient field of R. If S is a set of prime ideals of R, then an R-module M is *S-divisible* if $M = PM$ for all $P \in S$. The module M is *S-reduced* if it has no non-trivial S-divisible submodule. If $N \subseteq M$, then N is *S-pure* in M (denoted by $N \subseteq_S M$) if $IN = N \cap IM$ for all ideals I of R which are products of prime ideals from S.

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For example, $T := \bigoplus_{P\in S} R/P^{n_P}$ is an S-pure submodule of

$$\Pi := \prod_{P\in S} R/P^{n_P}$$

where the n_P are arbitrary positive integers. Both T and Π are S-reduced, and they are divisible by all prime ideals not in S. Note that Π/T is divisible: if $Q \in S$, then Π/T is a factor of the Q-divisible module $\Pi := \prod_{P\in S\setminus\{Q\}} R/P^{n_P}$ and hence it is Q-divisible. If $Q \notin S$, then Π/T is Q-divisible since Π is already Q-divisible.

Now we can reformulate the structure theory of E-rings from Schultz [5, p. 63, Theorem 6] for Dedekind domains.

Theorem 1.1. *Let A be a generalized $E(R)$-algebra. Then*

(i) *The primary components of the torsion part T of A are cyclic:*

$$T = \bigoplus_{P\in S} R/P^{n_P} \tag{1}$$

where S is a set of prime ideals of R and the n_P are positive integers. After Schultz, the elements of S are called the relevant *prime ideals.*

(ii) *Using the above notation, let D be the largest S-divisible submodule of A. Then D is an ideal, A/D is an S-reduced subalgebra of $\prod_{P\in S} R/P^{n_P}$ and*

$$T = \bigoplus_{P\in S} R/P^{n_P} \subseteq_S A/D \subseteq_S \prod_{P\in S} R/P^{n_P} =: \Pi. \tag{2}$$

Using the notation and assumptions of the previous theorem, we have a pull-back diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A/T \arrow[d] \\ A/D \arrow[r] & A/(D+T) \end{tikzcd} \tag{3}$$

This suggests that we can construct mixed $E(R)$-algebras as pull-backs; see Lemma 4.1. It is a fruitful method, as it gives the first example of an $E(R)$-module A for which D is not a direct summand as stated in the next theorem. Let R_S be the localization of R at the set S of prime ideals.

Theorem 1.2. *Let S be an infinite set of prime ideals of R. Suppose that S does not contain all the prime ideals and R_S is not a complete discrete valuation ring. Then for any family $\{n_P : P \in S\}$ of positive integers there exists an arbitrarily large $E(R)$-algebra A whose torsion submodule is given by* (1) *and whose S-divisible part D is not a direct summand.*

This indicates that the classification of mixed $E(R)$-modules is a difficult task. However, the classification of non-reduced E-rings generalizes to $E(R)$-algebras:

Theorem 1.3. *The non-reduced $E(R)$-modules over a Dedekind domain R are exactly the R-modules of the form $Q \oplus C$ where C is a torsion cyclic R-module. In particular they are all $E(R)$-algebras.*

2. Examples of $E(R)$-modules

2.1. **Torsion-free $E(R)$-modules.** Though most of the references below deal only with the case $R = \mathbb{Z}$, their methods generalize without problem to Dedekind domains R which are neither fields nor complete discrete valuation domains. Therefore we state the results for Dedekind domains.

If R is a field, then R is also the only $E(R)$-algebra. Let us note that all $E(R)$-algebras over a complete discrete valuation domain are $R, Q(R)$ and the direct sums $Q(R) \oplus R/p^nR$ for all $n \in \mathbb{N}$, so the exclusion of these rings is necessary. We also note that the generalized $E(R)$-algebras over a complete discrete valuation domain are $E(R)$-algebras; see [3]. Thus it is also necessary to exclude these rings.

Classical examples of torsion-free $E(R)$-algebras are the subalgebras of Q and the pure subalgebras of J_P, the algebra of P-adic integers over R where P is a prime ideal.

Arbitrarily large $E(R)$-algebras were constructed using the Black Box in Dugas, Mader, Vinsonhaler [1]. The proof can be simplified by use of the Strong Black Box; see [4] and also [3]. These $E(R)$-algebras are distorted polynomial algebras over R.

Finally, we mention some more constructions of E-rings. Faticoni [2] showed that every countable reduced torsion-free commutative ring is a pure subring of an E-ring. The same is true for rings of cardinality $\leq 2^{\aleph_0}$; see [3].

2.2. **$E(R)$-modules whose torsion part is a direct summand.** An R-module is an $E(R)$-module whose torsion part is a direct summand iff, for some non-zero ideal I of R, it is of the form $R/I \oplus N$ where N is a torsion-free I-divisible $E(R)$-module. (N is I-divisible means $N = IN$.) This follows directly from Theorem 1.1.

Thus this class of $E(R)$-modules is classified modulo torsion-free $E(R)$-modules. In particular, the torsion $E(R)$-modules are precisely the cyclic modules.

2.3. **Other mixed $E(R)$-modules.** The following example is due to Schultz. Let S be a set of prime ideals of R. Let A be a subalgebra of $\prod_{P\in S} R/P^{n_P}$ such that

$$\bigoplus_{P\in S} R/P^{n_P} \subseteq_S A \subseteq_S \prod_{P\in S} R/P^{n_P}. \tag{4}$$

Then A is an $E(R)$-algebra and S is the set of relevant prime ideals.

So far we have only seen $E(R)$-modules whose S-divisible part is a direct summand. But this is not true for our examples in Section 4:

Mixed $E(R)$-modules can be constructed as a pull-back of the last example and a torsion-free $E(R)$-module. See Lemma 4.1 for the precise construction.

3. The structure theory of $E(R)$-algebras

In this section we prove Theorems 1.1 and 1.3, which describe the structure of $E(R)$-algebras.

We will use $N \sqsubseteq M$ to denote that N is a direct summand of M as an R-module.

We recall some well-known facts about torsion submodules over Dedekind domains used below. Torsion modules decompose into primary components. Each torsion module M has a basic submodule i.e. a pure submodule B which is a direct sum of cyclic modules and M/B is divisible.

Basic submodules of M are not unique but the number of cyclic summands R/P^k is unique for all k. For example, this number is the largest cardinal κ for which $(R/P^k)^{(\kappa)}$ is a direct summand of M.

Divisible submodules and pure bounded submodules are direct summands in any module. Divisible modules are direct sums of modules R_{P^∞}'s and Q. The number of the summands R_{P^∞}, respectively Q is independent of the decomposition. Namely, in every decomposition of a divisible module M, the number of the summands Q is the largest cardinal κ for which $Q^{(\kappa)}$ is a direct summand of M. The same result holds for R_{P^∞}.

Recall that $J_P \cong \operatorname{End} R_{P^\infty}$ is the algebra of P-adic integers over R. This gives an embedding of R_{P^∞} in $\operatorname{Hom}(J_P, R_{P^\infty})$. As we already mentioned, $J_P \cong \operatorname{End} J_P$.

Recall that $\operatorname{Hom}(A, B)$ is torsion-free if B is divisible, and $\operatorname{Hom}(A, B)$ is divisible if A is torsion-free and B is divisible.

The proofs of the next two lemmas are obvious.

Lemma 3.1. *Let M be a torsion cyclic module or the quotient field Q. Then for every cardinal κ,*

$$\operatorname{Endo} M^{(\kappa)} \cong M^{(\lambda)} \tag{5}$$

for some λ. If $\kappa > 1$, then $\lambda > \kappa$.

Lemma 3.2. *Let A be an $E(R)$-module. Suppose that $A = B \oplus C$ is a direct sum of R-modules. Then* $\operatorname{Endo} B \oplus \operatorname{Endo} C$ *is a direct summand of A as an R-module. Moreover,*

$$A \cong \operatorname{Endo} A = \operatorname{Endo} B \oplus \operatorname{Hom}(B, C) \oplus \operatorname{Hom}(C, B) \oplus \operatorname{Endo} C. \tag{6}$$

3.1. **The main structure theorem.** We are ready to prove Theorem 1.1. The two parts of the theorem are proven separately.

Proof of Theorem 1.1(i). By the preceding remarks, a basic submodule of the P-component A_P of A is a direct sum of cyclic P-modules: $\bigoplus_{k=1}^{\infty} (R/P^k)^{(\kappa(k))}$ where $\kappa(k)$ is the largest cardinal κ for which $R_{P^k}^{(\kappa)}$ is a direct summand of A. First we prove that $\kappa(k)$ is at most 1.

From Lemmas 3.1 and 3.2 we have a direct summand $R_{P^k}^{(\lambda)}$ of A where $\lambda > \kappa(k)$ if $\kappa(k) > 1$. Hence $\kappa(k) \leq 1$.

Now we prove that the basic subgroup of A_P is cyclic. Otherwise $R_{P^k} \oplus R_{P^l}$ would be a direct summand of A for some $k < l$. Then

$$\operatorname{Endo}(R_{P^k} \oplus R_{P^l}) = \operatorname{Endo} R_{P^k} \oplus \operatorname{Hom}(R_{P^k}, R_{P^l}) \oplus \operatorname{Hom}(R_{P^l}, R_{P^k}) \oplus \operatorname{Endo} R_{P^l} \tag{7}$$

is a direct summand of A isomorphic to $R_{P^k}^3 \oplus R_{P^l}$, contradicting the previous paragraph. Hence the basic submodule of the P-component is cyclic.

It follows that A_P is a direct sum of a cyclic P-module (a basic submodule) and a divisible one $R_{P^\infty}^{(\kappa)}$, where κ is maximal with $R_{P^\infty}^{(\kappa)}$ being a direct summand of A. We show that $\kappa = 0$ and hence A_P is cyclic.

By a double application of Lemma 3.2,

$$\operatorname{Hom}(\operatorname{Endo} R_{P^\infty}^{(\kappa)}, R_{P^\infty}^{(\kappa)}) \sqsubseteq A. \tag{8}$$

Now $\operatorname{Endo} R_{P^\infty}^{(\kappa)}$ is torsion-free since $R_{P^\infty}^{(\kappa)}$ is divisible. Hence

$$\operatorname{Hom}(\operatorname{Endo} R_{P^\infty}^{(\kappa)}, R_{P^\infty}^{(\kappa)})$$

is divisible. Its P-divisible part is $R_{P^\infty}^{(\lambda)}$, where λ is the dimension of the P-socle:

$$\operatorname{Hom}(\operatorname{Endo} R_{P^\infty}^{(\kappa)}, R_{P^\infty}^{(\kappa)})[P] \cong \operatorname{Hom}((\operatorname{Endo} R_{P^\infty}^{(\kappa)}) \otimes R/P, R_P^{(\kappa)}) \sqsupseteq \operatorname{Endo} R_P^{(\kappa)}. \tag{9}$$

The dimension of $(\operatorname{Endo} R_{P^\infty}^{(\kappa)}) \otimes R/P$ over R/P is obviously at least κ (e.g. since the canonical projections to the coordinates are linearly independent), which justifies the last relation in (9). If $\kappa > 1$, then we have $\lambda > \kappa$ since the dimension of $\operatorname{Endo} R_P^{(\kappa)}$ is larger than κ by Lemma 3.1. But $\lambda > \kappa$ contradicts the maximality of κ. Thus $\kappa \leq 1$.

We also have to exclude the case $\kappa = 1$. So suppose that $R_{P^\infty} \sqsubseteq A$. Then $J_P \cong \operatorname{Endo} R_{P^\infty} \sqsubseteq A$. So $R_{P^\infty} \oplus J_P \sqsubseteq A$. By Lemma 3.2,

$$J_P \oplus J_P \cong \operatorname{Endo} R_{P^\infty} \oplus \operatorname{Endo} J_P \sqsubseteq A. \tag{10}$$

Hence $R_{P^\infty}^2 \sqsubseteq \operatorname{Hom}(J_P^2, R_{P^\infty}) \sqsubseteq A$, which is impossible.

So every P-component is reduced with cyclic basic submodule; hence all of them are cyclic. □

Now we are ready to prove the second part of the main structure theorem.

Proof of Theorem 1.1(ii). If $P \in S$, then the P-component R/P^{n_P} of A is cyclic and pure in A. Hence it is a direct summand:

$$A = R/P^{n_P} \oplus B_P. \tag{11}$$

Thus we have

$$A \cong \operatorname{Endo} A = \operatorname{Endo} R/P^{n_P} \oplus \operatorname{Hom}(R/P^{n_P}, B_P) \oplus \operatorname{Hom}(B_P, R/P^{n_P}) \oplus \operatorname{Endo} B_P. \tag{12}$$

The first three summands are all P-modules. But $\operatorname{Endo} R/P^{n_P} \cong R/P^{n_P}$, so this is the whole P-component of A.

In particular, we must have $\operatorname{Hom}(B_P, R/P^{n_P}) = 0$ so B_P is P-divisible.

It follows that B_P is the largest P-divisible submodule of A and hence is an ideal of A. Also, R/P^{n_P} is an ideal of A, being the P-torsion part of A. So $A = R/P^{n_P} \times B_P$ as R-algebras and this gives rise to an algebra epimorphism $\varphi_P \colon A \to R/P^{n_P}$.

Combining the algebra homomorphisms φ_P together for all $P \in S$, we obtain a homomorphism

$$\varphi \colon A \to \prod_{P \in S} R/P^{n_P}. \tag{13}$$

Obviously, φ restricted to the torsion part T of A is just the canonical embedding of $T = \bigoplus_{P \in S} R/P^{n_P}$ into $\prod_{P \in S} R/P^{n_P}$.

Let $D := \bigcap_{P \in S} B_P$. This is the kernel of φ. Since $\prod_{P \in S} R/P^{n_P}$ is S-reduced, every S-divisible submodule of A must be contained in D. We prove that D is S-divisible and so it is the largest S-divisible submodule of A.

If $P \in S$, then $B_P/D = \prod_{Q \in S \setminus \{P\}} R/Q^{n_Q}$ is P-divisible. In other words, $B_P = D + PB_P$ and hence the localization of P at D is B_P, and so P-divisible. Hence D itself is P-divisible. All in all, D is P-divisible for all $P \in S$ and so D is S-divisible.

So far we have proved (2) except that A/D is S-pure in $\prod_{P \in S} R/P^{n_P} = \Pi$. Since T is S-pure in Π and Π/T is S-divisible, this is equivalent to saying that $(A/D)/T$ is S-divisible. But already A/T is S-divisible as can be easily seen: for $P \in S$, the factor A/T is P-divisible, being a factor of the P-divisible B_P by (11). □

3.2. **Non-reduced $E(R)$-algebras.** We prove Theorem 1.3. The methods are similar to the proof of Theorem 1.1(i).

Proof of Theorem 1.3. Let A be a non-reduced $E(R)$-module. Its torsion part must be reduced by Theorem 1.1(i). It follows that the divisible part must be $Q^{(\kappa)}$ for some $\kappa \geq 1$. By Lemma 3.1, if $\kappa > 1$, then $Q^{(\lambda)}$ is a direct summand of A for some $\lambda > \kappa$. This is impossible, and therefore the divisible part of A is Q. So A splits as $A = Q \oplus C$ for some reduced module C. We have

$$A \cong \operatorname{Endo} A = \operatorname{Endo} Q \oplus \operatorname{Hom}(Q, C) \oplus \operatorname{Hom}(C, Q) \oplus \operatorname{Endo} C. \tag{14}$$

The first three summands are divisible and $\operatorname{Endo} Q \cong Q$. Since the divisible part of A is only Q, the second and third summands must be zero. In particular, $\operatorname{Hom}(C, Q) = 0$ means C is torsion, so C is the torsion part of A. Hence the torsion part C of A is a direct summand, so C is cyclic as shown in Subsection 2.2. □

4. Construction of mixed E-rings

The next lemma is our main tool for constructing mixed (generalized) $E(R)$-algebras out of torsion-free ones:

Lemma 4.1. *Let S be an infinite set of prime ideals and B an algebra such that*

$$T := \bigoplus_{P \in S} R/P^{n_P} \subseteq_S B \subseteq_S \prod_{P \in S} R/P^{n_P} =: \Pi. \tag{15}$$

Moreover, let C be an S-divisible, torsion-free generalized $E(R)$-algebra. Fix an R-algebra isomorphism $\phi \colon C \cong \operatorname{End} C$. Let D be an ideal of C such that $C/D \cong B/T$. Suppose that D is invariant under endomorphisms and that the induced map $C \cong \operatorname{End} C \to \operatorname{End} C/D$ maps every $c \in C$ to the multiplication by the residue class of c. (Note that $C/D \cong B/T$ is commutative.)

Then the pull-back algebra A defined by the next diagram is a generalized $E(R)$-algebra.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & B/T \cong C/D \end{tikzcd} \tag{16}$$

A is commutative exactly if C is commutative.

Proof. Note that the pull-back diagram (16) extends to a diagram of exact sequences of R-modules:

$$\begin{tikzcd} & 0 \arrow[r] \arrow[d] & D \arrow[r, equal] \arrow[d] & D \arrow[d] & \\ 0 \arrow[r] & T \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & T \arrow[r] & B \arrow[r] \arrow[d] & B/T \cong C/D \arrow[r] \arrow[d] & 0 \\ & & 0 & 0 & \end{tikzcd} \tag{17}$$

Therefore we can regard T and D as R-submodules of A; they are even ideals. Since $B \cong A/D$ is S-reduced and D is S-divisible (as an ideal of the S-divisible algebra C), D is the largest S-divisible submodule of A. Similarly, since T is torsion and $C \cong A/T$ is torsion-free, T is the torsion part of A.

We can look at diagram (16) as a pull-back diagram of algebras of endomorphisms: B resp. B/T act on themselves via multiplication and C acts on itself via ϕ. The conditions of the lemma guarantee the compatibility of these actions with the maps of (16). Hence, by pull-back, A is isomorphic to the subalgebra of $\operatorname{End} A$ consisting of those endomorphisms which induce an endomorphism on both B and C i.e. under which D and T are invariant. But this subalgebra is clearly the whole $\operatorname{End} A$ since every endomorphism of A must leave the torsion part T and the largest S-divisible submodule D invariant. Thus A is a generalized $E(R)$-algebra. It is obvious from the pull-back diagram (16) that A is commutative exactly if C is commutative. □

Proof of Theorem 1.2. We are going to apply Lemma 4.1 to obtain the required algebras.

By the *localization* of an R-module M at a set W of some prime ideals we mean the R_W-module $M_W := M \otimes_R R_W$. Note that M_W is an R_W-algebra if M is an R-algebra.

Let P be any prime ideal not in S. Choose any algebra B_0 as in (15), which is P-reduced and $E := B_0/T$ is P-cotorsion-free. (Take e.g. the B_0 determined by $B_0/T = R_S$.)

Since E is P-cotorsion-free, there is a set X, larger than any given cardinal, and an $E(R)$-algebra C such that

$$E[X] \subseteq C \subseteq E[X]_P = E_P[X], \tag{18}$$

where $E[X]$ denotes the polynomial R-algebra over E in the set of variables X. (Such a C can be constructed in the following way. Let us start with a polynomial algebra $C_0 := E[X_0]$. In the P-adic completion of C_0, we choose an appropriate set X_1 of independent elements over C_0. Let $X := X_0 \cup X_1$ and C be the purification of $C_0[X_1] = E[X]$ in the P-adic completion of C_0. See [1, 3, 4] for more details.)

We view the elements of C as polynomials in $E_P[X]$. Let D be the ideal of C consisting of polynomials with zero constant term, i.e. the kernel of the algebra homomorphism which substitutes 0 for every variable. Thus $E \subseteq C/D \subseteq E_P$. Since Π/T is a divisible torsion-free algebra, we can identify E_P, and hence C/D, as a subalgebra B/T of Π/T for some subalgebra B of Π.

Now let us apply Lemma 4.1 for the data B, T, C and D. The result is an $E(R)$-algebra A such that $A/D \cong B$ and $A/T \cong C$. We prove that D is not a direct summand of A. If it were a direct summand, then the projection onto D would be a multiplication by some $p \in A$ with $p^2 = p$ (since A is an $E(R)$-algebra). Since the image of multiplication by p is D, we have $p = p \cdot 1 \in D$. So, by $D \subseteq E[X]_P$, the element p would be a non-zero idempotent polynomial with zero constant term. But no such polynomial exists. □

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