

## A PURE SUBALGEBRA OF A FINITELY GENERATED ALGEBRA IS FINITELY GENERATED

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**ABSTRACT.** We prove the following. Let  $R$  be a Noetherian commutative ring,  $B$  a finitely generated  $R$ -algebra, and  $A$  a pure  $R$ -subalgebra of  $B$ . Then  $A$  is finitely generated over  $R$ .

In this paper, all rings are commutative. Let  $A$  be a ring and  $B$  an  $A$ -algebra. We say that  $A \rightarrow B$  is pure, or  $A$  is a pure subring of  $B$ , if for any  $A$ -module  $M$ , the map  $M = M \otimes_A A \rightarrow M \otimes_A B$  is injective. Considering the case  $M = A/I$ , where  $I$  is an ideal of  $A$ , we immediately have that  $IB \cap A = I$ .

There have been a number of cases where it has been shown that if  $B$  has a good property and  $A$  is a pure subring of  $B$ , then  $A$  has a good property. If  $B$  is a regular Noetherian ring containing a field, then  $A$  is Cohen-Macaulay [5], [4]. If  $k$  is a field of characteristic zero,  $A$  and  $B$  are essentially of finite type over  $k$ , and  $B$  has at most rational singularities, then  $A$  has at most rational singularities [1].

In this paper, we prove the following.

**Theorem 1.** *Let  $R$  be a Noetherian ring,  $B$  a finitely generated  $R$ -algebra, and  $A$  a pure  $R$ -subalgebra of  $B$ . Then  $A$  is finitely generated over  $R$ .*

The case that  $B$  is  $A$ -flat is proved in [3, Corollary 2.6]. This theorem is on the same line as the finite generation results in [3].

To prove the theorem, we need the following, which is a special case of a theorem of Raynaud-Gruson [7], [8].

**Theorem 2.** *Let  $A \rightarrow B$  be a homomorphism of Noetherian rings, and  $\varphi: X \rightarrow Y$  the associated morphism of affine schemes. Let  $U \subset Y$  be an open subset, and assume that  $\varphi: \varphi^{-1}(U) \rightarrow U$  is flat. Then there exists some ideal  $I$  of  $A$  such that  $V(I) \cap U = \emptyset$ , and such that the morphism  $\Phi: \text{Proj } R_B(BI) \rightarrow \text{Proj } R_A(I)$ , determined by the associated morphism of the Rees algebras  $R_A(I) := A[tI] \rightarrow R_B(BI) := B[tBI]$ , is flat.*

The morphism  $\Phi$  in the theorem is called a flattening of  $\varphi$ .

*Proof of Theorem 1.* Note that for any  $A$ -algebra  $A'$ , the homomorphism  $A' \rightarrow B \otimes_A A'$  is pure.

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Since  $B$  is finitely generated over  $R$ , it is Noetherian. Since  $A$  is a pure subring of  $B$ ,  $A$  is also Noetherian. So if  $A_{\text{red}}$  is finitely generated, then so is  $A$ . Replacing  $A$  by  $A_{\text{red}}$  and  $B$  by  $B \otimes_A A_{\text{red}}$ , we may assume that  $A$  is reduced.

Since  $A \rightarrow \prod_{P \in \text{Min}(A)} A/P$  is finite and injective, it suffices to prove that each  $A/P$  is finitely generated for  $P \in \text{Min}(A)$ , where  $\text{Min}(A)$  denotes the set of minimal primes of  $A$ . By the base change, we may assume that  $A$  is a domain.

There exists some minimal prime  $P$  of  $B$  such that  $P \cap A = 0$ . Assume the contrary. Then take  $a_P \in P \cap A \setminus \{0\}$  for each  $P \in \text{Min}(B)$ . Then  $\prod_P a_P$  must be nilpotent, which contradicts our assumption that  $A$  is a domain.

So by [6, (2.11) and (2.20)],  $A$  is a finitely generated  $R$ -algebra if and only if  $A_{\mathfrak{p}}$  is a finitely generated  $R_{\mathfrak{p}}$ -algebra for each  $\mathfrak{p} \in \text{Spec } R$ . So we may assume that  $R$  is a local ring.

By the descent argument [2, (2.7.1)],  $\hat{R} \otimes_R A$  is a finitely generated  $\hat{R}$ -algebra if and only if  $A$  is a finitely generated  $R$ -algebra, where  $\hat{R}$  is the completion of  $R$ . So we may assume that  $R$  is a complete local ring. We may lose the assumption that  $A$  is a domain (even if  $A$  is a domain,  $\hat{R} \otimes_R A$  may not be a domain). However, doing the same reduction argument as above if necessary, we may still assume that  $A$  is a domain.

Let  $\varphi: X \rightarrow Y$  be a morphism of affine schemes associated with the map  $A \rightarrow B$ . Note that  $\varphi$  is a morphism of finite type between Noetherian schemes. We denote the flat locus of  $\varphi$  by  $\text{Flat}(\varphi)$ . Then  $\varphi(X \setminus \overline{\text{Flat}(\varphi)})$  is a constructible set of  $Y$  not containing the generic point. So  $U = Y \setminus \overline{\varphi(X \setminus \text{Flat}(\varphi))}$  is a dense open subset of  $Y$ , and  $\varphi: \varphi^{-1}(U) \rightarrow U$  is flat. By Theorem 2, there exists some nonzero ideal  $I$  of  $A$  such that  $\Phi: \text{Proj } R_B(BI) \rightarrow \text{Proj } R_A(I)$  is flat.

If  $J$  is a homogeneous ideal of  $R_A(I)$ , then  $J$  can be expressed as  $J = \bigoplus_{n \geq 0} J_n t^n$  ( $J_n \subset I^n$ ). Since  $A$  is a pure subalgebra of  $B$ , we have  $J_n B \cap I^n = J_n$  for each  $n$ . Since  $JR_B(BI) = \bigoplus_{n \geq 0} (J_n B) t^n$ , we have that  $JR_B(BI) \cap R_A(I) = J$ . Namely, any homogeneous ideal of  $R_A(I)$  is contracted from  $R_B(BI)$ .

Let  $P$  be a homogeneous prime ideal of  $R_A(I)$ . Then there exists some minimal prime  $Q$  of  $PR_B(BI)$  such that  $Q \cap R_A(AI) = P$ . Assume the contrary. Then for each minimal prime  $Q$  of  $PR_B(BI)$ , there exists some  $a_Q \in (Q \cap R_A(AI)) \setminus P$ . Then  $\prod a_Q \in \sqrt{PR_B(BI)} \cap R_A(AI) \setminus P$ . However, we have

$$\sqrt{PR_B(BI)} \cap R_A(I) = \sqrt{PR_B(BI) \cap R_A(I)} = \sqrt{P} = P,$$

and this is a contradiction. Hence  $\Phi: \text{Proj } R_B(BI) \rightarrow \text{Proj } R_A(I)$  is faithfully flat.

Since  $\text{Proj } R_B(BI)$  is of finite type over  $R$  and  $\Phi$  is faithfully flat, we have that  $\text{Proj } R_A(I)$  is of finite type by [3, Corollary 2.6]. Note that the blow-up  $\text{Proj } R_A(I) \rightarrow Y$  is proper surjective. Since  $R$  is excellent,  $Y$  is of finite type over  $R$  by [3, Theorem 4.2]. Namely,  $A$  is a finitely generated  $R$ -algebra.  $\square$

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