PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 11, Pages 3295–3302
S 0002-9939(05)07895-0
Article electronically published on May 4, 2005

A NEW RESULT FOR HYPERGEOMETRIC POLYNOMIALS

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(Communicated by Carmen C. Chicone)

Abstract. In some recent investigations involving differential operators for generalized Laguerre polynomials, Herman Bavinck (1996) encountered and proved a certain summation formula for the classical Laguerre polynomials. The main object of this sequel to Bavinck's work is to prove a generalization of this summation formula for a class of hypergeometric polynomials. The demonstration, which is presented here in the general case, differs markedly from the earlier proof given for the known special case. The general summation formula is also applied to derive the corresponding result for the classical Jacobi polynomials.

1. Introduction and motivation

Recently, Bavinck [1] made use of the differential operator in order to prove the following summation formula for the classical Laguerre polynomials $L_n^{(\alpha)}(x)$:

$$\sum_{k=j}^{l} k^m \; L_{k-j}^{(\alpha+j)}(x)\, L_{l-k}^{(-\alpha-l-1)}(-x) = (-x)^m \;\; \delta_{l,j+2m} \tag{1}$$

$$\left(x,\alpha\in\mathbb{C};\;\; j,l,m\in\mathbb{N}_0 := \{0,1,2,\ldots\}\,;\;\; l \geqq j+2m\right),$$

where $\delta_{m,n}$ denotes the Kronecker symbol and

$$L_k^{(\alpha)}(x) := \sum_{j=0}^{k} \binom{\alpha+k}{k-j} \frac{(-x)^j}{j!} = \binom{\alpha+k}{k} {}_1F_1(-k;\alpha+1;x) \tag{2}$$

in terms of the confluent hypergeometric ${}_1F_1$ function. As already remarked by Bavinck [1, p. L279], the relationship (1) was encountered in connection with certain differential operators for generalized Laguerre polynomials.

The main object of this sequel to Bavinck's work [1] is to present an interesting generalization of the summation formula (1) to hold true for the classical hypergeometric polynomials ${}_2F_1(-k,\beta;\alpha;x)$ of degree k in x, defined by (*cf.*, *e.g.*, [3, p. 334 *et seq.*])

$${}_2F_1(-k,\beta;\alpha;x) := \sum_{j=0}^{k} \frac{(-k)_j\,(\beta)_j}{(\alpha)_j} \;\frac{x^j}{j!}, \tag{3}$$

Received by the editors February 26, 2004 and, in revised form, June 15, 2004.
2000 *Mathematics Subject Classification.* Primary 33C05, 33C45; Secondary 11B73.
Key words and phrases. Laguerre polynomials, generating functions, hypergeometric polynomials, Stirling numbers of the second kind, Jacobi polynomials, summation formula.

so that, obviously,

$$L_k^{(\alpha)}(x) = \binom{\alpha+k}{k} \lim_{\mu\to\infty} \left\{ {}_2F_1\left(-k, \beta+\mu; \alpha+1; \frac{x}{\mu}\right)\right\}, \tag{4}$$

for all β independent of μ.

2. Main results

For the hypergeometric polynomials $\mathcal{S}_k^{(\alpha,\beta)}(x)$ defined by [*cf.* Definition (3)]

$$\mathcal{S}_k^{(\alpha,\beta)}(x) := \binom{\alpha+k-1}{k} {}_2F_1(-k,\beta;\alpha;x) \qquad (x,\alpha,\beta\in\mathbb{C}), \tag{5}$$

the following linear generating function is well known (see, *e.g.*, [3, p. 293, Equation 5.2 (12)]):

$$\begin{aligned}\sum_{k=0}^{\infty} \mathcal{S}_k^{(\alpha,\beta)}(x)\, t^k &= (1-t)^{-\alpha}\left(1+\frac{xt}{1-t}\right)^{-\beta} \\ &= (1-t)^{\beta-\alpha}\left[1-(1-x)\,t\right]^{-\beta} \\ &\left(|t| < \min\left\{1, |1-x|^{-1}\right\}\right).\end{aligned} \tag{6}$$

Now, in light of the Cauchy Integral Formula, we find from (6) that

$$\mathcal{S}_k^{(\alpha,\beta)}(x) = \frac{1}{2\pi i}\oint_{\mathcal{C}_\varepsilon} \frac{(1-z)^{\beta-\alpha}\left[1-(1-x)\,z\right]^{-\beta}}{z^{k+1}}\, dz, \tag{7}$$

where the closed contour

$$\mathcal{C}_\varepsilon \qquad \left(0<\varepsilon<\min\left\{1, |1-x|^{-1}\right\}\right)$$

in the complex z-plane is a circle of radius ε (centred at $z=0$), which is described in the positive (counter-clockwise) direction.

Making use of (7), it is easily observed that

$$\begin{aligned}&\sum_{k=0}^{\infty}\binom{m+k}{k}\mathcal{S}_{m+k}^{(\alpha,\beta)}(x)\, t^k \\ &= \frac{1}{2\pi i}\oint_{\mathcal{C}_\varepsilon} \frac{(1-z)^{\beta-\alpha}\left[1-(1-x)\,z\right]^{-\beta}}{(z-t)^{m+1}}\, dz \\ &= \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{(1-z)^{\beta-\alpha}\left[1-(1-x)\,z\right]^{-\beta}}{(z-t)^{m+1}}\, dz \qquad (|t|<\varepsilon;\ m\in\mathbb{N}_0),\end{aligned} \tag{8}$$

where $\mathcal{C}$ is a circle (centred at $z=t$) lying interior to the circle $\mathcal{C}_\varepsilon$. Thus, by setting

$$1-z=(1-t)(1-\zeta) \qquad \text{and} \qquad dz=(1-t)\,d\zeta,$$

we can rewrite the last contour integral in (8) as follows:

$$\begin{aligned}\sum_{k=0}^{\infty}\binom{m+k}{k}\mathcal{S}_{m+k}^{(\alpha,\beta)}(x)\, t^k &= (1-t)^{\beta-\alpha-m}\left[1-(1-x)\,t\right]^{-\beta} \\ &\cdot \frac{1}{2\pi i}\oint_{\mathcal{C}^*} \frac{(1-\zeta)^{\beta-\alpha}}{\zeta^{m+1}}\left[1-\left(1-\frac{x}{1-(1-x)\,t}\right)\zeta\right]^{-\beta} d\zeta,\end{aligned} \tag{9}$$

where the closed contour $\mathcal{C}^*$ in the complex ζ-plane is a circle (centred at $\zeta=0$) of sufficiently small radius.

Finally, upon comparing (9) with (7), we obtain the following *extended* generating function for the hypergeometric polynomials $\mathcal{S}_k^{(\alpha,\beta)}(x)$ defined by (5):

$$\begin{aligned}(10)\qquad &\sum_{k=0}^{\infty}\binom{m+k}{k}\mathcal{S}_{m+k}^{(\alpha,\beta)}(x)\,t^k \\ &= (1-t)^{\beta-\alpha-m}\left[1-(1-x)\,t\right]^{-\beta}\ \mathcal{S}_m^{(\alpha,\beta)}\left(\frac{x}{1-(1-x)\,t}\right) \\ &\left(|t|<\min\left\{1,|1-x|^{-1}\right\};\ m\in\mathbb{N}_0\right),\end{aligned}$$

where we have enlarged the region of validity by appealing to the principle of analytic continuation on t.

Next we recall (as Theorem 1 below) some general results of Srivastava [2] on generating functions associated with the Stirling numbers $S(n,k)$ of the second kind, defined by

$$(11)\qquad S(n,k):=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n,$$

so that

$$(12)\qquad S(n,0)=\delta_{n,0}=\begin{cases}1 & (n=0)\\ 0 & (n\in\mathbb{N}:=\mathbb{N}_0\setminus\{0\})\end{cases}$$

and

$$(13)\qquad S(n,1)=S(n,n)=1\qquad\text{and}\qquad S(n,n-1)=\binom{n}{2}.$$

Theorem 1 (Srivastava [2, p. 754, Theorem 1]). *Let the sequence* $\{\mathbb{S}_n(x)\}_{n=0}^{\infty}$ *be generated by*

$$(14)\quad \sum_{k=0}^{\infty}\binom{m+k}{k}\mathbb{S}_{m+k}(x)\,t^k=f(x,t)\left[g(x,t)\right]^{-m}\ \mathbb{S}_m\big(h(x,t)\big)\qquad(m\in\mathbb{N}_0),$$

where f, g *and* h *are suitable functions of* x *and* t. *Then, in terms of the Stirling numbers* $S(n,k)$ *defined by* (11), *the following family of generating functions holds true*:

$$\begin{aligned}(15)\qquad &\sum_{k=0}^{\infty}k^m\ \mathbb{S}_k\big(h(x,-z)\big)\left(\frac{z}{g(x,-z)}\right)^k \\ &= \left[f(x,-z)\right]^{-1}\sum_{k=0}^{m}k!\ S(m,k)\,\mathbb{S}_k(x)\,z^k\qquad(m\in\mathbb{N}_0),\end{aligned}$$

provided that each member of (15) *exists.*

The generating function (10) obviously belongs to the family given by (14). Indeed, by comparing (10) with (14), it is easily observed that

$$\begin{aligned}f(x,t)&=(1-t)^{\beta-\alpha}\left[1-(1-x)\,t\right]^{-\beta},\\ g(x,t)&=1-t,\\ h(x,t)&=\frac{x}{1-(1-x)\,t},\end{aligned}$$

and

$$\mathbb{S}_k(x) \longmapsto \mathcal{S}_k^{(\alpha,\beta)}(x) \qquad (k \in \mathbb{N}_0).$$

Thus the assertion (15) of Theorem 1 yields the following generating function involving the Stirling numbers $S(n,k)$ defined by (11):

$$\sum_{k=0}^{\infty} k^m \, \mathcal{S}_k^{(\alpha,\beta)}\left(\frac{x}{1+(1-x)z}\right)\left(\frac{z}{1+z}\right)^k = (1+z)^{\alpha-\beta}\left[1+(1-x)z\right]^{\beta} \sum_{k=0}^{m} k! \, S(m,k)\, \mathcal{S}_k^{(\alpha,\beta)}(x)\, z^k \tag{16}$$

$$\left(m \in \mathbb{N}_0;\ \ |z| < \min\left\{1, |1-x|^{-1}\right\}\right),$$

which, for

$$z \longmapsto \frac{z}{1-z} \qquad \text{and} \qquad x \longmapsto \frac{x}{1-(1-x)z},$$

assumes the following form:

$$\sum_{k=0}^{\infty} k^m \, \mathcal{S}_k^{(\alpha,\beta)}(x)\, z^k = (1-z)^{\beta-\alpha}\left[1-(1-x)z\right]^{-\beta} \cdot \sum_{k=0}^{m} k! \, S(m,k)\, \mathcal{S}_k^{(\alpha,\beta)}\left(\frac{x}{1-(1-x)z}\right)\left(\frac{z}{1-z}\right)^k \tag{17}$$

$$\left(m \in \mathbb{N}_0;\ \ |z| < \min\left\{1, |1-x|^{-1}\right\}\right).$$

Our main summation formula involving the hypergeometric polynomials $\mathcal{S}_k^{(\alpha,\beta)}(x)$ is given by Theorem 2 below.

Theorem 2. *Let the hypergeometric polynomials $\mathcal{S}_k^{(\alpha,\beta)}(x)$ be defined by* (5). *Suppose also that $\mathcal{P}_m(x)$ is a polynomial of degree m in x. Then*

$$\sum_{k=j}^{l} \mathcal{P}_m(k)\, \mathcal{S}_{k-j}^{(\alpha,\beta)}(x)\, \mathcal{S}_{l-k}^{(-\alpha-n+2,-\beta-r+1)}(x) = \frac{\mathcal{P}_m^{(m)}(0)}{m!}\Big[(\alpha-\beta)_m\, x^{r-1}\delta_{n,m+r} + (\beta)_m\,(-x)^{n-r-1}(1-x)^{l-j+m-n+1}\,\delta_{r,m}\Big] \tag{18}$$

$$(j,l,m,n,r \in \mathbb{N}_0;\ l-j+1 \geqq n \geqq m+r;\ r \geqq m).$$

Proof. First of all, by applying the generating functions (6) [or (10) *with* $m=0$] and (17), we have

$$\left(\sum_{k=0}^{\infty} k^m \, \mathcal{S}_k^{(\alpha,\beta)}(x)\, z^k\right)\left(\sum_{k=0}^{\infty} \mathcal{S}_k^{(-\alpha-n+2,-\beta-r+1)}(x)\, z^k\right) = \sum_{k=0}^{m} Q_k(x,z) = \sum_{p=0}^{\infty} z^p \left(\sum_{k=0}^{p} k^m \, \mathcal{S}_k^{(\alpha,\beta)}(x)\, \mathcal{S}_{p-k}^{(-\alpha-n+2,-\beta-r+1)}(x)\right), \tag{19}$$

where, for convenience,

$$Q_k(x,z) := k!\ S(m,k)\, z^k (1-z)^{n-k-r-1} [1-(1-x)z]^{r-1} \cdot \mathcal{S}_k^{(\alpha,\beta)}\left(\frac{x}{1-(1-x)z}\right). \tag{20}$$

Observe that, if

$$0 \leqq k < m, \qquad r \geqq m, \qquad \text{and} \qquad n \geqq m+r,$$

then $Q_k(x,z)$ is a polynomial in z of degree at most $n-2$. For $p \geqq n-1$, it follows from (19) and (20) that

$$\begin{aligned}
\sum_{k=0}^{p} k^m\ \mathcal{S}_k^{(\alpha,\beta)}(x)\, \mathcal{S}_{p-k}^{(-\alpha-n+2,-\beta-r+1)}(x)
&= \frac{1}{2\pi i} \oint_{\mathcal{C}_\varepsilon} \frac{\sum_{k=0}^m Q_k(x,z)}{z^{p+1}}\, dz \\
&= \frac{m!}{2\pi i} \oint_{\mathcal{C}_\varepsilon} \frac{(1-z)^{n-m-r-1}[1-(1-x)z]^{r-1}}{z^{p-m+1}} \\
&\qquad \cdot \mathcal{S}_m^{(\alpha,\beta)}\left(\frac{x}{1-(1-x)z}\right) dz \\
&\qquad \left(0 < \varepsilon < \min\left\{1, |1-x|^{-1}\right\}\right),
\end{aligned} \tag{21}$$

where $\mathcal{C}_\varepsilon$ denotes the closed contour used earlier in (7).

After a change of variable given by

$$z = \frac{1}{\zeta} \qquad \text{and} \qquad dz = -\frac{1}{\zeta^2}\, d\zeta,$$

(21) readily yields

$$\begin{aligned}
\sum_{k=0}^{p} k^m\ \mathcal{S}_k^{(\alpha,\beta)}(x)\, \mathcal{S}_{p-k}^{(-\alpha-n+2,-\beta-r+1)}(x)
&= \frac{m!}{2\pi i} \oint_{\mathcal{C}^*_{1/\varepsilon}} \frac{\zeta^{p-n+1}(\zeta-1)^{n-m-r}[\zeta-(1-x)]^r}{(\zeta-1)[\zeta-(1-x)]} \\
&\qquad \cdot \mathcal{S}_m^{(\alpha,\beta)}\left(\frac{\zeta x}{\zeta-(1-x)}\right) d\zeta,
\end{aligned} \tag{22}$$

where the closed contour $\mathcal{C}^*_{1/\varepsilon}$ in the complex ζ-plane is a circle (centred at $\zeta = 0$) of radius $1/\varepsilon$.

Since

$$r \geqq m, \qquad n \geqq m+r, \qquad \text{and} \qquad p \geqq n-1,$$

the integrand in (22) *without* the clearly-exhibited denominator

$$(\zeta-1)[\zeta-(1-x)]$$

is a polynomial in ζ. Also, by the Chu-Vandermonde theorem [3, p. 30, Equation 1.2 (8)], we have

$$\mathcal{S}_m^{(\alpha,\beta)}(1) = \frac{(\alpha)_m}{m!} \cdot \frac{(\alpha-\beta)_m}{(\alpha)_m} = \frac{(\alpha-\beta)_m}{m!} \qquad (m \in \mathbb{N}_0). \tag{23}$$

Furthermore, for $r \geqq m$, it is easily seen that

$$\lim_{\zeta \to 1-x} \left\{ [\zeta - (1-x)]^r \; \mathcal{S}_m^{(\alpha,\beta)} \left(\frac{\zeta x}{\zeta - (1-x)} \right) \right\} \tag{24}$$

$$= \lim_{\zeta \to 1-x} \left\{ [\zeta - (1-x)]^r \; \frac{(\alpha)_m}{m!} \cdot \frac{(-m)_m \, (\beta)_m}{m! \, (\alpha)_m} \left(\frac{\zeta x}{\zeta - (1-x)} \right)^m \right\}$$

$$= \frac{(\beta)_m}{m!} (-x)^m (1-x)^m \;\; \delta_{r,m} \qquad (r \geqq m; \;\; m \in \mathbb{N}_0).$$

Thus, by the Cauchy Residue Theorem, we conclude that

$$\sum_{k=0}^{p} k^m \; \mathcal{S}_k^{(\alpha,\beta)}(x) \, \mathcal{S}_{p-k}^{(-\alpha-n+2,-\beta-r+1)}(x) \tag{25}$$

$$= (\alpha-\beta)_m \;\; x^{r-1} \;\; \delta_{n,m+r} + (\beta)_m \, (-x)^{n-r-1} \, (1-x)^{p+m-n+1} \;\; \delta_{r,m}$$

$$(m,n,p,r \in \mathbb{N}_0; \;\; p+1 \geqq n \geqq m+r; \;\; r \geqq m).$$

In particular, if $s < m$, (25) yields

$$\sum_{k=0}^{p} k^s \; \mathcal{S}_k^{(\alpha,\beta)}(x) \, \mathcal{S}_{p-k}^{(-\alpha-n+2,-\beta-r+1)}(x) = 0 \tag{26}$$

$$(m,n,p,r,s \in \mathbb{N}_0; \;\; p+1 \geqq n \geqq m+r; \;\; s < m \leqq r)$$

or, equivalently,

$$\sum_{k=j}^{l} (k-j)^s \;\; \mathcal{S}_{k-j}^{(\alpha,\beta)}(x) \, \mathcal{S}_{l-k}^{(-\alpha-n+2,-\beta-r+1)}(x) = 0 \tag{27}$$

$$(j,l,m,n,r,s \in \mathbb{N}_0; \;\; l-j+1 \geqq n \geqq m+r; \;\; s < m \leqq r).$$

Finally, the assertion (18) of Theorem 2 follows from (25) and (27), since every polynomial $\mathcal{P}_m(x)$ of degree m in x can be expressed as a linear combination of

$$(x-j)^m, (x-j)^{m-1}, \ldots, (x-j), 1.$$

□

For the classical Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$ of order (or indices) α, β and degree k in x, defined by

$$P_k^{(\alpha,\beta)}(x) := \sum_{j=0}^{k} \binom{\alpha+k}{k-j} \binom{\beta+k}{j} \left(\frac{x-1}{2} \right)^j \left(\frac{x+1}{2} \right)^{k-j} \tag{28}$$

$$= \binom{\alpha+k}{k} \, {}_2F_1 \left(-k, \alpha+\beta+k+1; \alpha+1; \frac{1-x}{2} \right),$$

it is easily seen from the definition (5) that

$$P_k^{(\alpha,\beta-k)}(x) = \mathcal{S}_k^{(\alpha+1,\alpha+\beta+1)} \left(\frac{1-x}{2} \right) \tag{29}$$

or, equivalently,

$$\mathcal{S}_k^{(\alpha,\beta)}(x) = P_k^{(\alpha-1,\beta-\alpha-k)}(1-2x). \tag{30}$$

Thus, by setting

$$\alpha \longmapsto \alpha + 1, \qquad \beta \longmapsto \alpha + \beta + 1, \qquad \text{and} \qquad x \longmapsto \frac{1-x}{2},$$

Theorem 2 yields the following result for the classical Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$.

Theorem 3. *For every polynomial $\mathcal{P}_m(x)$ of degree m in x,*

$$\sum_{k=j}^{l} \mathcal{P}_m(k)\, P_{k-j}^{(\alpha,\beta-k+j)}(x)\, P_{l-k}^{(-\alpha-n,-\beta+n-r+k-l-1)}(x) \tag{31}$$

$$= \frac{\mathcal{P}_m^{(m)}(0)}{m!}\left[\Big(-\beta\Big)_m \left(\frac{1-x}{2}\right)^{r-1} \delta_{n,m+r} + (\alpha+\beta+1)_m \cdot \left(\frac{x-1}{2}\right)^{n-r-1} \left(\frac{x+1}{2}\right)^{l-j+m-n+1} \delta_{r,m}\right]$$

$$(j,l,m,n,r \in \mathbb{N}_0;\ l-j+1 \geqq n \geqq m+r;\ r \geqq m).$$

3. Remarks and observations

Upon setting

$$\alpha \longmapsto \alpha + j + 1, \quad n = l - j + 1, \quad x \longmapsto \frac{x}{\beta}, \quad r = m + 1, \quad \text{and} \quad \mathcal{P}_m(k) = k^m$$

and using the limit relationship (4), the assertion (18) of Theorem 2 for $\beta \to \infty$ would reduce at once to Bavinck's result (1). Furthermore, upon setting

$$\alpha \longmapsto \alpha + 1, \qquad r = m, \qquad n = 2m + 1, \qquad \text{and} \qquad x \longmapsto \frac{x}{\beta}$$

and using the limit relationship (4) once again, the assertion (18) of Theorem 2 for $\beta \to \infty$ would immediately yield the following summation formula:

$$\sum_{k=j}^{l} \mathcal{P}_m(k)\, L_{k-j}^{(\alpha)}(x)\, L_{l-k}^{(-\alpha-2m-1)}(-x) = \frac{\mathcal{P}_m^{(m)}(0)}{m!}(-x)^m \tag{32}$$

$$(j,l,m \in \mathbb{N}_0;\ l \geqq j + 2m)$$

for every polynomial $\mathcal{P}_m(x)$ of degree m in x.

By applying the familiar limit relationship [3, p. 131, Equation 2.5 (1)]:

$$L_k^{(\alpha)}(x) = \lim_{\beta\to\infty}\left\{P_k^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right)\right\} \tag{33}$$

instead of (4), the assertion (31) of Theorem 3 can also be applied in order to deduce (1) as well as (32) in an analogous manner.

Acknowledgements

The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

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