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ALGEBRAS OF OPERATORS AND CLOSED RANGE

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Abstract. We consider operators T such that every operator in the norm closed algebra generated by T has closed range. Examples in a triangular AF algebra are constructed.

Introduction

In this note we consider closed range properties for certain families of Hilbert space operators. Our motivation is the following question raised by Zemanek. (See [Z, p. 379].)

Question 1. *If T is a quasinilpotent operator and if T^n has closed range for all $n \in \mathbb{N}$, must T be nilpotent?*

This question arose from results of Mbekhta and Zemanek, [MZ, Theorem 2], concerning uniform convergence of operator ergodic averages. An equivalent question is

Question 2. *If $\sigma(T)$ is finite and if $p(T)$ has closed range for every polynomial p, must T be algebraic?*

L. Burlando and the author independently produced examples (unpublished) that answer these questions in the negative.[1] However, it turns out that these questions had been answered in a strong way by Apostol in [A1]. (See also [A2].) Suppose that $\mathcal{H}$ is a complex Hilbert space and that $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded linear operators on $\mathcal{H}$. Apostol proved these theorems in [A1].

Theorem 3. *If $T \in \mathcal{B}(\mathcal{H})$ and if $T - \lambda$ has closed range for all $\lambda \in \mathbb{C}$, then $\sigma(T)$ is countable.*

Theorem 4. *If σ is a nonempty countable compact subset of $\mathbb{C}$, then there is an operator $T \in \mathcal{B}(\mathcal{H})$ (here $\dim \mathcal{H}$ is infinite) such that $\sigma(T) = \sigma$ and so that $p(T)$ has closed range for every polynomial p.*

Apostol's construction is subtle and intricate. However, for the case $\sigma = \{0\}$, one can check that the operator produced is not nilpotent.

In this note we prove analogs of Theorems 3 and 4. Given $T \in \mathcal{B}(\mathcal{H})$, let $\mathcal{A}(T)$ be the norm closed algebra generated by T. Thus $\mathcal{A}(T)$ is the norm closure of the

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[1] Added in proof: Burlando's example will appear in a paper, "On nilpotent operators", to appear in *Studia Math.*

polynomials in T. Clearly $\mathcal{A}(T)$ is finite dimensional precisely when T is algebraic. We will prove these theorems.

Theorem 5. *If $T \in \mathcal{B}(\mathcal{H})$ and if S has closed range for all $S \in \mathcal{A}(T)$, then $\sigma(T)$ is finite.*

Theorem 6. *If σ is a nonempty finite subset of $\mathbb{C}$, then there is a nonalgebraic operator $T \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) = \sigma$ and so that S has closed range for all $S \in \mathcal{A}(T)$.*

For the case that $\sigma = \{0\}$ in Theorem 6, we produce a countably generated noncommutative subalgebra $\mathcal{A}^0$ of a familiar triangular AF algebra such that S has closed range for all $S \in \mathcal{A}^0$.

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Results

We begin by proving Theorem 5. If T satisfies the hypotheses of that theorem, then $\sigma(T)$ is countable, by Theorem 3. If $\sigma(T)$ is also infinite, then we will construct $S \in \mathcal{A}(T)$ so that the range of S is not closed.

Every countable compact set has isolated points, so we can choose a sequence (λ_n) such that each λ_n is isolated in $\sigma(T)$. For each n, let P_n be the Riesz idempotent corresponding to $\{\lambda_n\}$ determined by the Riesz Functional Calculus (see [C, p. 210]). Then $P_n \in \mathcal{A}(T)$, $0 \neq P_n = P_n^2$, and $P_n P_m = 0$ if $m \neq n$.

Let $S = \sum_{n=1}^{\infty} \frac{P_n}{4^n \|P_n\|}$. Then $S \in \mathcal{A}(T)$. For all n, let x_n be a unit vector in $P_n\mathcal{H}$. Then

$$S\Big(\textstyle\sum_{n=1}^{N} 2^n \|P_n\| \, x_n\Big) = \textstyle\sum_{n=1}^{N} \dfrac{1}{2^n} x_n,$$

so $\sum_{n=1}^{\infty} \frac{1}{2^n} x_n$ is in the closure of the range of S. But if $y \in \mathcal{H}$ and if $Sy = \sum_{n=1}^{\infty} \frac{1}{2^n} x_n$, then

$$\frac{1}{2^n} x_n = P_n S y = \frac{P_n y}{4^n \|P_n\|},$$

so that

$$\|P_n y\| \geq 2^n \|P_n\| \quad \text{and} \quad \|y\| \geq 2^n.$$

This holds for all n, so we have a contradiction.

We proceed to the proof of Theorem 6. It will suffice to find a nonnilpotent quasinilpotent operator A such that S has closed range for all $S \in \mathcal{A}(A)$. For then, given $\sigma = \{\lambda_1, \lambda_2, \cdots, \lambda_N\} \subset \mathbb{C}$, the operator $T = \bigoplus_{n=1}^{N} (A - \lambda_n I)$ has the required properties. As noted in the introduction, we show that A can be chosen to be any nonnilpotent operator in a certain ideal $\mathcal{A}^0$ of the triangular AF algebra known as the refinement algebra.

We outline the construction of $\mathcal{A}^0$. ($\mathcal{A}^0$ was also considered in [PW, Remark 2.18] in another setting.) Let M_k denote the k by k complex matrices. For all $n \in \mathbb{N}$, $\{e_{i,j}^{(n)} : 1 \leq i.j \leq 2^n\}$ denotes the matrix units in M_{2^n}, and $\nu_n : M_{2^n} \to M_{2^{n+1}}$ denotes the refinement embedding given by

$$\nu_n\big(e_{i,j}^{(n)}\big) = e_{2i-1,2j-1}^{(n+1)} + e_{2i,2j}^{(n+1)}.$$

Let $\mathcal{T}_n$ be the upper triangular subalgebra of M_{2^n} and let $\mathcal{T}_n^0$ be the strictly upper triangular subalgebra. The inductive limit $\mathcal{T} = \varinjlim(\mathcal{T}_n, \nu_n)$ is called the refinement algebra. $\mathcal{T}$ is a norm closed subalgebra of the UHF 2^∞ C^*-algebra, or CAR algebra. (See [KR], p. 759, for example.) In what follows, we can consider $\mathcal{T}$ as faithfully represented as a subalgebra of $\mathcal{B}(\mathcal{H})$ for some $\mathcal{H}$. In [D], Donsig showed that $\mathcal{T}^0 = \varinjlim(\mathcal{T}_n^0, \nu_n)$ is the Jacobsen radical of $\mathcal{T}$. Thus $\sigma(T) = \{0\}$ if $T \in \mathcal{T}^0$.

For $n \in \mathbb{N}$, let $X_n = e^{(n)}_{2^n-1,2^n}$ and suppose that $\mathcal{A}_0$ is the norm closed algebra generated by $\{X_n\}_{n=1}^{\infty}$. Note that $\mathcal{A}^0 \subset \mathcal{T}^0$. Let $\mathcal{A} = \mathbb{C}I + \mathcal{A}^0$.

We analyze the structure of $\mathcal{A}$ in more detail. First we consider the matrix units in $\mathcal{A}$. Observe that $X_m X_n = 0$ if $m \geq n$. If $m < n$, then $X_m X_n$ is a matrix unit in M_{2^n}.

For $n \in \mathbb{N}$, let S_n be the set of all ordered subsets K of $\mathbb{N}$ such that $n = \max K$. So if $K \in S_n$, then $K = \{n\}$ or K has the form $\{k_1 < k_2 < \cdots < k_\ell < n\}$. In the latter case, let X_K denote $X_{k_1} X_{k_2} \cdots X_{k_\ell} X_n$, and let $X_{\{n\}} = X_n$. Also let $X_\emptyset = I$. Define $\theta : S_n \to \mathbb{N}$ by $\theta(\{n\}) = 2^n - 1$, and $\theta(K) = 2^n - \left(\sum_{j=1}^{\ell} 2^{n-k}\right) - 1$ if $K = \{k_1 < k_2 < \cdots < k_\ell < n\}$. A computation shows the following.

(7) If $K \in S_m$, then $X_K = e^{(n)}_{\theta(K),2^n}$. Thus $K \mapsto X_K$ sets up a 1–1 correspondence from S_m onto $\{e^{(n)}_{i,2^n} : 1 \leq i < 2^n,\ i \text{ odd}\}$.

It is now easy to check that if K_1 and K_2 are distinct finite subsets of $\mathbb{N}$, then X_{K_1} and X_{K_2} are disjoint in the following sense: For any n large enough that both X_{K_1} and X_{K_2} are in M_{2^n}, the matrix entry of X_{K_1} in M_{2^n} is 0 at each position for which the matrix entry of X_{K_2} is 1. It follows easily that each element of $\mathcal{A}$ is a series $\sum_K a_K X_K$, where K ranges over the finite subsets of $\mathbb{N}$. We show in Remark 12 that the coefficients (a_K) must be in ℓ^2.

Next we show that $\mathcal{A}$ has a fractal-like behavior; $\mathcal{A}$ replicates itself relative to each matrix unit X_J. To make this precise, fix X_J with $J \in S_n$. Then $d(X_J) = X_J^* X_J = e^{(n)}_{2^n,2^n}$ is the domain of X_J and $r(X_J) = X_J X_J^*$ is the range of X_J. Denote the compression of $\mathcal{A}$ to X_J by $\mathcal{A}_J = r(X_J)\,\mathcal{A}\,d(X_J)$. One can check that $X_K \in \mathcal{A}_J \Leftrightarrow J$ is the initial segment of K. (That is, either $K = J$ or $K = J \cup K'$ for some K' with m in $K' > n$.)

Finally, observe that the matrix units in $X_J^* \mathcal{A}_J$ have domain and range inside $D(X_J)$.

(8) $\mathcal{A}$ and $X_J^* \mathcal{A}_{J|d(X_J)}$ are isometrically isomorphic as unital operator algebras.

In fact, define $\psi : \mathcal{A} \to X_J^* \mathcal{A}_J$ by $\psi(\sum a_K X_K) = X_J^* \sum a_K X_J X_{K+\{n\}}$. Check that the matrix pictures for $A \in \mathcal{A}$ and $\psi(A)_{|d(X_J)}$ are identical.

Theorem 9. *Every $A \in \mathcal{A}$ has closed range.*

Proof. Note that if $A \in \mathcal{A}$, then $A = a_\emptyset I + A^0$ where $A^0 \in \mathcal{A}^0$. So if $a_\emptyset \neq 0$, then A is invertible. Thus it suffices to assume that $A \in \mathcal{A}^0$. Then $A = \sum a_K X_K$, where K ranges over the nonempty finite subsets of $\mathbb{N}$. We will need the following observation about $n \times n$ operator matrices with only one nontrivial column.

Lemma 10. *Suppose that $T = (T_{i,j})_{i,j=1}^n \in \mathcal{B}(\bigoplus_{i=1}^n \mathcal{H})$. Suppose that $T_{i,j} = 0$ unless $j = n$ and that $T_{i_0,n}$ is invertible for some i_0. Then T has closed range.*

Proof. $\ker T$ is the direct sum of the first $n-1$ copies of $\mathcal{H}$. Since $T_{i_0,n}$ is invertible, T is bounded below on $(\ker T)^\perp$, so T has closed range. $\square$

We apply this lemma to $A = \sum a_K X_K \in \mathcal{A}^0$. Choose the smallest n such that $a_J \neq 0$ for some $J \in S_n$. Then A has a 2^n by 2^n block matrix as in Lemma 10, and one block, $A_J = r(X_J) A\, d(X_J)$, is invertible. In fact, referring to (8), we have that

$\psi^{-1}(X_J^* A_J) = a_J I + B_0$ for some $B^0 \in \mathcal{A}^0$. $a_J \neq 0$, so A_J is invertible. Lemma 10 shows that A has closed range. □

Concluding remarks

Remark 11. To finish the proof of Theorem 6, it suffices to choose a nonnilpotent operator T in $\mathcal{A}^0$. For example, $T = \sum \frac{1}{2^n} X_n$ does the job.

Remark 12. There are many ways to represent the refinement algebra $\mathcal{T}$ as an algebra of operators. For example, let $\mathcal{H} = \ell^2(B)$ where B denotes the dyadic rationals in (0,1]. $\{f_r : r \in B\}$ is a basis for $\mathcal{H}$, where $f_r(s) = \delta_{rs}$, $s \in B$. If $e_{ij}^{(n)} \in M_{2^n}$, let

$$e_{ij}^{(n)} f_r = \begin{cases} f_{r+\frac{i-j}{2^n}}, & r \in \left(\frac{j-1}{2^n}, \frac{j}{2^n}\right], \\ 0, & \text{otherwise.} \end{cases}$$

Using (7), we see that the map $K \mapsto X_K f_1$ is 1–1 from S_n onto $\left\{ f_{\frac{i}{2^n}} : 1 \leq i < 2^n, i \text{ odd} \right\}$. This holds for all $n \in \mathbb{N}$, so the map $X_K \rightarrow X_K f_1$ carries the matrix units in $\mathcal{A}$ onto a basis for $\mathcal{H}$. Thus f_1 is a cyclic vector for $\mathcal{A}$. In addition, if $A = \sum a_K X_K \in \mathcal{A}$, then $\|Af_1\|^2 = \sum |a_K|^2$, so f_1 is a separating vector, and

$$\left\| \sum a_K X_K \right\|^2 \geq \sum |a_K|^2.$$

Denote the right-hand side of this inequality as the ℓ^2 norm of A.

It is unknown whether the operator norm and the ℓ^2 norm are equivalent on $\mathcal{A}$. The two norms agree on $A = \sum a_K X_K$ provided those X_K with $a_K \neq 0$ have pairwise orthogonal ranges. The equivalence of these norms would imply that every representation of $\mathcal{A}$ is closed in the weak operator topology.

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