

## OPEN BOOK DECOMPOSITIONS FOR CONTACT STRUCTURES ON BRIESKORN MANIFOLDS

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ABSTRACT. In this paper, we give an open book decomposition for the contact structures on some Brieskorn manifolds, in particular for the contact structures of Ustilovsky. The decomposition uses right-handed Dehn twists as conjectured by Giroux.

### 1. INTRODUCTION

At the ICM of 2002 Giroux announced some of his results concerning a correspondence between contact structures on manifolds and open book structures on them. In one direction this correspondence is relatively easy. We are given a compact Stein manifold  $P$  (i.e. a compact subset of a Stein manifold where the boundary is a level set of a plurisubharmonic function on it) and a symplectomorphism  $\psi$  of  $P$  that is the identity near the boundary of  $P$ . It can be shown that this symplectomorphism gives rise to a mapping torus that inherits a contact structure. Furthermore the boundary of the mapping torus will always look like  $S^1 \times \partial P$ , and a neighborhood of the binding,  $D^2 \times \partial P$  with the obvious contact structure, can be glued in to give a closed contact manifold.

Although Giroux announced much more than just this, it is already interesting to see how this construction turns out in a few simple cases. As a Stein manifold we will take  $T^*S^{n-1}$  with its canonical symplectic form. The symplectomorphisms used for the monodromy of the mapping torus will be so-called generalized Dehn twists. Seidel has shown [7] that these Dehn twists generate the symplectomorphism group of  $T^*S^2$  up to isotopy. Furthermore his results show that Dehn twists of  $T^*S^2$  are of order 2 diffeomorphically (relative to the boundary), but not symplectically. This means that many of these Dehn twists are isotopic to each other, but not symplectically so.

In the spirit of the above construction, we will show that the Brieskorn manifold  $W_k^{2n-1}$  (for notation and definition see Section 3) is supported by an open book whose monodromy is given by a  $k$ -fold Dehn twist. In particular this shows that the Ustilovsky spheres (special Brieskorn spheres with non-isomorphic contact structures) can all be written in terms of open book decompositions with Dehn twists as their monodromy. It also shows that Dehn twists cannot be of order 2 in all dimensions (this is well known for  $n$  even). Namely, among the Brieskorn spheres

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(these correspond to  $n$  and  $k$  odd) are exotic spheres as well as standard ones. As the binding is always glued in in the same way, the Dehn twists corresponding to a standard and an exotic sphere cannot be isotopic relative to the boundary.

## 2. NOTATION AND DEFINITIONS

**2.1. Open books.** The following definitions are taken from [3].

**Definition 2.1.** An **open book** on a closed manifold  $M$  is given by a codimension-2 submanifold  $B \hookrightarrow M$  with trivial normal bundle, and a bundle  $\vartheta : (M - B) \rightarrow S^1$ . The neighborhood of  $B$  should have a trivialization  $B \times D^2$ , where the angle coordinate on the disk agrees with the map  $\vartheta$ .

The manifold  $B$  is called the **binding** of the open book, and a fiber  $P = \vartheta^{-1}(\varphi_0)$  is called a **page**.

*Remark 2.2.* The open set  $M - B$  is a bundle over  $S^1$ , hence it is diffeomorphic to  $\mathbb{R} \times P / \sim$ , where  $\sim$  identifies  $(t, p) \sim (t + 1, \Phi(p))$  for some diffeomorphism  $\Phi$  of  $P$ .

**Definition 2.3.** A contact structure  $\xi$  on  $M$  is said to be **supported by an open book**  $(B, \vartheta)$  of  $M$ , if it admits a contact form  $\alpha$  with  $\ker \alpha = \xi$  such that

- (1)  $(B, \alpha|_{TB})$  is a contact manifold.
- (2) For every  $s \in S^1$ , the page  $P := \vartheta^{-1}(s)$  is a symplectic manifold with symplectic form  $d\alpha$ .
- (3) Denote the closure of a page  $P$  in  $M$  by  $\overline{P}$ . The orientation of  $B$  induced by its contact form  $\alpha|_{TB}$  should coincide with its orientation as the boundary of  $(\overline{P}, d\alpha)$ .

Such a contact form is said to be **adapted** to  $(B, \vartheta)$ .

**2.2. Dehn twists.** A **Dehn twist**  $\tau_k$  is a diffeomorphism from  $T^*S^{n-1}$  to itself constructed in the following way. Write points in  $T^*S^{n-1}$  as  $(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^{2n}$  with  $|\mathbf{q}| = 1$  and  $\mathbf{q} \perp \mathbf{p}$ .

Set

$$\tau_k(\mathbf{q}, \mathbf{p}) = \begin{pmatrix} \cos g_k(\mathbf{p}) & |\mathbf{p}|^{-1} \sin g_k(\mathbf{p}) \\ -|\mathbf{p}| \sin g_k(\mathbf{p}) & \cos g_k(\mathbf{p}) \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix}.$$

Here  $g_k(\mathbf{p}) = \pi k + f_k(|\mathbf{p}|)$  and  $f_k$  is a smooth function that increases monotonically from 0 to  $\pi k$  for  $k > 0$  on an interval that will be specified later. Outside this interval,  $f_k$  will be identically equal to 0 or  $\pi k$ . Though the details do not matter for the Dehn twist itself, our computations will turn out to put some constraints on  $f_k$ . For negative  $k$ ,  $f_k$  is required to decrease monotonically to  $\pi k$ , but we will be mainly concerned with positive  $k$ .

For small  $|\mathbf{p}|$ , the map  $\tau_k$  equals  $(-1)^k \text{id}$ , while for large  $|\mathbf{p}|$  it equals the identity map.

**Definition 2.4.** The map  $\tau_k$  ( $k \in \mathbb{N}$ ) is called a  **$k$ -fold right-handed Dehn twist**. The map  $\tau_{-k}$  is called a  **$k$ -fold left-handed Dehn twist**.

We will now construct a mapping torus of  $T^*S^{n-1}$  using right-handed Dehn twists following the construction of Giroux and Mohsen [4]. The canonical 1-form  $\lambda_{\text{can}} = \mathbf{p} \cdot d\mathbf{q}$  on  $T^*S^{n-1}$  transforms like

$$\tau_k^* \lambda_{\text{can}} = \lambda_{\text{can}} + |\mathbf{p}| d(f_k(|\mathbf{p}|)).$$

Note that the difference  $\lambda_{\text{can}} - \tau_k^* \lambda_{\text{can}}$  is exact. This implies in particular that the Dehn twists are symplectomorphisms of  $(T^*S^{n-1}, d\lambda_{\text{can}})$ . As a primitive of this difference  $\lambda_{\text{can}} - \tau_k^* \lambda_{\text{can}}$  we take

$$h_k(|\mathbf{p}|) := 1 - \int_0^{|\mathbf{p}|} s f'_k(s) ds.$$

Note that  $h_k$  can be assumed to be positive by choosing a suitable interval where  $f_k$  increases. To be more explicit, choose a smooth function  $f$  that is identically 0 on the interval  $[0, 1]$ ; on the interval  $[1, 2]$  it increases monotonically from 0 to 1 and  $f$  is identically 1 on the interval  $[2, \infty)$ . Furthermore, we may assume that the derivative  $f'$  is bounded by 2. Then we can take  $f_k(x) := k\pi f(c_k x)$  with  $c_k > 3k\pi$ . We have

$$\int_0^{|\mathbf{p}|} s f'_k(s) ds \leq \int_0^\infty k\pi c_k s f'(c_k s) ds \leq k\pi \int_0^\infty y f'(y) dy / c_k \leq \frac{k\pi}{c_k} \int_1^2 y 2 dy = \frac{3k\pi}{c_k},$$

where we have substituted  $y = c_k s$  and used that  $f'(y) = 0$  outside the interval  $[1, 2]$  and that  $f'$  is bounded by 2. Our choice of  $c_k$  ensures that this integral is indeed smaller than 1, so  $h_k$  is positive. Consider the map

$$\begin{aligned} \varphi_k : \mathbb{R} \times T^*S^{n-1} &\longrightarrow \mathbb{R} \times T^*S^{n-1}, \\ (t; \mathbf{q}, \mathbf{p}) &\longmapsto (t + h_k(|\mathbf{p}|); \tau_k(\mathbf{q}, \mathbf{p})). \end{aligned}$$

This map preserves the contact form  $dt + \lambda_{\text{can}}$  on  $\mathbb{R} \times T^*S^{n-1}$ , so we obtain an induced contact structure on  $\mathbb{R} \times T^*S^{n-1} / \varphi_k$ .

To make computations more convenient, we construct an additional intermediate mapping torus. Let  $\mathbb{R} \times T^*S^{n-1} / \sim_k$  be the mapping torus obtained by identifying  $(t; \mathbf{q}, \mathbf{p}) \sim_k (t + 1; \tau_k(\mathbf{q}, \mathbf{p}))$ . We can define a diffeomorphism

$$\mathbb{R} \times T^*S^{n-1} / \sim_k \rightarrow \mathbb{R} \times T^*S^{n-1} / \varphi_k$$

by sending  $(t; \mathbf{q}, \mathbf{p})$  to  $(h_k(|\mathbf{p}|)t; \mathbf{q}, \mathbf{p})$ . The pull-back  $\beta_k$  of the described contact form under this diffeomorphism is given by

$$\beta_k = h_k(|\mathbf{p}|)dt - t|\mathbf{p}|d(f_k(|\mathbf{p}|)) + \lambda_{\text{can}}.$$

Since we will introduce more mapping tori and maps between them, we have a diagram at the end of section 3.2 that indicates the relations between them.

### 3. OPEN BOOKS FOR THE BRIESKORN MANIFOLDS $W_k^{2n-1}$

The **Brieskorn manifolds**  $W_k^{2n-1} \subset \mathbb{C}^{n+1}$  (with  $k \in \mathbb{N}_0$ ) are defined as the intersection of the sphere  $S^{2n+1}$  with the zero set of the polynomial  $f(z_0, z_1, \dots, z_n) = z_0^k + z_1^2 + \dots + z_n^2$ . To make computations easier, assume that the radius of the  $(2n + 1)$ -sphere is  $\sqrt{2}$ .

The orthogonal group  $\text{SO}(n)$  acts linearly on  $\mathbb{C}^{n+1}$  by leaving the first coordinate of  $(z_0, z_1, \dots, z_n)$  fixed and multiplying the last  $n$  coordinates with  $\text{SO}(n)$  in its standard matrix representation, i.e.  $A \cdot (z_0, z_1, \dots, z_n) := (z_0, A \cdot (z_1, \dots, z_n))$ . This action restricts to  $W_k^{2n-1}$ , because the polynomial  $f$  can be written as  $z_0^k + \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 + 2i\langle \mathbf{x} | \mathbf{y} \rangle$  with  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbf{y} = (y_1, \dots, y_n)$ .

Finally, the  $\text{SO}(n)$ -invariant 1-form

$$\alpha_k := k \cdot (x_0 dy_0 - y_0 dx_0) + 2 \sum_{j=1}^n (x_j dy_j - y_j dx_j)$$

is of contact type on  $W_k^{2n-1}$  for all  $k \in \mathbb{N}$ , as was shown by Lutz and Meckert [6].

It is well known that all  $W_k^{2n-1}$  are  $(n-2)$ -connected and some of these Brieskorn manifolds are spheres [1], [5]. Ustilovsky [8] showed that among them there are diffeomorphic but non-contactomorphic manifolds. Namely if  $2n - 1 = 1 \pmod 4$ , then all  $W_k^{2n-1}$  with  $k = \pm 1 \pmod 8$  are standard spheres with inequivalent contact structures.

In the remainder of this paper will we show that the contact structures on Brieskorn manifolds  $W_k^{2n-1}$  are supported by an open book whose monodromy is given by a  $k$ -fold Dehn twist. We define the binding  $B$  of the open book by the set in  $W_k^{2n-1}$  with  $z_0 = 0$  and have the fibration  $\vartheta : (W_k^{2n-1} - B) \rightarrow S^1$ , given by  $(z_0, z_1, \dots, z_n) \mapsto z_0/|z_0|$ .

**3.1. The binding.** The only stabilizers of the  $SO(n)$ -action on the Brieskorn manifold that occur are  $SO(n - 1)$  and  $SO(n - 2)$ . The projection onto the orbit space is given by

$$\begin{aligned} W_k^{2n-1} &\longrightarrow D^2, \\ (z_0, z_1, \dots, z_n) &\longmapsto z_0. \end{aligned}$$

Points  $(z_0, \dots, z_n)$  lying over the interior of the disk (i.e.  $|z_0| \neq 1$ ) have principal stabilizer; points over  $\partial D^2$  lie on singular orbits. The binding of the open book is the orbit  $B = \text{Orb}(0, 1, i, 0, \dots, 0) \cong SO(n)/SO(n-2)$ . It is naturally contactomorphic to  $W_2^{2n-3}$ . In fact,  $W_2^{2n-3} = SO(n)/SO(n-2)$  is diffeomorphic to the unit sphere bundle  $S(T^*S^n)$ . This shows that part (1) of Definition 2.3 is satisfied.

The symplectic normal bundle of the binding is trivial, because for  $k \neq 1$  we have a symplectic basis

$$\frac{1}{\sqrt{2k}}(1, 0, \dots, 0), \frac{1}{\sqrt{2k}}(i, 0, \dots, 0),$$

and for  $k = 1$  we have the basis

$$\sqrt{\frac{2}{5}}(1, -\frac{\bar{z}_1}{4}, \dots, -\frac{\bar{z}_n}{4}), \sqrt{\frac{2}{5}}(i, -\frac{i\bar{z}_1}{4}, \dots, -\frac{i\bar{z}_n}{4}).$$

The neighborhood theorem for contact submanifolds [2] then shows that there is a neighborhood of the binding that is contactomorphic to  $(B \times D^2, \alpha_k|_B + r^2 d\vartheta)$ , where  $(r, \vartheta)$  are polar coordinates on the disk.

**3.2. The pages.** In this section, we will prove that  $W_k^{2n-1} - B$  is contactomorphic to  $\mathbb{R} \times T^*S^{n-1}/\sim_k$ , the mapping torus of a  $k$ -fold Dehn twist.

The  $\mathbb{R}$ -action on  $W_k^{2n-1} - B$ , given by

$$e^{it}(z_0, z_1, \dots, z_n) = (e^{it}z_0, e^{\frac{ki}{2}t}z_1, \dots, e^{\frac{ki}{2}t}z_n),$$

induces a diffeomorphism between the pages  $\vartheta^{-1}(1)$  and  $\vartheta^{-1}(e^{it})$ .

Let us define an auxiliary mapping torus to make computations more convenient. Define

$$M_k := \mathbb{R} \times T^*S^{n-1}/\sigma_k,$$

where

$$\sigma_k(t, \mathbf{q}, \mathbf{p}) = (t + 1, (-1)^k \mathbf{q}, (-1)^k \mathbf{p}).$$

We will now give an explicit map to show that  $P = \vartheta^{-1}(1)$  is diffeomorphic to  $T_{|\mathbf{p}|<1}^*S^{n-1}$ . Here  $T_{|\mathbf{p}|<1}^*S^{n-1}$  denotes the open unit disk bundle associated with

the cotangent bundle of  $S^{n-1}$ . A point  $(\mathbf{q}, \mathbf{p}) \in T^*S^{n-1} \subset \mathbb{R}^n \times \mathbb{R}^n$  with  $|\mathbf{q}| = 1$ ,  $|\mathbf{p}| \leq 1$ , and  $\mathbf{q} \perp \mathbf{p}$  is mapped to

$$(\mathbf{q}, \mathbf{p}) \mapsto \left(1 - |\mathbf{p}|^2, F(|\mathbf{p}|)\mathbf{p} + iG(|\mathbf{p}|)\mathbf{q}\right)$$

with  $F(r) = \sqrt{\frac{2-(1-r^2)^2-(1-r^2)^k}{2r^2}}$  and  $G(r) = \sqrt{\frac{2-(1-r^2)^2+(1-r^2)^k}{2}}$ .

Together with the  $\mathbb{R}$ -action this gives a map

$$\begin{aligned} \Phi_k : \mathbb{R} \times T^*_{|\mathbf{p}| < 1} S^{n-1} &\longrightarrow W_k^{2n-1}, \\ (t, \mathbf{q}, \mathbf{p}) &\longmapsto \left(e^{2\pi i t}(1 - |\mathbf{p}|^2), e^{\pi k i t}(F(|\mathbf{p}|)\mathbf{p} + iG(|\mathbf{p}|)\mathbf{q})\right). \end{aligned}$$

This descends to a diffeomorphism of the subset of  $M_k$  with  $|\mathbf{p}| < 1$  to  $W_k^{2n-1} - B$ . For  $k$  even, one obtains  $\Phi_k(t+1, \mathbf{q}, \mathbf{p}) = \Phi_k(t, \mathbf{q}, \mathbf{p})$ , so that  $W_k^{2n-1} - B \cong S^1 \times T^*_{|\mathbf{p}| < 1} S^{n-1}$ , and for  $k$  odd, one obtains  $\Phi_k(t+1, \mathbf{q}, \mathbf{p}) = \Phi_k(t, -\mathbf{q}, -\mathbf{p})$ , so that  $W_k^{2n-1} - B$  is a non-trivial  $T^*_{|\mathbf{p}| < 1} S^{n-1}$ -bundle over  $S^1$ .

The pull-back of the contact form  $\alpha_k$  to  $M_k$  under  $\Phi_k$  gives

$$\Phi_k^* \alpha_k = 2\pi k \left( (1 - |\mathbf{p}|^2)^2 + |\mathbf{p}|^2 F^2 + G^2 \right) dt + 4FG \lambda_{\text{can}} = 4\pi k dt + 4FG \lambda_{\text{can}}.$$

Next, we construct a diffeomorphism  $\Psi_k$  from  $M_k$  to the mapping torus  $\mathbb{R} \times T^*S^{n-1} / \sim_k$  by defining

$$\begin{aligned} \Psi_k(t; \mathbf{q}, \mathbf{p}) &= \left[ t; \mathbf{q} \cdot \cos(tf_k(|\mathbf{p}|)) + \frac{\mathbf{p}}{|\mathbf{p}|} \cdot \sin(tf_k(|\mathbf{p}|)), \right. \\ &\quad \left. \mathbf{p} \cdot \cos(tf_k(|\mathbf{p}|)) - |\mathbf{p}|\mathbf{q} \cdot \sin(tf_k(|\mathbf{p}|)) \right]. \end{aligned}$$

The map is well defined, because  $\Psi_k \circ \sigma_k(t; \mathbf{q}, \mathbf{p})$  is identified with  $\Psi_k(t; \mathbf{q}, \mathbf{p})$  in the mapping torus  $\mathbb{R} \times T^*S^{n-1} / \sim_k$ . In order to show that  $(W_k^{2n-1} - B, \alpha_k)$  and  $(\mathbb{R} \times T^*S^{n-1} / \sim_k, \beta_k)$  are contactomorphic, we will show that the pull-back of  $\alpha_k$  under  $\Phi_k$  is contactomorphic to the pull-back of  $\beta_k$  under  $\Psi_k$ .

We now compute the pull-back of  $\beta_k$  under  $\Psi_k$ , noting that the norm of  $\mathbf{p}$  is invariant under  $\Psi_k$  (we do not write the dependence of  $h_k$  and  $f_k$  on  $|\mathbf{p}|$ ):

$$\begin{aligned} \Psi_k^* \beta_k &= h_k dt - t|\mathbf{p}|df_k + (\mathbf{p} \cos(tf_k) - |\mathbf{p}|\mathbf{q} \sin(tf_k)) \cdot (d\mathbf{q} \cos(tf_k) \\ &\quad - \mathbf{q} \sin(tf_k)(f_k dt + tdf_k) + \left(\frac{d\mathbf{p}}{|\mathbf{p}|} - \frac{\mathbf{p}d|\mathbf{p}|}{|\mathbf{p}|^2}\right) \sin(tf_k) + \frac{\mathbf{p}}{|\mathbf{p}|} \cos(tf_k)(f_k dt + tdf_k). \end{aligned}$$

Since we have  $\mathbf{p} \cdot \mathbf{q} = 0$  and  $|\mathbf{q}|^2 = 1$ , it follows that  $\mathbf{p}d\mathbf{q} = -\mathbf{q}d\mathbf{p}$  (recall that  $\mathbf{p}d\mathbf{q} = \lambda_{\text{can}}$ ) and  $\mathbf{q}d\mathbf{q} = 0$ . We now use the standard trigonometric equalities and the fact that  $h_k(y) = 1 - yf_k(y) + \int_0^y f_k(s)ds$  to find

$$\Psi_k^* \beta_k = \left(1 + \int_0^{|\mathbf{p}|} f_k(s)ds\right) dt + \lambda_{\text{can}}.$$

Note that  $\Phi_k^* \alpha_k$  has a very similar form. We make the following ansatz for a contactomorphism of  $(M_k|_{|\mathbf{p}| < 1}, \Phi_k^* \alpha_k)$  to  $(M_k, \Psi_k^* \beta_k)$ :

$$S_k : (t, \mathbf{q}, \mathbf{p}) \mapsto \left(t, \mathbf{q}, \frac{g(|\mathbf{p}|)}{|\mathbf{p}|} \mathbf{p}\right).$$

With this ansatz we find what  $\mathbf{p}$  should map to in order to be a contactomorphism. Note that we just rescale  $\mathbf{p}$ . The pull-back under this map of  $\Psi_k^*\beta_k$  is given by

$$\left( 1 + \int_0^{g(|\mathbf{p}|)} f_k(s) ds \right) dt + \frac{g(|\mathbf{p}|)}{|\mathbf{p}|} \lambda_{\text{can}}.$$

Since we want this to be a multiple of  $\Phi_k^*\alpha_k$ , we need to solve the following equation:

$$\frac{g(|\mathbf{p}|)}{1 + \int_0^{g(|\mathbf{p}|)} f_k(s) ds} = \frac{|\mathbf{p}|FG}{k\pi}.$$

Define an auxiliary function

$$h(y) := \frac{y}{1 + \int_0^y f_k(s) ds}.$$

The above equation becomes

$$h(g(|\mathbf{p}|)) = \frac{|\mathbf{p}|FG}{k\pi}.$$

We will solve for  $g(|\mathbf{p}|)$  by inverting  $h$ . This can be done by the following considerations. The derivative of  $h$  is given by

$$h'(y) = \frac{1 - \int_0^y s f'_k(s) ds}{\left(1 + \int_0^y f_k(s) ds\right)^2} = \frac{h_k(y)}{\left(1 + \int_0^y f_k(s) ds\right)^2}$$

and is positive by our choice of  $h_k$  in Section 2.2. Since this shows that  $h$  is strictly increasing, we also observe that the function  $h$  maps  $[0, \infty)$  to  $[0, \frac{1}{k\pi})$ . This can be seen by noting that  $f_k(s) = k\pi$  for  $s$  sufficiently large, again due to our choice of  $h_k$ . It also means that  $h$  can be inverted when restricted to a suitable range. One easily checks that the right-hand side of the above equation,  $\frac{|\mathbf{p}|FG}{k\pi}$ , has positive derivative and is therefore strictly increasing on the interval  $[0, 1)$ . Moreover it has the same range as  $h$ , namely  $[0, \frac{1}{k\pi})$ . Therefore we can find a smooth solution to  $g(|\mathbf{p}|)$  by applying the inverse of  $h$  to  $\frac{|\mathbf{p}|FG}{k\pi}$ .

This shows that the open book  $(B, \vartheta)$  on  $W_k^{2n-1}$  has page  $T^*S^{n-1}$  with monodromy given by a  $k$ -fold Dehn twist. The contactomorphism that achieves this is

$$C_k := \Phi_k \circ S_k^{-1} \circ \Psi_k^{-1} : (\mathbb{R} \times T^*S^{n-1} / \sim_k, \beta_k) \rightarrow (W_k^{2n-1} - B, \alpha_k).$$

Note that this contactomorphism also respects the projection to  $S^1$ , because the  $S^1$ -coordinate is invariant under  $C_k$ . For the sake of convenience, we summarize our results in the following diagram:

$$\begin{array}{ccc} (\mathbb{R} \times T^*S^{n-1} / \varphi_k, dt + \lambda_{\text{can}}) & & \\ \parallel & & \\ (\mathbb{R} \times T^*S^{n-1} / \sim_k, \beta_k) & \xleftarrow{\psi_k} M_k \xleftarrow{S_k} M_k|_{|\mathbf{p}|<1} \xrightarrow{\Phi_k} & (W_k^{2n-1}, \alpha_k). \end{array}$$

**3.3. The contact structure on  $W_k^{2n-1}$  is supported by the open book.** Part (1) of Definition 2.3 was already checked in Section 3.1. Note that the Reeb field  $R_{\alpha_k}$  is transverse to the pages, as its flow even provides a diffeomorphism from one page to another. If we denote a page by  $P$ , this implies in particular that the rank of  $d\alpha_k|_P$  is maximal, or in other words that  $d\alpha_k$  is a symplectic form when restricted to  $P$ . This shows part (2).

For property (3), take a page  $P$  and  $w_0 \in \mathbb{C} - \{0\}$  such that  $P = \vartheta^{-1}(w_0/|w_0|)$ . Then define  $B_{w_0} := \{(z_0, z_1, \dots, z_n) \in W_k^{2n-1} \mid z_0 = w_0\}$ . Note that  $B_{w_0}$  is contactomorphic to  $B$  if  $|w_0|$  is small enough and has the same orientation induced by its contact form.  $B_{w_0}$  can be thought of as a copy of the binding that is obtained by pushing the binding into the page.

We can now regard  $B_{w_0}$  as the boundary of the compact page  $P_{w_0}$  by cutting off a part of  $P$ . The computations are a little easier if we consider the inverse images of  $B_{w_0}$  and  $P_{w_0}$  under the map  $\Phi_k$ . The inverse image of  $B_{w_0}$  is the set

$$\left\{ \frac{w_0}{|w_0|} \right\} \times S_c T^* S^{n-1} \subset \mathbb{R} \times T_{|\mathbf{p}| < 1}^* S^{n-1} / \sigma_k,$$

where  $S_c T^* S^{n-1}$  denotes the associated circle bundle with radius  $c = \sqrt{1 - |w_0|}$ . Similarly, the trimmed page  $P_{w_0}$  corresponds to

$$\left\{ \frac{w_0}{|w_0|} \right\} \times T_{|\mathbf{p}| \leq c}^* S^{n-1}.$$

Because the contact condition is an open condition,  $\tilde{\alpha} := FG\lambda_{\text{can}}$  is a contact form on  $S_c T^* S^{n-1}$  for small  $|w_0|$ . This means that it suffices to check that the orientation of  $(S_c T^* S^{n-1}, \tilde{\alpha})$  induced by the contact form coincides with its orientation as the boundary of the symplectic manifold  $(T_{|\mathbf{p}| \leq c}^* S^{n-1}, d\tilde{\alpha})$ . Observe that  $X = \mathbf{p} \frac{\partial}{\partial \mathbf{p}}$  is a vector field transverse to  $B_{w_0}$  pointing outwards. The volume form on  $P_{w_0}$  induced by the symplectic form is given by  $\Omega = d\tilde{\alpha}^{n-1}$ . The induced orientation on  $B_{w_0}$  can then be obtained by restricting  $\iota_X \Omega$  to  $B_{w_0}$ . This form is equal to

$$\begin{aligned} \iota_X \Omega &= (n-1)(\iota_X d\tilde{\alpha}) \wedge d\tilde{\alpha}^{n-2} = (n-1) \left( 1 + \frac{\mathcal{L}_X FG}{FG} \right) \tilde{\alpha} \wedge d\tilde{\alpha}^{n-2} \\ &= (n-1)(1 + X \log(FG)) \tilde{\alpha} \wedge (d\tilde{\alpha})^{n-2}. \end{aligned}$$

The term  $X \log(FG)$  can be checked to be larger than  $-1$  for  $|w_0| > 0$  and hence the orientations of  $B_{w_0}$  as the boundary of the page and as a contact manifold coincide.

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