PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 4, Pages 1189–1195
S 0002-9939(05)08057-3
Article electronically published on July 20, 2005

HYPERELLIPTIC SURFACES ARE LOEWNER

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(Communicated by Jon G. Wolfson)

Abstract. We prove that C. Loewner's inequality for the torus is satisfied by conformal metrics on hyperelliptic surfaces X as well. In genus 2, we first construct the Loewner loops on the (mildly singular) companion tori, locally isometric to X away from Weierstrass points. The loops are then transplanted to X, and surgered to obtain a Loewner loop on X. In higher genus, we exploit M. Gromov's area estimates for ε-regular metrics on X.

1. Hermite constant and Loewner surfaces

The systole, $\mathrm{sys}\pi_1(\mathcal{G})$, of a compact nonsimply connected Riemannian manifold $(X, \mathcal{G})$ is the least length of a noncontractible loop $\gamma \subset X$:

$$\mathrm{sys}\pi_1(\mathcal{G}) = \min_{[\gamma]\neq 0 \in \pi_1(X)} \mathrm{length}(\gamma). \tag{1.1}$$

This notion of systole is apparently unrelated to the systolic arrays of [Ku78]. We will be concerned with comparing this Riemannian invariant to the total area of the metric, as in the Loewner inequality (1.3) for the torus.

The Hermite constant, denoted γ_n, can be defined as the optimal constant in the inequality

$$\mathrm{sys}\pi_1(\mathbb{T}^n)^2 \leq \gamma_n \,\mathrm{vol}(\mathbb{T}^n)^{2/n}, \tag{1.2}$$

over the class of all *flat* tori $\mathbb{T}^n$. Here γ_n is asymptotically linear in n; *cf.* [LLS90, pp. 334, 337]. The precise value is known for small n, *e.g.* one has $\gamma_2 = \frac{2}{\sqrt{3}}$, $\gamma_3 = 2^{\frac{1}{3}}$. An inequality of type (1.2) remains valid in the class of *all* metrics, and in fact on a more general manifold, but with a nonsharp constant on the order of n^{4n} [Gr83].

Around 1949, Charles Loewner proved the first systolic inequality; *cf.* [Pu52]. He showed that every Riemannian metric $\mathcal{G}$ on the torus $\mathbb{T}^2$ satisfies the inequality

$$\mathrm{sys}\pi_1(\mathcal{G})^2 \leq \gamma_2 \,\mathrm{area}(\mathcal{G}), \tag{1.3}$$

while a metric satisfying the boundary case of equality in (1.3) is necessarily flat, and is homothetic to the quotient of $\mathbb{C}$ by the lattice spanned by the cube roots of unity.

Received by the editors March 18, 2004 and, in revised form, October 26, 2004.
2000 *Mathematics Subject Classification.* Primary 53C23; Secondary 30F10.
Key words and phrases. ε-regular metrics, Hermite constant, hyperelliptic involution, Loewner inequality, Pu's inequality, systole, Weierstrass point.
The first author was supported by the Israel Science Foundation (grants no. 620/00-10.0 and 84/03).

Higher-dimensional optimal generalisations of the Loewner inequality are studied in [BK03, Ka03, BK04, IK04, BCIK04, Sa05]. The defining text for this material is [Gr99], with more details in [Gr83, Gr96], complemented by [Bab93]. See also the recent survey [CK03], as well as [KL04, KR04].

Definition 1.1. We say that a metric $\mathcal{G}$ on a surface is *Loewner* if it satisfies the Loewner inequality (1.3).

Question 1.2. It follows from Gromov's estimate (2.1) that every metric on an orientable surface Σ_g of genus g is Loewner if $g > 50$. This result is improved in [KS04] to $g \geq 20$. Can the genus assumption be removed altogether?

Note that a similar question for Pu's inequality [Pu52] has an affirmative answer. The generalisation is immediate from Gromov's inequality (2.2). Namely, every surface X which is not a 2-sphere satisfies the inequality

$$\mathrm{sys}\pi_1(X)^2 \leq \frac{\pi}{2}\,\mathrm{area}(X), \tag{1.4}$$

where the boundary case of equality in (1.4) is attained precisely when, on one hand, the surface X is a real projective plane, and on the other, the metric is of constant Gaussian curvature.

We will prove the following result toward answering Question 1.2; *cf.* Theorem 3.1.

Theorem 1.3. *Every metric on an orientable surface is Loewner if it lies in a hyperelliptic conformal class. In particular, every metric on the genus* 2 *surface is Loewner.*

While the precise value of the systolic ratio in genus 2 is unknown in the class of all metrics, an optimal systolic inequality for CAT(0) metrics does exist [KS05]. The relevant literature and basic estimates are reviewed in Section 2. We will state the main theorem in more detail in Section 3 and prove it for genus 3 or more. We will complete the proof in the genus 2 case in Section 4.

2. Basic estimates

Definition 2.1. The *systolic ratio* of a metric $\mathcal{G}$ on a closed n-manifold is defined as

$$\mathrm{SR}(\mathcal{G}) = \frac{\mathrm{sys}\pi_1(\mathcal{G})^n}{\mathrm{vol}_n(\mathcal{G})}.$$

The *conformal systolic ratio*, denoted $\mathrm{SR}_c(X)$, of a closed n-manifold X with a chosen conformal class of metrics, is defined as

$$\mathrm{SR}_c(X) = \sup_{\mathcal{G}} \{\mathrm{SR}(\mathcal{G}) : \mathcal{G} \text{ a conformal metric on } X\}.$$

The supremum of the conformal systolic ratio over all the conformal structures of X is called the *optimal systolic ratio*. It is denoted by $\mathrm{SR}(X)$.

M. Gromov [Gr83, p. 50] (*cf.* [Kod87, Theorem 4, part (1)]) proved a general estimate which implies that if Σ_g is a compact orientable surface of genus g with a Riemannian metric, then

$$\mathrm{SR}(\Sigma_g) < \frac{64}{4\sqrt{g}+27}. \tag{2.1}$$

Thus the optimal systolic ratio tends to 0 as the genus increases without bound.

Remark 2.2. It was shown in [Gr83] (see also [KS04] and [Bal04]) that asymptotically the optimal systolic ratio of a surface of genus g behaves as $C\frac{(\log g)^2}{g}$, as $g \to \infty$.

Another helpful estimate is found in [Gr83, Corollary 5.2.B]. Namely, every aspherical compact surface $(\Sigma, \mathcal{G})$ admits a metric ball $B = B_p\left(\frac{1}{2}\operatorname{sys}\pi_1(\mathcal{G})\right) \subset \Sigma$ of radius $\frac{1}{2}\operatorname{sys}\pi_1(\mathcal{G})$, which satisfies

$$\operatorname{sys}\pi_1(\mathcal{G})^2 \leq \frac{4}{3}\operatorname{area}(B). \tag{2.2}$$

Furthermore, whenever a point $x \in \Sigma$ lies on a two-sided loop which is minimizing in its free homotopy class, the metric ball $B_x(r) \subset \Sigma$ of radius $r \leq \frac{1}{2}\operatorname{sys}\pi_1(\mathcal{G})$ satisfies the estimate

$$2r^2 < \operatorname{area}\left(B_x(r)\right). \tag{2.3}$$

3. Hyperelliptic surfaces and ε-regularity

Recall that a Riemann surface X is called hyperelliptic if it admits a degree 2 meromorphic function; *cf.* [Mi95, pp. 60-61] as well as [Mi95, Proposition 4.11, p. 92]. The associated ramified double cover

$$Q : X \to S^2 \tag{3.1}$$

over the sphere S^2 is conformal away from the $2g+2$ ramification points, where g is the genus of X. Its deck transformation $J : X \to X$ is called the *hyperelliptic involution*. Such a holomorphic involution, if it exists, is uniquely characterized by the property of having precisely $2g+2$ fixed points. The fixed points of J are called Weierstrass points. Their images under the map Q of (3.1) will be referred to as *ramification points*.

Theorem 3.1. *Let $(X, \mathcal{G})$ be an orientable surface, where the metric $\mathcal{G}$ belongs to a hyperelliptic conformal class. Then $(X, \mathcal{G})$ is Loewner.*

Since every genus 2 surface is hyperelliptic [FK92, Proposition III.7.2, page 100], we obtain the following corollary.

Corollary 3.2. *Every metric on the genus* 2 *surface is Loewner.*

Note that this is the first improvement, known to the authors, on Gromov's 4/3 bound (2.2) in over 20 years, for surfaces of genus below 50; *cf.* Question 1.2. No extremal metric has as yet been conjectured in this genus, but it cannot be flat with conical singularities [Sa04]. The best available lower bound for the optimal systolic ratio in genus 2 can be found in [CK03, section 2.2] and [KS05].

For genus $g \geq 3$, our Theorem 3.1 follows from Proposition 3.6; *cf.* Remark 2.2 and [Kon03]. Let us go over the definitions of [Gr83, 5.1].

Definition 3.3. Given a Riemannian metric $\mathcal{G}$ on X, the *tension* of a noncontractible loop γ based at $x \in X$ is the upper bound of all $\delta > 0$ such that there exists a free homotopy of γ which diminishes the length of γ by δ. The tension is denoted $\operatorname{tens}_{\mathcal{G}}(\gamma)$.

Definition 3.4. The height $h_{\mathcal{G}}(x)$ of $x \in X$ is defined as the lower bound of the tensions of the noncontractible loops based at x. Note that the height function is 2-Lipschitz.

A metric $\mathcal{G}$ is said to be ε-regular with $\varepsilon < \operatorname{sys}\pi_1(\mathcal{G})$ if its height function $h_{\mathcal{G}}$ is bounded from above by ε.

Lemma 3.5. *Given any conformal metric $\mathcal{G}$ on (Σ_g, J) and a real number $\delta > 0$, there exists an ε-regular, J-invariant, conformal metric $\bar{\mathcal{G}}$ with a systolic ratio at least $\mathrm{SR}(\mathcal{G}) - \delta$.*

Proof. The proof appears in [Gr83, 5.6.C''] in the general case, and proceeds by a (finite) sequence of modifications of the metric in suitable small disks, while staying in the same conformal class. Note that averaging the metric by J improves the systolic ratio and does not change the conformal class. Thus we can assume that $\mathcal{G}$ is J-invariant. To adapt the proof to our situation, we perform the modifications in a J-invariant way. □

Proposition 3.6. *Every hyperelliptic surface (Σ_s, J) of genus g satisfies the estimate*

$$\mathrm{SR}_c(\Sigma_g) \leq \frac{4}{g+1}.$$

Proof. Let $\mathcal{G}$ be a conformal metric on the surface. By Lemma 3.5, we can assume that the metric $\mathcal{G}$ is J-invariant and ε-regular. Since the metric is J-invariant, the distance between any pair of the $2g+2$ Weierstrass points is at least $\frac{1}{2}\operatorname{sys}\pi_1(\mathcal{G})$. Thus, the disks of radius $R = \frac{1}{4}\operatorname{sys}\pi_1(\mathcal{G})$ centered at the Weierstrass points are disjoint. From [Gr83, 5.1.B] (see also [He81]), the area of such a disk on an ε-regular surface is at least

$$2R^2 - \varepsilon = \frac{1}{8}\operatorname{sys}\pi_1(\mathcal{G})^2 - \varepsilon;$$

cf. (2.3). Therefore, we have

$$\operatorname{area}(\mathcal{G}) \geq (2g+2)\left(\tfrac{1}{8}\operatorname{sys}\pi_1(\mathcal{G})^2 - \varepsilon\right).$$

Since this is true for all $\varepsilon > 0$, the proposition is proved. □

4. Proof of Theorem 3.1 in genus two

Let X be a genus 2 surface. Recall that X has a hyperelliptic involution J with 6 Weierstrass points.

The idea of the proof of Theorem 3.1 in genus 2 is to apply the Loewner inequality to certain *companion tori* of X, and to surger the resulting loops so as to obtain a Loewner loop on X. We may need the following lemma.

Lemma 4.1. *Let $\mathbb{T}^2$ be a torus endowed with a metric invariant under its hyperelliptic involution J_{T^2}, with conical singularities with total angle less than 2π around each. Then the image of a systolic loop of $\mathbb{T}^2$ in S^2 under the hyperelliptic projection is a simple loop.*

Proof. Let $\gamma \subset \mathbb{T}^2$ be a systolic loop. Since J_{T^2} induces minus the identity homomorphism on $\pi_1(\mathbb{T}^2)$, the loops γ and $-J_{T^2}(\gamma)$ are homotopic. Under the hypotheses of our lemma, two homotopic systolic loops are necessarily disjoint. Hence the image of γ on S^2 is simple. □

Definition 4.2. A *companion torus* $\mathbb{T}(a,b,c,d)$ of X is a torus whose ramification locus $\{a,b,c,d\} \subset S^2$ is a subset of the ramification locus of X.

As in the proof of Proposition 3.6, we can assume that the metric on X is invariant under J (see [BCIK05]). Therefore $\mathcal{G}$ descends to a metric $\mathcal{G}_0$, of half the area, on S^2. Let us choose four of the 6 ramification points, say $a, b, c, d \in S^2$. Choose a double cover with ramification locus $\{a, b, c, d\}$, denoted

$$\mathbb{T}^2(a, b, c, d) \to S^2.$$

Pulling back the metric $\mathcal{G}_0$ to the torus $\mathbb{T}^2(a, b, c, d)$, we obtain a metric of the same area as the surface X itself. This metric on the torus is smooth away from the two remaining points, where it has a conical singularity with total angle π around each. Consider a Loewner loop

$$L_L \subset \mathbb{T}^2(a, b, c, d)$$

on this torus, *e.g.* a systolic loop realizing (1.3). Let L be the projection of L_L to S^2. The simple loop $L \subset S^2$ separates the four points a, b, c, d into two pairs, say a, b on one side and c, d, on the other. If the lift of L to X closes up, we obtain a Loewner loop on X and the theorem is proved. Thus, we may assume that the following three equivalent conditions are satisfied:

(1) the lift of L to X does not close up;
(2) the inverse image $Q^{-1}(L) \subset X$ under Q of (3.1) is connected;
(3) the loop L surrounds precisely 3 ramification points of Q.

The last condition is equivalent to the first two since every based loop is homotopic, in the complement of the ramification points, to a composition of some loops from a finite collection of "standard" simple based loops, circling each of the ramification points. Meanwhile, going once around such a standard loop clearly switches the two sheets of the cover.

Definition 4.3. The simple loop L partitions the sphere into two hemispheres, H_+ and H_-, with $a, b, e \in H_+$ and $c, d, f \in H_-$, where a, b, c, d, e, f are the 6 ramification points of Q.

Using a pair of companion tori, we will construct two loops on the sphere, defining two distinct partitions of the ramification locus into a pair of triples. The basic example to think of is the case of a centrally symmetric 6-tuple of points, corresponding for instance to the curve

$$y^2 = x^5 - x,$$

and a pair of generic great circles, such that each of the four digons contains at least one ramification point. We now construct a companion torus $\mathbb{T}(a, b, e, f)$.

Consider a Loewner loop $L_L{}' \subset \mathbb{T}^2(a, b, e, f)$, and its projection $L' \subset S^2$. If its lift to X closes up, the theorem is proved. Therefore assume that the lift of L' to X does not close up, *i.e.* L' surrounds exactly 3 ramification points. Now L' separates the four points a, b, e, f into two pairs. Hence it defines a different splitting of the six points into two triples. The connected components of $L' \cap H_+$ form a nonempty finite collection of disjoint nonselfintersecting arcs α.

Each arc α divides H_+ into a pair of regions homeomorphic to disks. Such regions are partially ordered by inclusion. A minimal element for the partial order is necessarily a digon. Such a digon must contain at least one ramification point of Q (otherwise exchange the two sides of the digon between the loops L and L', so as to decrease the total number of intersections, or else argue as in Lemma 4.1). It is clear that there are at least two such digons in H_+.

Hence one of them, denoted $D \subset H_+$, must contain precisely one of the 3 ramification points of H_+. We now exchange the two sides of D between the loops L, L', obtaining two new loops M, M'. Each of the new loops surrounds a nonzero even number of ramification points. Since

$$\operatorname{length}(M) + \operatorname{length}(M') = \operatorname{length}(L) + \operatorname{length}(L'),$$

one of the loops M or M' is no longer than Loewner. Moreover, its lift to X closes up, producing a Loewner loop on X, as required.

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