PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 4, Pages 943–948
S 0002-9939(05)08066-4
Article electronically published on July 25, 2005

$HSP \neq SHPS$ FOR COMMUTATIVE RINGS WITH IDENTITY

JOHN LAWRENCE AND BOŽA TASIĆ

(Communicated by Lance W. Small)

Abstract. Let I, H, S, P, P_s be the usual operators on classes of rings: I and H for isomorphic and homomorphic images of rings and S, P, P_s respectively for subrings, direct, and subdirect products of rings. If $\mathcal{K}$ is a class of commutative rings with identity (and in general of any kind of algebraic structures), then the class $HSP(\mathcal{K})$ is known to be the variety generated by the class $\mathcal{K}$. Although the class $SHPS(\mathcal{K})$ is in general a proper subclass of the class $HSP(\mathcal{K})$ for many familiar varieties $HSP(\mathcal{K}) = SHPS(\mathcal{K})$. Our goal is to give an example of a class $\mathcal{K}$ of commutative rings with identity such that $HSP(\mathcal{K}) \neq SHPS(\mathcal{K})$. As a consequence we will describe the structure of two partially ordered monoids of operators.

1. Introduction

The operators we will consider are assumed to be defined on classes of algebras of the same type (e.g. groups, rings, lattices, etc.). Further on $\mathcal{K}$ will stand for an arbitrary class of algebras of the same type. Operators X, Y are *equal*, in symbols $X = Y$, if and only if $X(\mathcal{K}) = Y(\mathcal{K})$ for every $\mathcal{K}$. On the set of operators we define a *partial order* by $X \leq Y$ if and only if $X(\mathcal{K}) \subseteq Y(\mathcal{K})$ for all $\mathcal{K}$, and the *composition of operators* X and Y by $XY(\mathcal{K}) = X(Y(\mathcal{K}))$, for an arbitrary $\mathcal{K}$. It is easy to see that the composition of operators is an associative operation, i.e., $X(YZ) = (XY)Z$ for any class operators X, Y, Z. An operator X is a *closure operator* if it is *extensive* ($\mathcal{K} \subseteq X(\mathcal{K})$), *monotone* ($\mathcal{K}_1 \subseteq \mathcal{K}_2 \Rightarrow X(\mathcal{K}_1) \subseteq X(\mathcal{K}_2)$), and *idempotent* ($X(X(\mathcal{K})) = X(\mathcal{K})$).

Given a class $\mathcal{K}$ we let $I(\mathcal{K})$ and $H(\mathcal{K})$ denote the classes of all isomorphic images and the class of all homomorphic images of algebras in $\mathcal{K}$, respectively. Let $S(\mathcal{K})$, denote the class of all algebras isomorphic to subalgebras of algebras in $\mathcal{K}$, and let $P(\mathcal{K})$, $P_u(\mathcal{K})$, $P_s(\mathcal{K})$ denote the classes of all algebras isomorphic to direct, ultra and subdirect products of algebras in $\mathcal{K}$ respectively.

I, H, S, P, P_u, P_s are closure operators and all the composites (e.g. HS, P_sH, HSP, $SHHP$) can be thought of as forming a monoid $\mathcal{M}$ with the composition of operators as multiplication and I as the identity element. We will be interested in two special cases. Namely, we will consider the monoid $\mathcal{M}$ generated by the operators H, S, P, as well as the monoid $\mathcal{M}_s$ generated by the operators H, S, P, P_s.

Received by the editors November 29, 2001 and, in revised form, October 28, 2004.
2000 *Mathematics Subject Classification.* Primary 06F05, 68Q99.
Key words and phrases. Class operators, commutative rings with identity, partially ordered monoid.

The structure of $\mathcal{M}$, which is in fact a partially ordered monoid (in the remainder of the text po-monoid) is determined by Pigozzi in [8]. There are at most eighteen possible composites one can get starting with I, H, S, P. The corresponding partial ordering is shown in Figure 1.

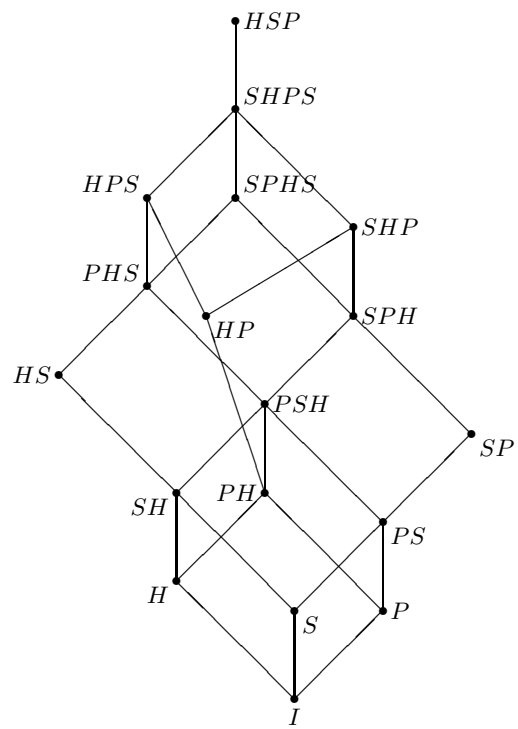

FIGURE 1. The partial ordering of $\mathcal{M}$

The structure of $\mathcal{M}_s$ is determined in [10]. In this case there are at most 22 operators one can get as composites starting with I, H, S, P, P_s. The corresponding partial ordering is shown in Figure 2.

We will consider commutative rings with identity as structures in the language $\mathbf{R} = \langle R, +, \cdot, -, 0, 1\rangle$. Let $\mathcal{R}_c$ denote the variety of all commutative rings with identity. When we restrict domains of the operators in $\mathcal{M}$ and $\mathcal{M}_s$ to be subclasses of $\mathcal{R}_c$, we get the so-called monoids of the variety of commutative rings with identity generated by the corresponding operators. The monoid of the variety $\mathcal{R}_c$ generated by H, S, P is a homomorphic image of $\mathcal{M}$, and it is denoted by $\mathcal{M}(\mathcal{R}_c)$. Similarly, we will consider $\mathcal{M}_s(\mathcal{R}_c)$ the monoid of the variety $\mathcal{R}_c$ generated by H, S, P, P_s. The monoids of operators H, S, P are described for the following varieties: groups in [7], metabelian groups in [1], commutative semigroups in [8], pseudocomplemented lattices in [5], implicative semilattices in [6], varieties generated by a quasiprimal algebra in [9]. The account of the various results is also given in [4].

For many varieties there is actually a great deal of collapsing among the operators which implies that for such a variety its monoid of operators H, S, P is a proper

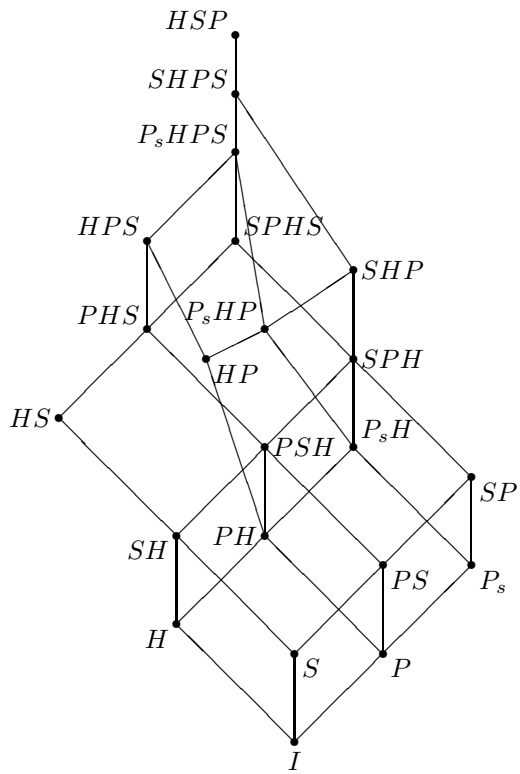

FIGURE 2. The partial ordering of $\mathcal{M}_s$

homomorphic image of $\mathcal{M}$. For example, if $\mathcal{A}$ is the variety of Abelian groups, then for every $\mathcal{K} \subseteq \mathcal{A}$ we have $HS(\mathcal{K}) = SH(\mathcal{K})$, i.e., $HS = SH$ on $\mathcal{A}$. Hence in case of Abelian groups we have $HSP = SHPS$. For commutative rings we will see that $HSP \neq SHPS$. So, on one hand, we have that in general $HSP \neq SHPS$, but if we restrict our attention to the Abelian groups we have $HSP = SHPS$. On the other hand, there are relations which are true in general, such as $HSP = HP_s$ and $SPHS = P_sHS$.

2. Results on commutative rings with identity

We recall the following result from [8].

Theorem 2.1 (see [8, Theorem 4])**.** *Let $\mathcal{V}$ be a variety. A necessary and sufficient condition for the monoid of operators H, S, P of the variety $\mathcal{V}$ to be isomorphic to $\mathcal{M}$ is that there exist classes $\mathcal{K}_1$, $\mathcal{K}_2 \subseteq \mathcal{V}$ satisfying the non-inclusions*

$$HSP(\mathcal{K}_1) \not\subseteq SHPS(\mathcal{K}_1), \tag{1}$$

$$HP(\mathcal{K}_2) \not\subseteq SPHS(\mathcal{K}_2). \tag{2}$$

Now we are ready to show

Theorem 2.2. *There exists a class $\mathcal{K}$ of commutative rings with identity satisfying the non-inclusions* (1) *and* (2)*. Consequently, the monoid of operators H, S, P is full, i.e., $\mathcal{M}(\mathcal{R}_c) \cong \mathcal{M}$.*

Proof. Let $\mathcal{K} = \{\mathbf{Z}_p \mid p \in P\}$ where P is the set of all primes. Let $\mathcal{U}$ be a nonprincipal ultrafilter over P and let $\mathbf{F} = \prod_{p\in P} \mathbf{Z}_p/\mathcal{U}$. Since $\mathbf{Z}_p$ is a field for $p \in P$, by Łoš's theorem $\mathbf{F}$ is also a field. So we have $\mathbf{F} \in P_u(\mathcal{K}) \subseteq HP(\mathcal{K})$. On the other hand, since the only subrings and homomorphic images of $\mathbf{Z}_p$'s are trivial (either $\mathbf{Z}_p$ or a zero ring up to isomorphism) we have $SPHS(\mathcal{K}) = SP(\mathcal{K})$. The ring $\mathbf{F}$ being a field is simple and therefore subdirectly irreducible. Hence, if $\mathbf{F}$ were in $SPHS(\mathcal{K})$, then $\mathbf{F}$ would be in $S(\mathcal{K})$ which is a contradiction. So, $\mathbf{F} \in HP(\mathcal{K})$ and $\mathbf{F} \notin SPHS(\mathcal{K})$ which shows (2).

To prove (1) we need the following results.

Claim 2.3. $HSP(\mathcal{K}) = \mathcal{R}_c$.

Claim 2.4. If $\mathbf{R}$ is a ring such that $Char(\mathbf{R}) = 2$ and $\mathbf{R} \in SHP(\mathcal{K})$, then $\mathbf{R} \in HSP(\mathbf{Z}_2)$.

To prove Claim 2.3 suppose that $HSP(\mathcal{K})$ is a proper subvariety of $\mathcal{R}_c$. Then there is an identity $p = q$ where $p, q \in Z[x_1, \ldots, x_l]$ for some l such that $\mathcal{R}_c \nvDash p = q$ and $HSP(\mathcal{K}) \vDash p = q$ or equivalently there is a polynomial $0 \neq f \in Z[x_1, \ldots, x_l]$ such that $\mathcal{R}_c \nvDash f = 0$ and $HSP(\mathcal{K}) \vDash f = 0$. Since $\mathbf{Z}_p \vDash f = 0$ for all $p \in P$ we have $\prod_{p\in P} \mathbf{Z}_p/\mathcal{U} \vDash f = 0$. Therefore we have the field $\prod_{p\in P} \mathbf{Z}_p/\mathcal{U}$ of characteristic zero satisfying $f = 0$ which implies that f is to be the zero polynomial. This contradiction proves Claim 2.3.

To prove Claim 2.4 assume that $\mathbf{R} \in SHP(\mathcal{K})$ and let $\mathbf{R} \leq \mathbf{M} \in HP(\mathcal{K})$. Since $Char(\mathbf{R}) = 2$ and our rings are with identity, we will have $Char(\mathbf{M}) = 2$. Let $\alpha : \prod_{i\in I} \mathbf{R}_i \to \mathbf{M}$ be an epimorphism, where $\mathbf{R}_i \in \mathcal{K}$ for all $i \in I$. Let us denote by $J = \{i \in I \mid \mathbf{R}_i = \mathbf{Z}_2\}$ and let $\overline{\mathbf{R}}_i = \mathbf{R}_i$ if $i \notin J$ and $\overline{\mathbf{R}}_i = \mathbf{0}$, a zero ring, if $i \in J$. We claim that $\prod_{i\in I} \overline{R}_i \subseteq Ker(\alpha)$. If $x \in \prod_{i\in I} \overline{R}_i$, then $\alpha(2x) = 2\alpha(x) = 0$ and hence $2x \in Ker(\alpha)$. Since $\mathbf{R}_i \in \mathcal{K} - \{\mathbf{Z}_2\}$ are fields, in $\mathbf{R}_i$ we will have 2^{-1}, the inverse of 2. Now define an element $\mathbf{2}^{-1} \in \prod_{i\in I} R_i$ as follows: if $i \in J$, then $\mathbf{2}^{-1}(i) = 0$, otherwise $\mathbf{2}^{-1}(i) = 2^{-1}$. Hence $\mathbf{2}^{-1}(2x) = x \in Ker(\alpha)$ because $Ker(\alpha)$ is an ideal. Therefore, $\prod_{i\in I} \overline{R}_i \subseteq Ker(\alpha)$. Since $\prod_{i\in I} \overline{R}_i$ is also an ideal of the ring $\prod_{i\in I} \mathbf{R}_i$, we have

$$\mathbf{M} \cong \prod_{i\in I} \mathbf{R}_i/\mathbf{Ker}(\alpha) \cong (\prod_{i\in I} \mathbf{R}_i/\prod_{i\in I} \overline{\mathbf{R}}_i)/(\mathbf{Ker}(\alpha)/\prod_{i\in I} \overline{\mathbf{R}}_i)$$

and hence

$$\mathbf{M} \in H(\prod_{i\in I} \mathbf{R}_i/\prod_{i\in I} \overline{\mathbf{R}}_i) = H(\prod_{i\in I} (\mathbf{R}_i/\overline{\mathbf{R}}_i)) = H(\mathbf{Z}_2^J) \subseteq HP(\mathbf{Z}_2).$$

Now $\mathbf{R} \in S(\mathbf{M})$ implies $\mathbf{R} \in SHP(\mathbf{Z}_2) = HSP(\mathbf{Z}_2)$ which proves the claim.

For $n \geq 2$, the Galois field $\mathbf{GF}(2^n) \in HSP(\mathcal{K})$, and if $\mathbf{GF}(2^n)$ were in $SHPS(\mathcal{K})$, then $\mathbf{GF}(2^n) \in HSP(\mathbf{Z}_2)$. Since $HSP(\mathbf{Z}_2) \vDash x^2 = x$, we would have $\mathbf{GF}(2^n)$ is idempotent for $n \geq 2$ which is not true. Finally, we have that $\mathbf{GF}(2^n) \in HSP(\mathcal{K})$ for $n \geq 2$, and $\mathbf{GF}(2^n) \notin SHPS(\mathcal{K})$. $\square$

To prove that the monoid of operators H, S, P, P_s of the variety $\mathcal{R}_c$ is full we will use the following.

Lemma 2.5 (see [10, Lemma 3.1])**.** *Let $\mathcal{K}$ be any variety. Then $\mathcal{M}_s(\mathcal{K}) \cong \mathcal{M}_s$ if and only if there exist classes of algebras $\mathcal{K}_1$, $\mathcal{K}_2 \subseteq \mathcal{K}$ satisfying the non-inclusions*

$$HSP(\mathcal{K}_1) \nsubseteq SHPS(\mathcal{K}_1), \tag{3}$$

$$SHP(\mathcal{K}_2) \nsubseteq P_sHPS(\mathcal{K}_2). \tag{4}$$

Now we can prove

Theorem 2.6. *There exist classes $\mathcal{K}_1$ and $\mathcal{K}_2$ of commutative rings with identity satisfying the non-inclusions* (3) *and* (4)*. Consequently, the monoid of operators H, S, P, P_s is full, i.e., $\mathcal{M}_s(\mathcal{R}_c) \cong \mathcal{M}$.*

Proof. We have already showed the non-inclusion (3) in Theorem 2.2. To show (4) we will use the following class: let $P^* = \{p \in P \mid -1 \text{ is a square in } Z_p\}$ and let $\mathcal{K}^* = \{\mathbf{Z}_p \mid p \in P^*\}$. The set P^* is infinite because odd primes p such that $p \equiv 1(mod 4)$ satisfy $p = a^2 + b^2$ for some a and b and hence $1 + (b/a)^2 = 0$ in Z_p. Since P^* is an infinite subset of primes $HP(\mathcal{K}^*)$ contains fields of characteristic zero. This implies that $\mathbf{Q} \in SHP(\mathcal{K}^*)$, where $\mathbf{Q}$ is the field of rationals. On the other hand, $P_sHPS(\mathcal{K}^*) = P_sHP(\mathcal{K}^*)$ and the property "-1 is a square" is preserved under direct products and homomorphic images. Now, if $\mathbf{F}$ is a field and $\mathbf{F} \in P_sHP(\mathcal{K}^*)$, then $\mathbf{F} \in HP(\mathcal{K}^*)$ because $\mathbf{F}$ being a field is a subdirectly irreducible ring. So, it will also hold that "-1 is a square in F" which implies that $\mathbf{Q}$ cannot be in $P_sHPS(\mathcal{K}^*) = P_sHP(\mathcal{K}^*)$. This shows the non-inclusion (4) and completes the proof. □

Appendix

The following result was obtained by the second author and added to the paper on October 25, 2004.

The aim of this appendix is to give a simpler example for the inequality $HSP \neq SHPS$. To show that $HSP \neq SHPS$ in Theorem 2.2 we used an infinite class of finite rings. An advantage of that example is that it simultaneously showed both non-inclusions required in Theorem 2.1. We will prove that one can take the ring $\mathbf{Z}_8$ to show that $HSP(\mathbf{Z}_8) \neq SHPS(\mathbf{Z}_8)$.

Proposition 2.7. *If $\mathbf{Z}_8 = \langle Z_8, +, \cdot, -, 0, 1\rangle$ denotes the commutative ring with identity of integers mod 8, then $HSP(\mathbf{Z}_8) \neq SHPS(\mathbf{Z}_8)$.*

Proof. The Variety generated by $\mathbf{Z}_8$ is defined by the following identities:

$$\text{Axioms for Commutative rings with identity,} \tag{5}$$

$$(x^2 - x)(y^2 - y)(z^2 - z) = 0, \tag{6}$$

$$(x^2 - x)^2 = 2(x^2 - x). \tag{7}$$

This variety is residually large and Willard described in [11] all countable subdirectly irreducibles in $\mathcal{V} = HSP(\mathbf{Z}_8)$. We will use the fact that $\mathcal{V}$ has arbitrary large finite subdirectly irreducibles. Let $\mathbf{R} \in \mathcal{V}$ be a finite subdirectly irreducible ring such that $|R| > 8$. If $\mathbf{R} \in SHPS(\mathbf{Z}_8) = SHP(\mathbf{Z}_8)$, then $\mathbf{R} \in SHP_{fin}(\mathbf{Z}_8)$, where P_{fin} denotes the operator of taking finite products. Therefore, $\mathbf{R} \leq (\mathbf{Z}_8)^n/I$ for some $n \in N$, and some ideal I of $(\mathbf{Z}_8)^n$. Since commutative rings with identity have the Fraser Horn property (FHP for short) we have that $I = I_1 \times I_2 \times \cdots \times I_n$

for some ideals $I_1, \ldots, I_n$ of $\mathbf{Z}_8$. This gives us $(\mathbf{Z}_8)^n/I \cong \mathbf{Z}_8/I_1 \times \cdots \times \mathbf{Z}_8/I_n \in P_{fin}H(\mathbf{Z}_8) = P_{fin}(\{\mathbf{Z}_8, \mathbf{Z}_4, \mathbf{Z}_2, \mathbf{Z}_1\})$. Finally, $\mathbf{R} \in SP_{fin}(\{\mathbf{Z}_8, \mathbf{Z}_4, \mathbf{Z}_2, \mathbf{Z}_1\})$ and $\mathbf{R}$ being subdirectly irreducible give us $\mathbf{R} \in S(\{\mathbf{Z}_8, \mathbf{Z}_4, \mathbf{Z}_2, \mathbf{Z}_1\}) = I(\{\mathbf{Z}_8, \mathbf{Z}_4, \mathbf{Z}_2, \mathbf{Z}_1\})$. Contradiction, since we assumed $|R| > 8$. $\square$

The previous proposition can easily be generalized to

Proposition 2.8. *Let* $\mathbf{A}$ *be a finite algebra. If the variety* $HSP(\mathbf{A})$ *satisfies the FHP and contains a finite subdirectly irreducible algebra* $\mathbf{B}$ *such that* $|B| > |A|$, *then* $HSP(\mathbf{A}) \neq SHPS(\mathbf{A})$. *In particular, if* $\mathbf{A}$ *generates a residually large variety satisfying the FHP, then* $HSP(\mathbf{A}) \neq SHPS(\mathbf{A})$.

References

1. G. M. Bergman, *$HSP \neq SHPS$ for metabelian groups, and related results*, Algebra Universalis **26** (1989), 267-283. MR1044848 (91m:08007)
2. G. M. Bergman, *Partially ordered sets, and minimal systems of counterexamples*, Algebra Universalis **32** (1994), 13-30. MR1287014 (95j:06001)
3. S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer-Verlag, New York, 1981. MR0648287 (83k:08001)
4. S. D. Comer and J. S. Johnson, *The standard semigroup of operators of a variety*, Algebra Universalis **2** (1972), 77-79. MR308012 (46:7127)
5. G. Grätzer and H. Lakser, *The structure of pseudocomplemented distributive lattices. II: Congruence extension and amalgamation*, Trans. Amer. Math. Soc. **156** (1971), 343-358. MR0274359 (43:124)
6. W. Nemitz and T. Whaley, *Varieties of implicative semilattices*, Pacific J. Math. **37** (1971), 759-769. MR0311522 (47:84)
7. P. M. Neumann, *The inequality of $SQPS$ and QSP as operators on classes of groups*, Bull. Amer. Math. Soc. **76** (1970), 1067-1069. MR0272875 (42:7756)
8. D. Pigozzi, *On some operations on classes of algebras*, Algebra Universalis **2** (1972), 346-353. MR0316354 (47:4901)
9. R. W. Quackenbush, *Structure theory for equational classes generated by quasi-primal algebras*, Trans. Amer. Math. Soc. **187** (1974), 127-145. MR0327619 (48:5961)
10. B. Tasić, *On the partially ordered monoid generated by the operators H, S, P, P_s on classes of algebras*, Journal of Algebra **245** (2001), 1-19. MR1868180 (2002i:08001)
11. R. Willard, Three Lectures on the RS Problem, B. T. Hart et al. (eds.), Algebraic Model Theory, 231-254, 1997 Kluwer Academic Publishers. MR1481448 (99d:03029)

University of Waterloo, Department of Pure Mathematics, Waterloo, Ontario, Canada N2L 3G1

E-mail address: jwlawren@math.uwaterloo.ca

University of Waterloo, Department of Pure Mathematics, Waterloo, Ontario, Canada N2L 3G1

E-mail address: btasic@math.uwaterloo.ca