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# UNIFORMLY COMPLETE QUOTIENT SPACE $UCQ(G)$ AND COMPLETELY ISOMETRIC REPRESENTATIONS OF $UCQ(G)^*$ ON $\mathcal{B}(L_2(G))$

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ABSTRACT. The uniformly complete quotient space $UCQ(G)$ of a locally compact group $G$ is introduced. It is shown that the operator space dual $UCQ(G)^*$ is a completely contractive Banach algebra, which contains the completely bounded Fourier multiplier algebra $M_{cb}A(G)$ as a completely contractively complemented Banach subalgebra. A natural completely isometric representation of $UCQ(G)^*$ on $\mathcal{B}(L_2(G))$ is studied and some equivalent amenability conditions associated with $UCQ(G)$ are proved.

## 1. INTRODUCTION

Let $G$ be a locally compact group. It was shown by Størmer [23], Ghahramani [9], and Neufang [17] that there exists a natural completely isometric homomorphism $\Theta = \Theta_l$ (which is induced by the left regular representation $\lambda$) from the measure algebra $M(G)$ into the completely contractive Banach algebra $\mathcal{CB}^\sigma(\mathcal{B}(L_2(G)))$ of all normal completely bounded maps on $\mathcal{B}(L_2(G))$. We note that one may analogously obtain a completely isometric homomorphism $\Theta = \Theta_r$ from $M(G)$ into $\mathcal{CB}^\sigma(\mathcal{B}(L_2(G)))$ induced by the right regular representation $\rho$ (see [18]). As a corresponding duality result, it was also shown by Haagerup [11] and Spronk [22] that there exists a natural completely isometric homomorphism $\hat{\Theta}$ from the completely bounded Fourier multiplier algebra $M_{cb}A(G)$ into the same Banach algebra $\mathcal{CB}^\sigma(\mathcal{B}(L_2(G)))$. Moreover, the range spaces of $\Theta$ and $\hat{\Theta}$ in $\mathcal{CB}^\sigma(\mathcal{B}(L_2(G)))$ have been successfully characterized in the recent works of Neufang [17], and Neufang, the second author and Spronk [18]. Indeed, if we take $\Theta = \Theta_r$, then we can characterize these range spaces in the following perfect duality form:

(1.1)
$$\Theta_r(M(G)) = \mathcal{CB}^{\sigma, L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G))) \text{ and } \hat{\Theta}(M_{cb}A(G)) = \mathcal{CB}^{\sigma,\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G))),$$

where $L_\infty(G)$ is the commutative von Neumann algebra of all essentially bounded measurable functions on $G$ and $\mathcal{L}(G)$ is the group von Neumann algebra generated by the left regular representation $\lambda$ on $L_2(G)$. Throughout this paper, if we are given von Neumann algebras $\mathcal{R}$ and $\mathcal{M}$ on $L_2(G)$, we let $\mathcal{CB}^{\mathcal{M}}_{\mathcal{R}}(\mathcal{B}(L_2(G)))$ denote the space of all completely bounded $\mathcal{R}$-bimodule morphisms on $\mathcal{B}(L_2(G))$, which map

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$\mathcal{M}$ into $\mathcal{M}$, and we let $\mathcal{CB}_{\mathcal{R}}^{\sigma,\mathcal{M}}(\mathcal{B}(L_2(G)))$ denote the space of all normal morphisms in $\mathcal{CB}_{\mathcal{R}}^{\mathcal{M}}(\mathcal{B}(L_2(G)))$.

Let $LUC(G)$ denote the space of all left uniformly continuous functions on $G$. Then $LUC(G)^*$ is a completely contractive Banach algebra and $M(G)$ can be identified with a completely contractively complemented Banach subalgebra of $LUC(G)^*$. Neufang proved in [17] that $\Theta$ can be naturally extended to a completely isometric homomorphism $\tilde{\Theta}$ from $LUC(G)^*$ into $\mathcal{CB}_{\mathcal{L}(G)}^{L_\infty(G)}(\mathcal{B}(L_2(G)))$. However, it is still an open question whether $\tilde{\Theta}$ is onto $\mathcal{CB}_{\mathcal{L}(G)}^{L_\infty(G)}(\mathcal{B}(L_2(G)))$. To consider the duality of $LUC(G)^*$, we would first like to consider the $C^*$-algebra $UCB(\hat{G})$ introduced by Granirer [10]. This $C^*$-algebra is defined to be the norm closure of $A(G) \cdot \mathcal{L}(G)$ in $\mathcal{L}(G)$. It was shown by Lau [15] that the dual space $UCB(\hat{G})^*$ is a Banach algebra. Actually $UCB(\hat{G})^*$ with the canonical dual operator space structure is a completely contractive Banach algebra (see [18]). If $G$ is an amenable group, we may consider $UCB(\hat{G})^*$ as the dual object of $LUC(G)^*$. In this case, $M_{cb}A(G) = B(G)$ can be identified with a completely contractive Banach subalgebra of $UCB(\hat{G})^*$ and $\hat{\Theta}$ can be extended to a completely isometric homomorphism $\tilde{\hat{\Theta}}$ from $UCB(\hat{G})^*$ into $\mathcal{CB}_{L_\infty(G)}^{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$ (see [18, §6]). However, this may fail when $G$ is non-amenable. Lau proved in [15] that for any discrete group $G$, we have $UCB(\hat{G}) = C_\lambda^*(G)$ and thus we get

$$UCB(\hat{G})^* = B_\lambda(G) \subseteq B(G) \subseteq M_{cb}A(G). \tag{1.2}$$

Therefore, if $G$ is a non-amenable discrete group, the last two containing relations in (1.2) are proper and their norms are distinct. This is against our expectation to identify $M_{cb}A(G)$ with a completely contractive Banach subalgebra of $UCB(\hat{G})^*$. So it is interesting to find the appropriate dual object of $LUC(G)^*$ for general locally compact groups. This is the goal of this paper.

Motivated by [18, Lemma 6.1], we introduce the uniformly complete quotient space $UCQ(G)$ in §2. This space is defined to be a complete quotient of $\mathcal{T}(L_2(G))\hat{\otimes}\mathcal{B}(L_2(G))$, where $\mathcal{T}(L_2(G))$ is the operator predual of $\mathcal{B}(L_2(G))$. If $G$ is an amenable group, we have $UCQ(G) = UCB(\hat{G})$. In general, $UCQ(G)$ is not necessarily a Banach algebra. There is a natural operator space left $\mathcal{T}(L_2(G))$-module structure on $UCQ(G)$, with which we can define a completely contractive Banach algebra structure on its dual space $UCQ(G)^*$. We show in Theorem 2.3 that for any locally compact group $G$, there is a weak*-weak* continuous completely isometric monomorphism $\tilde{\hat{\Theta}}$ from $UCQ(G)^*$ into $\mathcal{CB}_{L_\infty(G)}^{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$.

It is known from [13] and [2] that $M_{cb}A(G)$ is a dual space with a predual $Q(G)$. Moreover, it was shown in [14] (for general Kac algebras) that there exists a canonical operator space structure on $Q(G)$ inherited from $M_{cb}A(G)^*$ such that $M_{cb}(A(G))$ is completely isometric to the dual operator space $Q(G)^*$. We study the connection between $Q(G)$ and $UCQ(G)$ in §3. We show in Theorem 3.2 that for any locally compact group $G$, the regular representation $\lambda$ induces a completely isometric injection $\pi_\lambda$ from $Q(G)$ into $UCQ(G)$ and there is a canonical completely isometric inclusion $\iota : M_{cb}A(G) \hookrightarrow UCQ(G)^*$ such that $\pi_\lambda^* \circ \iota = id_{M_{cb}A(G)}$. Therefore we may identify $M_{cb}A(G)$ with a completely contractively complemented Banach subalgebra in $UCQ(G)^*$. With this identification we have $\hat{\Theta} = \tilde{\hat{\Theta}}|_{M_{cb}A(G)}$. This shows that $UCQ(G)^*$ acts perfectly as the dual of $LUC(G)^*$.

Finally, we discuss some equivalent amenability conditions related to $UCQ(G)$ in §4. We show in Theorem 4.1 that $G$ is amenable if and only if $UCQ(G)$ is (Banach/operator space) isomorphic to a norm closed subspace of $\mathcal{L}(G)$, and show in Theorem 4.2 that $G$ is amenable if and only if $UCB(\hat{G})$ is (Banach/operator space) isomorphic to a norm closed subspace of $UCQ(G)$.

We note that operator space theory plays an important role in this paper. Readers are referred to the recent books [5], [19] and [20] for the fundamental results in operator spaces and are referred to papers [1], [3], [4], [6], [18] and [21] for the extended and normal Haagerup tensor products of von Neumann algebras $\mathcal{R} \subseteq \mathcal{B}(L_2(G))$ and their connections with the completely bounded $\mathcal{R}'$-bimodule morphisms on $\mathcal{B}(L_2(G))$.

## 2. Uniformly complete quotient space $UCQ(G)$

It is known from the Kac algebra theory (see [7]) that for any locally compact group $G$, there is an important *fundamental unitary operator* $W$ on $L_2(G \times G)$ defined by $W\zeta(s,t) = \zeta(s,st)$ for $\zeta \in L_2(G \times G)$. The operator $W$ is contained in $L_\infty(G)\bar{\otimes}\mathcal{L}(G)$. For any $f \in L_1(G)$, we may define a *right slice map* $R_f$ : $L_\infty(G)\bar{\otimes}\mathcal{L}(G) \to \mathcal{L}(G)$ given by

$$R_f(x \otimes y) = f(x)y = \langle f \otimes id, x \otimes y\rangle.$$

With this notation, we may write

$$\lambda(f) = R_f(W^*) = \langle f \otimes id, W^*\rangle \quad \text{and} \quad \hat{\kappa}(\lambda(f)) = R_f(W) = \langle f \otimes id, W\rangle,$$

where we let $\hat{\kappa}$ denote the *co-involution* $\hat{\kappa}(\lambda(s)) = \lambda(s^{-1})$ on $\mathcal{L}(G)$. The operator $W$ satisfies the *pentagonal relation*

$$W_{23}W_{13}W_{12} = W_{12}W_{23}, \tag{2.1}$$

where we let $W_{12} = W \otimes 1$, $W_{23} = 1 \otimes W$ and $W_{13} = (\sigma \otimes 1)W_{23}(\sigma \otimes 1)$, and we let $\sigma$ be the *flip map* $\sigma\zeta(s,t) = \zeta(t,s)$ on $L_2(G \times G)$.

We denote by $\hat{W} = \sigma W^* \sigma$ the *dual fundamental unitary operator* of $W$. Then $\hat{W}$ also satisfies the pentagonal relation (2.1) and we may define a normal unital completely isometric *-homomorphism

$$\hat{\Gamma}(x) = \hat{W}(1 \otimes x)\hat{W}^* \tag{2.2}$$

from $\mathcal{B}(L_2(G))$ into $\mathcal{B}(L_2(G))\bar{\otimes}\mathcal{B}(L_2(G))$. The pentagonal relation implies that $\hat{\Gamma}$ is *co-associative*, i.e., it satisfies

$$(\hat{\Gamma} \otimes id) \circ \hat{\Gamma} = (id \otimes \hat{\Gamma}) \circ \hat{\Gamma}. \tag{2.3}$$

The preadjoint of $\hat{\Gamma}$ defines an associative completely contractive multiplication

$$m_{\hat{\Gamma}} = \hat{\Gamma}_* : \mathcal{T}(L_2(G)) \times \mathcal{T}(L_2(G)) \to \mathcal{T}(L_2(G))$$

on $\mathcal{T}(L_2(G))$. This also determines a completely contractive $\mathcal{T}(L_2(G))$-bimodule action on $\mathcal{B}(L_2(G))$ given by

$$\langle \omega \cdot x, \omega'\rangle = \langle x, m_{\hat{\Gamma}}(\omega' \otimes \omega)\rangle \text{ and } \langle x \cdot \omega, \omega'\rangle = \langle x, m_{\hat{\Gamma}}(\omega \otimes \omega')\rangle \tag{2.4}$$

for $\omega, \omega' \in \mathcal{T}(L_2(G))$ and $x \in \mathcal{B}(L_2(G))$. Equivalently, we can write

$$\omega \cdot x = \langle id \otimes \omega, \hat{\Gamma}(x)\rangle \quad \text{and} \quad x \cdot \omega = \langle \omega \otimes id, \hat{\Gamma}(x)\rangle.$$

If we restrict $\hat{\Gamma}$ to $\mathcal{L}(G)$, we obtain the *co-multiplication*

$$\hat{\Gamma}(\lambda(s)) = \lambda(s) \otimes \lambda(s) \tag{2.5}$$

on $\mathcal{L}(G)$ and $m_{\hat{\Gamma}}$ induces the (pointwise) completely contractive multiplication on $A(G)$.

It was observed in [18] that since $\hat{W} \in \mathcal{L}(G)\bar{\otimes}L_\infty(G)$, the left $\mathcal{T}(L_2(G))$-module action

$$\begin{aligned} \omega \otimes x \in \mathcal{T}(L_2(G)) \otimes \mathcal{B}(L_2(G)) &\mapsto \omega \cdot x \\ &= \langle id \otimes \omega, \hat{\Gamma}(x)\rangle = \langle id \otimes \omega, \hat{W}(1 \otimes x)\hat{W}^*\rangle \in \mathcal{B}(L_2(G)) \end{aligned} \tag{2.6}$$

defines a complete contraction $\hat{\mathcal{S}}$ from $\mathcal{T}(L_2(G))\hat{\otimes}\mathcal{B}(L_2(G))$ into $\mathcal{L}(G)$. We define $UCQ(G)$ to be the range space $\hat{\mathcal{S}}(\mathcal{T}(L_2(G))\hat{\otimes}\mathcal{B}(L_2(G)))$ in $\mathcal{L}(G)$ and assume that $UCQ(G)$ is equipped with the complete quotient norm from $\mathcal{T}(L_2(G))\hat{\otimes}\mathcal{B}(L_2(G))/\ker\hat{\mathcal{S}}$. In general, the norm on $UCQ(G)$ is bigger than the norm on $\mathcal{L}(G)$ and $UCQ(G)$ may not be a norm closed subspace of $\mathcal{L}(G)$.

It was shown in [18, Lemma 6.1] that if $G$ is an amenable group, $\hat{\mathcal{S}}$ is a complete quotient from $\mathcal{T}(L_2(G))\hat{\otimes}\mathcal{B}(L_2(G))$ onto $UCB(\hat{G})$ and thus we have $UCQ(G) = UCB(\hat{G})$. In this case, the norm on $UCQ(G)$ coincides with the norm on $\mathcal{L}(G)$. Conversely, we show in Theorem 4.1 that if $UCQ(G)$ is a closed subspace of $\mathcal{L}(G)$ (i.e. the norm on $UCQ(G)$ is equivalent to the norm on $\mathcal{L}(G)$), then $G$ must be an amenable group. We also show in Theorem 4.2 that $G$ is amenable if and only if $UCB(\hat{G})$ is (Banach/operator space) isomorphic to a norm closed subspace of $UCQ(G)$.

We note from (2.3) and (2.6) that the map $\hat{\mathcal{S}}$ induces a completely contractive left $\mathcal{T}(L_2(G))$-module action

$$\omega \otimes x \in \mathcal{T}(L_2(G))\hat{\otimes}UCQ(G) \mapsto \omega \cdot x \in UCQ(G) \tag{2.7}$$

on $UCQ(G)$. With this action, we can define a completely contractive Banach algebra structure on the dual space $UCQ(G)^*$. More precisely, given any $x \in \mathcal{B}(L_2(G))$ and $n \in UCQ(G)^*$ we may define an operator $n \diamond x$ on $L_2(G)$ by letting

$$\langle n \diamond x, \omega\rangle = \langle n, \omega \cdot x\rangle \tag{2.8}$$

for all $\omega \in \mathcal{T}(L_2(G))$. Then for each $n \in UCQ(G)^*$,

$$x \in \mathcal{B}(L_2(G)) \mapsto n \diamond x \in \mathcal{B}(L_2(G))$$

defines a completely bounded map, which is denoted by $\tilde{\hat{\Theta}}(n)$, on $\mathcal{B}(L_2(G))$. It is easy to see from (2.8) that $\tilde{\hat{\Theta}} = \hat{\mathcal{S}}^*$. Since $\hat{\mathcal{S}}$ is a complete quotient from $\mathcal{T}(L_2(G))\hat{\otimes}\mathcal{B}(L_2(G))$ onto $UCQ(G)$, it is clear that $\tilde{\hat{\Theta}}$ is a weak*-weak* continuous complete isometry from $UCQ(G)^*$ into $\mathcal{CB}(\mathcal{B}(L_2(G))) = (\mathcal{T}(L_2(G))\hat{\otimes}\mathcal{B}(L_2(G)))^*$.

**Lemma 2.1.** *Given $n \in UCQ(G)^*$, we have*

$$n \diamond (\omega \cdot x) = \omega \cdot (n \diamond x) \tag{2.9}$$

*for all $\omega \in \mathcal{T}(L_2(G))$ and $x \in \mathcal{B}(L_2(G))$. Therefore, $\tilde{\hat{\Theta}}(n)$ is a completely bounded left $\mathcal{T}(L_2(G))$-module homomorphism on $\mathcal{B}(L_2(G))$ which maps $UCQ(G)$ into $UCQ(G)$.*

*Proof.* Given $n \in UCQ(G)^*$, $\omega \in \mathcal{T}(L_2(G))$ and $x \in \mathcal{B}(L_2(G))$, we have

$$\begin{aligned}\langle n \diamond (\omega \cdot x), \omega' \rangle &= \langle n, \omega' \cdot (\omega \cdot x) \rangle = \langle n, (\omega' \omega) \cdot x \rangle \\ &= \langle n \diamond x, \omega' \omega \rangle = \langle \omega \cdot (n \diamond x), \omega' \rangle\end{aligned}$$

for all $\omega' \in \mathcal{T}(L_2(G))$, where we used (2.4) in the last equality. This shows that $n \diamond (\omega \cdot x) = \omega \cdot (n \diamond x)$, and thus $\tilde{\hat{\Theta}}(n)$ is a completely bounded left $\mathcal{T}(L_2(G))$-module morphism on $\mathcal{B}(L_2(G))$.

Let $x$ be an arbitrary element in $UCQ(G)$ with $\|x\|_{UCQ(G)} < 1$. It is known from the definition that there exist contractive $\alpha = [\alpha_{ik}] \in M_{1,\infty \times \infty}(\mathbb{C})$, $[\omega_{ij}] \in \mathcal{K}_\infty(\mathcal{T}(L_2(G)))$, $[y_{kl}] \in \mathcal{K}_\infty(\mathcal{B}(L_2(G)))$ and $\beta = [\beta_{jl}] \in M_{\infty \times \infty, 1}(\mathbb{C})$ such that

$$x = \hat{\mathcal{S}}\left(\alpha\, [\omega_{ij} \otimes y_{kl}]\, \beta\right) = \alpha\, [\omega_{ij} \cdot y_{kl}]\, \beta = \sum_{i,j,k,l=1}^{\infty} \alpha_{ik} (\omega_{ij} \cdot y_{kl}) \beta_{jl},$$

where the sum converges in the norm topology. Therefore, we can conclude that

$$\tilde{\hat{\Theta}}(n)(x) = \tilde{\hat{\Theta}}(n) \left( \sum_{i,j,k,l=1}^{\infty} \alpha_{ik} (\omega_{ij} \cdot y_{kl}) \beta_{jl} \right) = \sum_{i,j,k,l=1}^{\infty} \alpha_{ik} \left( \omega_{ij} \cdot \tilde{\hat{\Theta}}(n)\,(y_{kl}) \right) \beta_{jl}$$

is an element in $UCQ(G)$. Therefore, $\tilde{\hat{\Theta}}(n)$ maps $UCQ(G)$ into $UCQ(G)$. $\square$

As a consequence of Lemma 2.1, it is easy to see that

$$n \otimes x \in UCQ(G)^* \otimes UCQ(G) \to n \diamond x \in UCQ(G)$$

extends to a complete contraction from $UCQ(G)^* \hat{\otimes} UCQ(G)$ into $UCQ(G)$. Now given $m, n \in UCQ(G)^*$, we can define $m \diamond n \in UCQ(G)^*$ by letting

$$\langle m \diamond n,\, x \rangle = \langle m,\, n \diamond x \rangle \tag{2.10}$$

for all $x \in UCQ(G)$. In the following we show that this defines a completely contractive Banach algebra structure on $UCQ(G)^*$. Let us first consider the following lemma.

**Lemma 2.2.** *For any $m, n \in UCQ(G)^*$ and $x \in \mathcal{B}(L_2(G))$, we have*

$$(m \diamond n) \diamond x = m \diamond (n \diamond x). \tag{2.11}$$

*This shows that*

$$\tilde{\hat{\Theta}}(m \diamond n) = \tilde{\hat{\Theta}}(m) \circ \tilde{\hat{\Theta}}(n). \tag{2.12}$$

*Proof.* The lemma follows from the calculation

$$\begin{aligned}\langle (m \diamond n) \diamond x, \omega \rangle &= \langle m \diamond n, \omega \cdot x \rangle = \langle m, n \diamond (\omega \cdot x) \rangle \\ &= \langle m, \omega \cdot (n \diamond x) \rangle = \langle m \diamond (n \diamond x), \omega \rangle\end{aligned}$$

for all $\omega \in \mathcal{T}(L_2(G))$. $\square$

Now we can obtain the following theorem, which generalizes [18, Theorem 6.2] to arbitrary locally compact groups.

**Theorem 2.3.** *Let $G$ be a locally compact group. Then $UCQ(G)^*$ is an associative completely contractive Banach algebra and the map $\tilde{\hat{\Theta}}$ is a weak*-weak* continuous completely isometric monomorphism from $UCQ(G)^*$ into $\mathcal{CB}^{\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G)))$.*

*Proof.* It follows from Lemma 2.2 that for any $n_i \in UCQ(G)^*$ $(i = 1, 2, 3)$ and $x \in UCQ(G)$, we have

$$\begin{aligned}\langle (n_1 \diamond n_2) \diamond n_3, x\rangle &= \langle n_1 \diamond n_2, n_3 \diamond x\rangle = \langle n_1, n_2 \diamond (n_3 \diamond x)\rangle \\ &= \langle n_1, (n_2 \diamond n_3) \diamond x\rangle = \langle n_1 \diamond (n_2 \diamond n_3), x\rangle.\end{aligned}$$

Therefore, $m \otimes n \mapsto m \diamond n$ defines an associative completely contractive multiplication on $UCQ(G)^*$ and $\tilde{\hat{\Theta}}$ defines a weak*-weak* continuous completely isometric monomorphism from $UCQ(G)^*$ into $\mathcal{CB}(\mathcal{B}(L_2(G)))$.

Since $\hat{W}$ is a unitary operator contained in $\mathcal{L}(G)\bar{\otimes}L_\infty(G)$, for any $f, g \in L_\infty(G)$ and $x \in \mathcal{B}(L_2(G))$, we have

$$\hat{W}(1 \otimes fxg)\hat{W}^* = (1 \otimes f)\hat{W}(1 \otimes x)\hat{W}^*(1 \otimes g).$$

Then for any $n \in UCQ(G)^*$, we have

$$\begin{aligned}\langle \tilde{\hat{\Theta}}(n)(fxg), \omega\rangle &= \langle n, \langle id \otimes \omega, \hat{W}(1 \otimes fxg)\hat{W}^*\rangle\,\rangle \\ &= \langle n, \langle id \otimes \omega, \Big((1 \otimes f)\hat{W}(1 \otimes x)\hat{W}^*(1 \otimes g)\Big)\rangle\,\rangle \\ &= \langle f\, \tilde{\hat{\Theta}}(n)(x)\, g, \omega\rangle\end{aligned}$$

for all $\omega \in \mathcal{T}(L_2(G))$. This shows that

$$\tilde{\hat{\Theta}}(n)(fxg) = f\tilde{\hat{\Theta}}(n)(x)g,$$

i.e. $\tilde{\hat{\Theta}}(n)$ is a completely bounded $L_\infty(G)$-bimodule morphism on $\mathcal{B}(L_2(G))$.

Moreover, we claim that for any $n \in UCQ(G)^*$, $\tilde{\hat{\Theta}}(n)$ maps $\mathcal{L}(G)$ into $\mathcal{L}(G)$. Indeed, if $x \in \mathcal{L}(G)$, then $\hat{\Gamma}(x)$ is an operator contained in $\mathcal{L}(G)\bar{\otimes}\mathcal{L}(G)$ and thus $\tilde{\hat{\Theta}}(n)(x) = n \diamond x$ is an operator in $\mathcal{B}(L_2(G))$ which is null on $\mathcal{L}(G)_\perp = \{\omega \in \mathcal{T}(L_2(G)) : \omega(x) = 0 \text{ for all } x \in \mathcal{L}(G)\}$. This implies that $\tilde{\hat{\Theta}}(n)(x) = n \diamond x \in \mathcal{L}(G) = (\mathcal{L}(G)_\perp)^\perp$. □

## 3. The connection between $Q(G)$ and $UCQ(G)$

A function $\varphi : G \to \mathbb{C}$ is called a *multiplier* of $A(G)$ if the induced (pointwise) multiplication

$$m_\varphi(\psi) = \varphi\psi$$

maps $A(G)$ into $A(G)$. It is known from the closed graph theorem that $m_\varphi$ is automatically bounded on $A(G)$. A multiplier $\varphi$ is said to be *completely bounded* if $\|m_\varphi\|_{cb} < \infty$. We let $M_{cb}A(G)$ denote the space of all completely bounded multipliers on $A(G)$. Then $M_{cb}A(G)$ is a completely contractive Banach algebra (contained in $\mathcal{CB}(A(G))$). It is known from Herz [13] and De Cannière and Haagerup [2] that $M_{cb}A(G)$ is actually a dual space with a canonical predual $Q(G)$, which is defined to be the closure of $L_1(G)$ under the norm

$$\|f\|_{Q(G)} = \left\{\left|\int_G f(s)\varphi(s)ds\right| : \varphi \in M_{cb}A(G), \|\varphi\|_{cb} \le 1\right\}.$$

Moreover, Kraus and the second author proved in [14] that (even for general Kac algebras) there exists a natural operator space matrix norm on $Q(G)$ inherited from $M_{cb}A(G)^*$ with which $M_{cb}A(G)$ is completely isometric to $Q(G)^*$. Therefore, $Q(G)$ is an operator predual of $M_{cb}A(G)$.

Given $\varphi \in M_{cb}A(G)$, we let $u_\varphi(s,t) = \varphi(st^{-1})$ denote the continuous *right invariant Schur multiplier* associated with $\varphi$. Then $u_\varphi$ can be identified with a *right invariant* element (i.e. $u_\varphi(sg,tg) = u_\varphi(s,t)$ for all $g \in G$) in the extended Haagerup tensor product $L_\infty(G) \otimes^{eh} L_\infty(G)$ and thus can be written as $u_\varphi = \sum_{k\in I} v_k \otimes w_k$ for some index set $I$ and $v_k, w_k \in L_\infty(G)$ (see details in [18, §4]). In general, every $u = \sum_{k\in I} v_k \otimes w_k \in L_\infty(G) \otimes^{eh} L_\infty(G)$ uniquely corresponds to a normal completely bounded $L_\infty(G)$-bimodule morphism $T(u) \in \mathcal{CB}^\sigma_{L_\infty(G)}(\mathcal{B}(L_2(G)))$, which is defined by

$$T(u)(x) = \sum_{k\in I} v_k x w_k, \qquad x \in \mathcal{B}(L_2(G)). \tag{3.1}$$

Here $T$ is a weak*-weak* continuous completely isometric isomorphism from $L_\infty(G) \otimes^{eh} L_\infty(G)$ onto $\mathcal{CB}^\sigma_{L_\infty(G)}(\mathcal{B}(L_2(G)))$ (see [6], [11], [18] and [21]). It was shown in [18, §4] that $\hat{\Theta}(\varphi) = T(u_\varphi)$ defines a weak*-weak* continuous completely isometric isomorphism

$$\hat{\Theta} : M_{cb}A(G) \cong \mathcal{CB}^{\sigma,\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G))).$$

Since we have the completely isometric (linear) isomorphism $L_\infty(G) \otimes^{eh} L_\infty(G) \cong (L_1(G) \otimes^h L_1(G))^*$ and the completely isometric inclusion

$$\mathcal{CB}^\sigma_{L_\infty(G)}(\mathcal{B}(L_2(G))) \hookrightarrow \mathcal{CB}(\mathcal{K}(L_2(G)), \mathcal{B}(L_2(G))) \cong (\mathcal{T}(L_2(G)) \hat{\otimes} \mathcal{K}(L_2(G)))^*,$$

the preadjoint of $T$ determines a complete quotient map

$$T_* : \mathcal{T}(L_2(G)) \hat{\otimes} \mathcal{K}(L_2(G)) \to L_1(G) \otimes^h L_1(G).$$

The map $T_*$ can be expressed as follows. If we are given any $\omega_{\xi,\eta} \in \mathcal{T}(L_2(G))$ and any rank one operator $x_{\xi',\eta'}$ in $\mathcal{K}(L_2(G))$ (which is defined by $x_{\xi',\eta'}(\zeta) = \langle \zeta \mid \eta' \rangle \xi'$), then we get

$$T_*(\omega_{\xi,\eta} \otimes x_{\xi',\eta'}) = f \otimes g \tag{3.2}$$

with $f(s) = \xi'(s)\bar{\eta}(s)$ and $g(t) = \xi(t)\bar{\eta}'(t)$ in $L_1(G)$. To see this, let us assume that $u = \sum_{k\in I} v_k \otimes w_k$ is an element in $L_\infty(G) \otimes^{eh} L_\infty(G) \cong (L_1(G) \otimes^h L_1(G))^*$. Then

$$\begin{aligned}
\langle u, T_*(\omega_{\xi,\eta} \otimes x_{\xi',\eta'}) \rangle &= \langle T(u)(x_{\xi',\eta'}), \omega_{\xi,\eta} \rangle \\
&= \langle \sum_{k\in I} v_k x_{\xi',\eta'} w_k \xi \mid \eta \rangle = \sum_{k\in I} \langle v_k \xi' \mid \eta \rangle \langle w_k \xi \mid \eta' \rangle \\
&= \sum_{k\in I} \int_G v_k(s)\xi'(s)\bar{\eta}(s) ds \int_G w_k(t)\xi(t)\bar{\eta}'(t) dt \\
&= \sum_{k\in I} \langle v_k, f \rangle \langle w_k, g \rangle = \langle u, f \otimes g \rangle.
\end{aligned}$$

This proves (3.2).

Let $\kappa$ denote the *co-involution* $\kappa(f)(t) = f(t^{-1})$ on $L_\infty(G)$. Then $\kappa_*$ is an isometry on $L_1(G)$ such that $\kappa_*(g)(t) = g(t^{-1})\Delta(t^{-1})$ for all $g \in L_1(G)$. It was shown in [18, §4] that

$$m \circ (id \otimes \kappa_*) : f \otimes g \in L_1(G) \otimes L_1(G) \to f * \kappa_*(g) \in L_1(G)$$

can be extended to a complete quotient $m_{id\otimes\kappa_*}$ from $L_1(G) \otimes^h L_1(G)$ onto $Q(G)$. We note that Spronk also observed this result in [22, §6] by taking a quite different argument. We also note that since for any $[f_{ij}] \in M_n(L_1(G))$, we have

$$\|[\lambda(f_{ij})]\| \le \|[f_{ij}]\|_{M_n(Q(G))},$$

the regular representation $\lambda : L_1(G) \to C^*_\lambda(G)$ induces a complete contraction

$$\pi_\lambda : Q(G) \to C^*_\lambda(G) \subseteq \mathcal{L}(G).$$

Then the composition map

$$\pi_\lambda \circ m_{id\otimes\kappa_*} : L_1(G) \otimes^h L_1(G) \to C^*_\lambda(G) \subseteq \mathcal{L}(G)$$

is the natural extension of $m \circ (\lambda \otimes \hat{\kappa} \circ \lambda) = \lambda \circ m \circ (id \otimes \kappa_*)$. We recall that $\hat{\kappa}$ is the co-involution on $\mathcal{L}(G)$ given by $\hat{\kappa}(\lambda(s)) = \lambda(s^{-1})$. For any $\omega_{\xi,\eta} \in \mathcal{T}(L_2(G))$ and $x_{\xi',\eta'} \in \mathcal{K}(L_2(G))$, we let $f(s) = \xi'(s)\bar{\eta}(s)$ and $g(t) = \xi(t)\bar{\eta}'(t)$ and thus obtain

$$\begin{aligned}
\omega_{\xi,\eta} \cdot x_{\xi',\eta'} &= \hat{\mathcal{S}}(\omega_{\xi,\eta} \otimes x_{\xi',\eta'}) \\
&= \langle id \otimes \omega_{\xi,\eta}, \hat{W}(1 \otimes x_{\xi',\eta'})\hat{W}^*\rangle \\
&= \Big((id \otimes \eta^*)\hat{W}(id \otimes \xi')\Big)\Big((id \otimes {\eta'}^*)\hat{W}^*(id \otimes \xi)\Big) \\
&= ((\eta^* \otimes id)W^*(\xi' \otimes id))\big(({\eta'}^* \otimes id)W(\xi \otimes id)\big) \\
&= \lambda(f)\hat{\kappa}(\lambda(g)) = \pi_\lambda \circ m_{id\otimes\kappa_*}(f \otimes g).
\end{aligned}$$

It follows from (3.2) that

$$\omega_{\xi,\eta} \cdot x_{\xi',\eta'} = \hat{\mathcal{S}}(\omega_{\xi,\eta} \otimes x_{\xi',\eta'}) = \pi_\lambda \circ m_{id\otimes\kappa_*} \circ T_*(\omega_{\xi,\eta} \otimes x_{\xi',\eta'}). \tag{3.3}$$

This shows that

$$\hat{\mathcal{S}} = \pi_\lambda \circ m_{id\otimes\kappa_*} \circ T_*,$$

i.e. we have the following commutative diagram of complete contractions

$$\begin{array}{ccc}
\mathcal{T}(L_2(G))\hat{\otimes}\mathcal{K}(L_2(G)) & \xrightarrow{\hat{\mathcal{S}}} & \mathcal{L}(G) \\
{\scriptstyle T_*}\big\downarrow & & \big\uparrow{\scriptstyle \pi_\lambda} \\
L_1(G) \otimes^h L_1(G) & \xrightarrow{m_{id\otimes\kappa_*}} & Q(G).
\end{array} \tag{3.4}$$

Therefore, $\pi_\lambda(Q(G)) = \hat{\mathcal{S}}(\mathcal{T}(L_2(G))\hat{\otimes}\mathcal{K}(L_2(G)))$ is actually a subspace of $UCQ(G)$. In the following we show that $\pi_\lambda$ is a complete isometry from $Q(G)$ into $UCQ(G)$ and $\hat{\mathcal{S}}$ is a complete quotient from $\mathcal{T}(L_2(G))\hat{\otimes}\mathcal{K}(L_2(G))$ onto $\pi_\lambda(Q(G))$.

Given $\varphi \in M_{cb}A(G)$, we may define a bounded linear functional $\tilde{\varphi}$ on $UCQ(G)$ by letting

$$\langle \tilde{\varphi}, \omega \cdot x\rangle = \langle \hat{\Theta}(\varphi)(x), \omega\rangle.$$

Since $M_{cb}A(G)$ is completely isometrically isomorphic to $\mathcal{CB}^{\sigma,\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G)))$, we can conclude from Theorem 2.3 (also see [6, Proposition 5.6]) that $\iota : \varphi \in M_{cb}A(G) \mapsto \tilde{\varphi} \in UCQ(G)^*$ is a well-defined completely isometric injection and thus we may identify $M_{cb}A(G)$ with a completely contractive Banach subalgebra of $UCQ(G)^*$.

**Lemma 3.1.** *For every $\varphi \in M_{cb}A(G)$, we have*

$$\tilde{\varphi} \circ \pi_\lambda = \varphi.$$

*Therefore, $\pi_\lambda$ is a completely contractive injection from $Q(G)$ into $UCQ(G)$.*

*Proof.* Given any $\varphi \in M_{cb}A(G)$, then $u_\varphi(s,t) = \varphi(st^{-1})$ is a right invariant element in $L_\infty(G)\otimes^{eh}L_\infty(G)$ such that $u_\varphi = \sum_{k\in I} v_k\otimes w_k$ for some index set $I$ and $v_k, w_k \in$

$L_\infty(G)$. For any $f, g \in L_1(G)$, we can write $f(s) = \xi'(s)\bar{\eta}(s)$ and $g(s) = \xi(s)\bar{\eta}'(s)$ for some $\xi, \eta, \xi', \eta' \in L_2(G)$. Then using the calculation for (3.3), we obtain

$$\begin{aligned}
\langle \tilde{\varphi} \circ \pi_\lambda, f * \kappa_*(g)\rangle &= \langle \tilde{\varphi}, \pi_\lambda \circ m_{id\otimes\kappa_*}(f \otimes g)\rangle = \langle \tilde{\varphi}, \omega_{\xi,\eta} \cdot x_{\xi',\eta'}\rangle \\
&= \langle \hat{\Theta}(\varphi)(x_{\xi',\eta'}), \omega_{\xi,\eta}\rangle = \langle \sum_{k\in I} v_k x_{\xi',\eta'} w_k \xi \mid \eta\rangle = \sum_{k\in I} \langle v_k, f\rangle\langle w_k, g\rangle \\
&= \int_G \int_G \varphi(st^{-1}) f(s) g(t) ds dt \\
&= \int_G \varphi(t) \left( \int_G f(s) g(t^{-1}s) \Delta(t^{-1}s) ds \right) dt \\
&= \langle \varphi, f * \kappa_*(g)\rangle.
\end{aligned}$$

This shows that $\tilde{\varphi} \circ \pi_\lambda = \varphi$ on the dense subspace $\operatorname{span}\{f * \kappa_*(g) : f, g \in L_1(G)\}$ in $Q(G)$ and thus we obtain the first statement.

If $f \in Q(G)$ such that $\pi_\lambda(f) = 0$, we must have $f = 0$ since

$$\langle \varphi, f\rangle = \langle \tilde{\varphi}, \pi_\lambda(f)\rangle = 0$$

for all $\varphi \in M_{cb}A(G) = Q(G)^*$. □

**Theorem 3.2.** *$\pi_\lambda$ is a completely isometric injection from $Q(G)$ into $UCQ(G)$ and $\pi_\lambda^*$ is a complete quotient from $UCQ(G)^*$ onto $M_{cb}A(G) = Q(G)^*$ such that $\pi_\lambda^* \circ \iota = id_{M_{cb}A(G)}$.*

*Therefore, we can completely isometrically identify $Q(G)$ with the operator subspace $\pi_\lambda(Q(G)) = \hat{\mathcal{S}}(\mathcal{T}(L_2(G)) \hat{\otimes} \mathcal{K}(L_2(G)))$ in $UCQ(G)$.*

*Proof.* Let us first prove that $\pi_\lambda^*$ is a complete quotient from $UCQ(G)^*$ onto $M_{cb}A(G) = Q(G)^*$. It is obvious that $\pi_\lambda^*$ is a complete contraction. Given any $\varphi \in M_{cb}A(G)$, we have

$$\langle \pi_\lambda^* \circ \iota(\varphi), f\rangle = \langle \tilde{\varphi}, \pi_\lambda(f)\rangle = \langle \varphi, f\rangle$$

for all $f \in Q(G)$. This shows that $\pi_\lambda^* \circ \iota = id_{M_{cb}A(G)}$ and thus $\pi_\lambda^*$ is a complete quotient from $UCQ(G)^*$ onto $M_{cb}A(G) = Q(G)^*$.

By duality, it is known that $\pi_\lambda$ must be a completely isometric injection from $Q(G)$ into $UCQ(G)$. □

From (3.4) and Theorem 3.2, we obtain the following diagrams of complete quotients

$$\begin{array}{ccc}
\mathcal{T}(L_2(G)) \hat{\otimes} \mathcal{K}(L_2(G)) & \xrightarrow{\hat{\mathcal{S}}} & \pi_\lambda(Q(G)) \\
{\scriptstyle T_*}\downarrow & & \Big\| {\scriptstyle \pi_\lambda} \\
L_1(G) \otimes^h L_1(G) & \xrightarrow{m_{id\otimes\kappa_*}} & Q(G).
\end{array} \tag{3.5}$$

It was shown in [18, Proposition 6.5] that for any amenable group $G$, the map

$$m \circ (\lambda \otimes \hat{\kappa} \circ \lambda) : f \otimes g \in L_1(G) \otimes L_1(G) \mapsto \lambda(f)\hat{\kappa}(\lambda(g)) \in \mathcal{L}(G) \tag{3.6}$$

can be extended to a complete contraction $m_{\lambda\otimes\hat{\kappa}\circ\lambda}$ from $L_1(G) \otimes^{eh} L_1(G)$ onto $UCQ(G) = UCB(\hat{G})$. The following result shows that this still holds for general locally compact groups.

**Proposition 3.3.** *Let $G$ be a locally compact group. Then the map $m \circ (\lambda \otimes \hat{\kappa} \circ \lambda)$ in (3.6) can be extended to a complete quotient $\tilde{m}_{\lambda\otimes\hat{\kappa}\circ\lambda}$ from $L_1(G) \otimes^{eh} L_1(G)$ onto $UCQ(G)$.*

*Proof.* The proof here is essentially the same as that given in [18, Proposition 6.5]. If we let $L_\infty(G) \otimes^{\sigma h} L_\infty(G) \cong (L_1(G) \otimes^{eh} L_1(G))^*$ be the *normal Haagerup tensor product* introduced by Effros and Kishimoto [3], then the map $T$ can be extended to a weak*-weak* continuous completely isometric isomorphism

$$\tilde{T} : L_\infty(G) \otimes^{\sigma h} L_\infty(G) \to \mathcal{CB}_{L_\infty(G)}(\mathcal{B}(L_2(G))).$$

Then $\Gamma_{\tilde{\tilde{\Theta}}} = \tilde{T}^{-1} \circ \tilde{\tilde{\Theta}}$ is a weak*-weak* continuous completely isometric isomorphism from $UCQ(G)^*$ onto $L_\infty(G) \otimes^{\sigma h} L_\infty(G)$ and $(\Gamma_{\tilde{\tilde{\Theta}}})_*$ is a complete quotient from $L_1(G) \otimes^{eh} L_1(G)$ onto $UCQ(G)$ which extends $m \circ (\lambda \otimes \hat{\kappa} \circ \lambda)$. Hence we have $\tilde{m}_{\lambda\otimes\hat{\kappa}\circ\lambda} = (\Gamma_{\tilde{\tilde{\Theta}}})_*$. □

To end this section we note that $\tilde{T} : L_\infty(G) \otimes^{\sigma h} L_\infty(G) \hookrightarrow \mathcal{CB}(\mathcal{B}(L_2(G)))$ induces a complete quotient

$$\tilde{T}_* : \mathcal{T}(L_2(G)) \hat{\otimes} \mathcal{B}(L_2(G)) \to L_1(G) \otimes^{eh} L_1(G).$$

This implies that $\hat{\mathcal{S}} = \tilde{m}_{\lambda\otimes\hat{\kappa}\circ\lambda} \circ \tilde{T}_*$.

## 4. Some equivalent amenability conditions associated with $UCQ(G)$

The amenability of locally compact groups has played an important role in the study of harmonic analysis and some related fields. Many equivalent amenability conditions have been studied in the literature. In the following we prove some new equivalent amenability conditions associated with $UCQ(G)$.

**Theorem 4.1.** *Let $G$ be a locally compact group. Then the following are equivalent:*

(1) *$G$ is an amenable group;*
(2) *$UCQ(G)$ is (Banach/operator space) isomorphic to a closed subspace of $\mathcal{L}(G)$;*
(3) *$\pi_\lambda$ is a (Banach/operator space) isomorphism from $Q(G)$ onto a closed subspace of $\mathcal{L}(G)$.*

*Proof.* (1) ⇒ (2) This is obvious by [18, Lemma 6.1], i.e. if $G$ is amenable, then $UCQ(G)$ is completely isometric to the $C^*$-subalgebra $UCB(\hat{G})$ of $\mathcal{L}(G)$.

(2) ⇒ (3) This is known by Theorem 3.2.

(3) ⇒ (1) Let $f \in L_1(G)$ with $f \geq 0$. Since the constant function $1_G \in M_{cb}A(G)$ with $\|1_G\|_{cb} = 1$, we have

$$\|f\|_{L_1(G)} = \int_G f(s)ds = \langle f, 1_G \rangle \leq \|1_G\|_{cb} \|f\|_{Q(G)} = \|f\|_{Q(G)} \leq \|f\|_{L_1(G)}.$$

This shows that $\|f\|_{Q(G)} = \|f\|_{L_1(G)}$. If $\varphi \in A(G)$, we may choose vectors $\xi, \eta \in L_2(G)$ such that

$$\varphi(s) = \omega_{\xi,\eta}(\lambda(s)) = \langle \lambda(s)\xi \mid \eta \rangle.$$

Then we can write

$$\langle \lambda(f), \varphi \rangle = \langle \lambda(f)\xi \mid \eta \rangle = \int_G f(s)\varphi(s)ds$$

and thus obtain

$$\|\lambda(f)\|_{C^*_\lambda(G)} = \sup\left\{\left|\int_G f(s)\varphi(s)ds\right| : \varphi \in A(G), \|\varphi\|_{A(G)} \le 1\right\}.$$

Suppose that $\pi_\lambda$ is a Banach space isomorphism from $Q(G)$ onto a closed subspace of $\mathcal{L}(G)$, i.e. the norm on $\pi_\lambda(Q(G))$ is equivalent to the norm on $\mathcal{L}(G)$. Then there exists a constant $0 < c \le 1$ such that

$$c\|x\|_{\pi_\lambda(Q(G))} \;\le\; \|x\|_{\mathcal{L}(G)} \;\le\; \|x\|_{\pi_\lambda(Q(G))}$$

for all $x \in \pi_\lambda(Q(G))$. Then every bounded linear functional $n \in \pi_\lambda(Q(G))^*$ can be extended to a bounded linear functional $\tilde{n} \in \mathcal{L}(G)^*$ with $\|\tilde{n}\|_{\mathcal{L}(G)^*} \le \frac{1}{c}\|n\|_{\pi_\lambda(Q(G))^*}$. In this case, every $\varphi \in M_{cb}A(G) \hookrightarrow \pi_\lambda(Q(G))^*$ of norm one can be extended to a bounded linear functional $\tilde{\varphi}$ on $\mathcal{L}(G)$ with $\|\tilde{\varphi}\|_{\mathcal{L}(G)^*} \le \frac{1}{c}$. We may choose a net of $\varphi_\alpha \in A(G)$ such that $\|\varphi_\alpha\|_{A(G)} \le \frac{1}{c}$ and $\varphi_\alpha(x) \to \tilde{\varphi}(x)$ for all $x \in \mathcal{L}(G)$. It follows that we can write

$$\begin{aligned}\|f\|_{Q(G)} &= \left\{\left|\int_G f(s)\varphi(s)ds\right| : \varphi \in M_{cb}A(G), \|\varphi\|_{cb} \le 1\right\}\\ &\le \sup\left\{\left|\int_G f(s)\varphi(s)ds\right| : \varphi \in A(G), \|\varphi\|_{A(G)} \le \frac{1}{c}\right\} = \frac{1}{c}\|\lambda(f)\|.\end{aligned}$$

Therefore, for every $f \in L_1(G)^+$, we have

$$c\|f\|_{L_1(G)} = c\|f\|_{Q(G)} \le \|\lambda(f)\|$$

and thus $G$ is amenable (see [16] for locally compact groups, and see [14] for general Kac algebras). $\square$

Given $\varphi \in A(G)$, we may write $\varphi(s) = \omega_{\xi,\eta}(\lambda(s))$ (i.e. $\varphi = \bar{\eta} * \check{\xi}$) for some $\xi, \eta \in L_2(G)$. Since the co-multiplication $\hat{\Gamma}$ on $\mathcal{L}(G)$ satisfies $\hat{\Gamma}(\lambda(s)) = \lambda(s) \otimes \lambda(s)$, it is easy to see that for any $x \in \mathcal{L}(G)$,

$$\varphi \cdot x = (id \otimes \varphi)\hat{\Gamma}(x) = (id \otimes \omega_{\xi,\eta})\hat{\Gamma}(x) = \omega_{\xi,\eta} \cdot x \in UCQ(G).$$

This shows that $A(G) \cdot \mathcal{L}(G)$ can be identified with a subspace of $UCQ(G)$. If $G$ is amenable (or equivalently, $UCQ(G)$ is a closed subspace of $\mathcal{L}(G)$), then $UCB(\hat{G})$ is contained in $UCQ(G)$ (in this case, we actually have $UCQ(G) = UCB(\hat{G})$). The following result shows that if $UCB(\hat{G})$ can be canonically identified with a closed subspace of $UCQ(G)$, then $G$ is amenable.

**Theorem 4.2.** *Let $G$ be a locally compact group. Then the following are equivalent:*

1. *$G$ is an amenable group;*
2. *the canonical inclusion $\iota : A(G) \cdot \mathcal{L}(G) \hookrightarrow UCQ(G)$ extends to a (Banach/operator space) isomorphism from $UCB(\hat{G})$ onto a closed subspace of $UCQ(G)$.*

*Proof.* (1) $\Rightarrow$ (2) This is known by [18, Lemma 6.1] and the definition of $UCQ(G)$. This is also an immediate consequence of Theorem 4.1.

(2) $\Rightarrow$ (1) Let us assume that the canonical inclusion $\iota : A(G) \cdot \mathcal{L}(G) \hookrightarrow UCQ(G)$ extends to a (Banach space) isomorphism from $UCB(\hat{G})$ onto a closed subspace of $UCQ(G)$. Since $C^*_\lambda(G)$ is a $C^*$-subalgebra of $UCB(\hat{G})$ (see [15]), $C^*_\lambda(G)$ is also

(Banach space) isomorphic to a subspace of $UCQ(G)$. Moreover, since the range space of

$$\tilde{m}_{\lambda\otimes\hat{\kappa}\circ\lambda} : f \otimes g \in L_1(G) \otimes L_1(G) \mapsto \lambda(f * \kappa_*(g)) = \lambda(f)\hat{\kappa}(\lambda(g))$$

is norm dense in $C^*_\lambda(G)$, we can conclude from Theorem 3.2 that $C^*_\lambda(G)$ is linearly isomorphic to a closed subspace of $\pi_\lambda(Q(G))$. Then every $\varphi \in M_{cb}A(G) = Q(G)^*$ restricts to a bounded linear functional on $C^*_\lambda(G)$. In particular, the constant function $1_G \in M_{cb}A(G)$ restricts to a bounded linear functional on $C^*_\lambda(G)$, which is actually the unital element in $B_\lambda(G) = C^*_\lambda(G)^*$. Therefore, $G$ is an amenable group. $\square$

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