PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 5, Pages 1315–1317
S 0002-9939(05)08416-9
Article electronically published on October 25, 2005

CHARACTERIZATION OF QUASI-BANACH SPACES WHICH COARSELY EMBED INTO A HILBERT SPACE

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(Communicated by David Preiss)

Abstract. We show that a quasi-Banach space coarsely embeds into a Hilbert space if and only if it is isomorphic to a linear subspace of $L_0(\mu)$ for some probability space $(\Omega, \mathcal{B}, \mu)$.

A (not necessarily continuous) map f between two metric spaces (X, d) and (Y, δ) is called a *coarse embedding* (see [G, 7.G]) if there exist two nondecreasing functions $\varphi_1 : [0, \infty) \to [0, \infty)$ and $\varphi_2 : [0, \infty) \to [0, \infty)$ such that

(1) $\varphi_1(d(x, y)) \le \delta(f(x), f(y)) \le \varphi_2(d(x, y))$,
(2) $\varphi_1(t) \to \infty$ as $t \to \infty$.

It was proved in [JR] that ℓ_p does not coarsely embed into a Hilbert space when $p > 2$. The present article is a strengthening of that result by giving a full characterization of quasi-Banach spaces that coarsely embed into a Hilbert space. This result, as well as its proof, mirrors the theorem in [AMM] which characterizes spaces that uniformly embed into a Hilbert space. The combination of Theorem 1 and the theorem in [AMM] yield that a quasi-Banach space uniformly embeds into a Hilbert space if and only if it coarsely embeds into a Hilbert space. This is counterintuitive in that a uniform embedding gives information only on small distances while a coarse embedding gives information only on large distances.

Theorem 1. *A quasi-Banach space X coarsely embeds into a Hilbert space if and only if there is a probability space $(\Omega, \mathcal{B}, \mu)$ such that X is linearly isomorphic to a subspace of $L_0(\mu)$.*

To prove Theorem 1, we use a result essentially contained in [JR] as is recalled in the following proposition.

Proposition 2. *Let X be a quasi-Banach space which coarsely embeds into a Hilbert space. Then there exists on X a continuous negative definite function g which satisfies $g(0) = 0$ and $\phi_1(\|x\|) \le g(x) \le \|x\|^{2\alpha}$ where $\phi_1 : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $\phi_1(t) \to \infty$ as $t \to \infty$, and $\alpha > 0$.*

Received by the editors November 17, 2004.
2000 *Mathematics Subject Classification.* Primary 46B20; Secondary 51F99.
Key words and phrases. Coarse embedding, uniform embedding.
The author was supported in part by NSF 0200690 and Texas Advanced Research Program 010366-0033-20013.
This paper represents a portion of the author's dissertation being prepared at Texas A&M University under the direction of William B. Johnson.

Proof. Steps 0, 1, 2 (and a piece of Step 3 on the continuity of g) in [JR] extend to the case of quasi-Banach spaces. □

Proof of Theorem 1. Let X be a quasi-Banach space. A theorem of Aoki and Rolewicz ([A] and [R]) gives an equivalent quasi-norm $\|\cdot\|$ on X which is also p-subadditive for some $0 < p \leq 1$, i.e., $\|x+y\|^p \leq \|x\|^p + \|y\|^p$ for all x, y in X. In particular, X under this norm has type p.

Say X is linearly isomorphic to a subspace of $L_0(\mu)$ for some probability space $(\Omega, \mathcal{B}, \mu)$. Then since X has type p, X is isomorphic to a subspace of $L_r(\mu)$ for every $r < p$. (See [Ni] and also [BL, Theorem 8.15].)

Now since $r < 2$, Nowak's result in [No] implies that X coarsely embeds into a Hilbert space. In fact, Nowak notices that the negative definite function $x \in L_r \mapsto \|x\|_{L_r}^r$ for $0 < r < 2$ gives, via Schoenberg's classical work [S], a coarse embedding of L_r into a Hilbert space. This explicit representation of a negative definite function in L_r appears in several works by Guerre-Delabrière; see [GD] for example. Mendel and Naor in [MN] actually give an explicit formula for a coarse embedding of L_r into L_q when $r < q$ given by $T : L_r(\mathbb{R}) \to L_q(\mathbb{R} \times \mathbb{R})$:

$$T(f)(s,t) = \frac{1 - e^{itf(s)}}{|t|^{(r+1)/q}}.$$

Conversely, let X be a quasi-Banach space which coarsely embeds into a Hilbert space. Let g be the negative definite function on X given by Proposition 2, and let f be the continuous positive definite function given by $f = e^{-g}$. Use Lemma 4.2 in [AMM] (see also Lemma 1 in [BDK, page 238] and Proposition 8.7 in [BL]) to get a probability space $(\Omega, \mathcal{B}, \mu)$ and a continuous linear operator $U : X \to L_0(\mu)$ such that the characteristic function $\mathbb{E}\exp(itUx)$ of Ux is equal to $f(tx)$ for every $x \in X$ and $t \in \mathbb{R}$. We show that U is an isomorphism into.

Let $(x_n)_n$ be a sequence in X such that $U(x_n) \to 0$ in $L_0(\mu)$, i.e., in measure. Then $f(tx_n) = \mathbb{E}(\exp(itUx_n)) \to 1$ for each fixed t in $\mathbb{R}$.

If $(x_n)_n$ does not converge to 0, then by passing to a subsequence we can assume without loss of generality that $\|x_n\| \geq \epsilon$ for all n and for some $\epsilon > 0$. But since ϕ_1 is nondecreasing, we get for every $t > 0$,

$$e^{-\phi_1(t\|x_n\|)} \leq e^{-\phi_1(t\epsilon)}.$$

Since $\phi_1(s) \to \infty$ as $s \to \infty$, we can pick $t_0 > 0$ so that $e^{-\phi_1(t_0\epsilon)} < \frac{1}{2}$. For that t_0, we have for every n,

$$f(t_0 x_n) \leq e^{-\phi_1(t_0\epsilon)} < \frac{1}{2}.$$

This contradicts the fact that $f(t_0 x_n) \to 1$.

Thus $x_n \to 0$, and hence U is one-to-one and its inverse is continuous. □

Acknowledgement

I would like to thank William B. Johnson for the constructive conversations that led to the production of this article.

References

[A] T. Aoki, *Locally bounded linear topological spaces*, Proc. Imp. Acad. Tokyo **18** (1942), 588–594. MR0014182 (7:250d)

[AMM] I. Aharoni, B. Maurey, and B. S. Mityagin, *Uniform embeddings of metric spaces and of Banach spaces into Hilbert spaces*, Israel J. Math. **52** (1985), no. 3, 251–265. MR0815815 (87b:46011)

[BDK] J. Bretagnolle, D. Dacunha Castelle, J. L. Krivine, *Lois stables et espaces L^p*, Ann. Inst. H. Poincaré Sect. B (N. S.) **2** (1965/1966), 231–259. MR0203757 (34:3605)

[BL] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, Coll. Pub. 48, Amer. Math. Soc., Providence, RI, 2000. MR1727673 (2001b:46001)

[G] M. Gromov, *Asymptotic invariants for infinite groups, Geometric Group Theory*, G. A. Niblo and M. A. Roller, eds., Cambridge University Press, Cambridge, 1993. MR1253544 (95m:20041)

[GD] S. Guerre-Delabrière, *Types et suites symétriques dans $L^p, 1 \leq p < +\infty, p \neq 2$*, Israel J. Math. **53** (1986), no. 2, 191–208. MR0845871 (87m:46047)

[JR] W. B. Johnson and N. L. Randrianarivony, *ℓ_p $(p > 2)$ does not coarsely embed into a Hilbert space*, Proc. Amer. Math. Soc. (to appear).

[MN] M. Mendel, A. Naor, *Euclidian quotients of finite metric spaces*, Adv. Math. **189** (2004), 451–494. MR2101227 (2005h:54031)

[Ni] E. M. Nikišin, *Resonance theorems and expansions in eigenfunctions of the Laplace operator* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 795–813, English translation: Math. USSR-Izv. **6** (1972), 788–806.

[No] P. Nowak, *Coarse embeddings of metric spaces into Banach spaces*, Proc. Amer. Math. Soc. **133** (2005), 2589–2596. MR2146202

[R] S. Rolewicz, *On a certain class of linear metric spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astrono. Phys. **5** (1957), 471–473. MR0088682 (19:562d)

[S] I. J. Schoenberg, *Metric spaces and positive definite functions*, Trans. Amer. Math. Soc. **44** (1938), 522–536. MR1501980

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