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FIBER PRODUCTS, POINCARÉ DUALITY AND A_∞-RING SPECTRA

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Abstract. For a Poincaré duality space X^d and a map $X \to B$, consider the homotopy fiber product $X \times^B X$. If X is orientable with respect to a multiplicative cohomology theory E, then, after suitably regrading, it is shown that the E-homology of $X \times^B X$ has the structure of a graded associative algebra. When $X \to B$ is the diagonal map of a manifold X, one recovers a result of Chas and Sullivan about the homology of the unbased loop space LX.

1. Introduction

Let M^d be an orientable, closed manifold. Using intersection theory on singular chains, M. Chas and D. Sullivan [C-S] have constructed an operation

$$\bullet\colon H_p(LM) \otimes H_q(LM) \to H_{p+q-d}(LM)\,,$$

called the *loop product*, where $LM = \mathrm{map}(S^1, M)$ denotes the free loop space of M. If we set $H_*(LM)[d] := H_{*+d}(LM)$, then the loop product gives $H_*(LM)[d]$ the structure of a graded associative ring. The action of the circle on LM given by rotating loops gives rise to a differential $\Delta\colon H_*(LM)[d] \to H_{*+1}(LM)[d]$. Chas and Sullivan prove that the pair $(\bullet, \Delta)$ gives $H_*(LM)[d]$ the structure of a *Batalin-Vilkovisky* algebra, i.e., a graded commutative algebra with differential such that the Leibniz rule fails up to a term which is bilinear. Recently, R. Cohen and J. Jones [C-J] gave a spectrum level description of the loop product in terms of a Pontryagin construction. Shortly thereafter, using equivariant Spanier-Whitehead duality, W. Dwyer and H. Miller and the author (independently, unpublished) described the loop product via the identification of $H_*(LM)$ with topological Hochschild homology. A spin-off of the last approach is that the loop product exists in the more general setting when M is an orientable Poincaré duality space.

This paper has two goals. The first is to exhibit algebra structures on the homology of a wider class of spaces. To each space in the class we will associate a certain Thom spectrum. The algebra structures then follow by showing that the Thom spectra are A_∞-ring spectra. The second goal of this paper is to illustrate how some of the already known examples (based loop spaces, orientable closed manifolds and free loop spaces) fit into the wider class.

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To describe our class, fix a Poincaré duality space X of formal dimension d. Let $X \to B$ be a map of based spaces, which for the purposes of exposition we take to be a Serre fibration. Then we consider the fiber product

$$X \times^B X .$$

Let $-\tau$ denote the Spivak normal fibration of X. This is a stable spherical fibration over X. By desuspending, our convention will be that the fiber of $-\tau$ is a $(-d)$-sphere spectrum. Let

$$(X \times^B X)^{-\tau}$$

be the effect of taking the Thom spectrum of the pullback of $-\tau$ along the first factor projection $X \times^B X \to X$.

The main theorem of this paper is

Theorem A. *Assume X is connected. Then*

$$(X \times^B X)^{-\tau}$$

is an A_∞-ring spectrum.

Consequently, if E denotes a multiplicative cohomology theory for which X is E-orientable, then

$$E_*(X \times^B X)[d] := E_{*+d}(X \times^B X)$$

has the structure of a graded associative ring.

(The assumption that X is connected is an artifact of our method of proof: Theorem A is indeed true when X is disconnected, but we will not address this issue in the paper.)

Theorem A is actually a corollary of our next result which identifies the above Thom spectrum with a certain ring spectrum of endomorphisms. To formulate the result, we give X a basepoint. The map $X \to B$ then becomes a map of based spaces. Let F denote its homotopy fiber. Then F may be equipped with an action by ΩB, where the latter is a suitable topological group version of the based loop space of B (the Borel construction of this action recovers the map $X \to B$ up to homotopy). Let F_+ be the disjoint union of F with a basepoint. Then the suspension spectrum $\Sigma^\infty F_+$ comes equipped with a (naive) ΩB-action.

Theorem B. *Assume X is connected. Then there is a weak homotopy equivalence of spectra*

$$(X \times^B X)^{-\tau} \;\simeq\; \mathrm{end}_{\Omega B}(\Sigma^\infty F_+)\,,$$

where the right side denotes the endomorphism spectrum of ΩB-equivariant stable self-maps of F_+.

Examples. (1) $X \to B$ is the diagonal map of X.

Then $(X \times^B X)^{-\tau}$ is weak equivalent to $(LX)^{-\tau}$, where the latter is formed by taking the Thom spectrum with respect to pulling back the Spivak fibration of X along the evaluation map $LX \to X$. The endomorphism spectrum appearing in the statement of Theorem B is the *topological Hochschild cohomology* of the A_∞-ring $S[\Omega X] := \Sigma^\infty(\Omega X_+)$. In their solution to Deligne's Hochschild cohomology conjecture, J. McClure and J. Smith have shown that the topological Hochschild cohomology of an A_∞-ring always admits an action by an operad which is weak equivalent to the little 2-disks operad [M-S]. Consequently, Theorem B implies that $(LX)^{-\tau}$ admits an action by an operad which is weak equivalent to the little 2-disks.

In particular, $(LX)^{-\tau}$ is a homotopy commutative ring spectrum (an observation first made by Cohen and Jones in the manifold case).

(2) $X = B$ *and the map* $X \to B$ *is the identity.*

Since X is E-orientable, we have by Poincaré duality

$$E_*(X)[d] \;\cong\; E^{-*}(X)\,,$$

and this isomorphism defines the multiplication on $E_*(X)[d]$.

In particular, when E_* is a singular homology, the multiplication is given by the intersection pairing of X.

(3) X *is a point.*

In this instance, $X \times^B X$ is identified with the based loop space ΩB. Theorem A says that $E_*(\Omega B)$ is a graded ring for any multiplicative theory E. The ring structure in this case is given by Pontryagin product, i.e., the homomorphism induced by loop multiplication.

(4) B *is a point.*

If E is a singular homology and X is an orientable manifold, then the ring structure on $H_*(X \times X)[d]$ can be described on the chain level as follows: if $\sigma \otimes \tau \in C_p(X) \otimes C_q(X)$ is a cycle and $\sigma' \otimes \tau' \in C_{p'}(X) \otimes C_{q'}(X)$ is another cycle, then

$$(\sigma \otimes \tau) \bullet (\sigma' \otimes \tau') \;:=\; \epsilon(\tau, \sigma')\sigma \otimes \tau' \in C_p(X) \otimes C_{q'}(X)\,,$$

where $\epsilon(\tau, \sigma')$ is trivial unless $q + p' = n$, in which case $\epsilon(\tau, \sigma')$ denotes the intersection number $\tau \cdot \sigma'$.

(5) $B = \mathrm{map}(K, X)$ *for a CW complex* K *and* $X \to \mathrm{map}(K, X)$ *is the inclusion of the constant maps.*

Then $X \times^B X$ is identified with

$$\mathrm{map}(SK, X)\,,$$

where SK denotes the unreduced suspension of K. Hence, $H_*(\mathrm{map}(SK, X))[d]$ has a ring structure.

The topological monoid $\mathrm{Aut}(SK)$ of self homotopy equivalences of SK acts by composition on $\mathrm{map}(SK, X)$. On singular homology, it would be interesting to understand how the action intertwines with the ring structure, since in the special case when $K = S^0$, one knows that this information encodes the Batalin-Vilkovisky structure on $H_*(LX)[d]$.

The next result of this paper concerns the compatibility of the A_∞-ring structures with respect to base change. It is clear that a map of spaces $B \to C$ gives rise to a map of Thom spectra

$$(X \times^B X)^{-\tau} \to (X \times^C X)^{-\tau}$$

where $X \to C$ is given by the composition $X \to B \to C$.

Theorem C. *Assume X is connected. Then the map*

$$(X \times^B X)^{-\tau} \to (X \times^C X)^{-\tau}$$

is a morphism of A_∞-rings.

In the special case when $B \to C$ is given by the projection of $X \times X$ onto its first factor, Theorem C reduces to an observation already made by Cohen and Jones [C-J, Th. 1(1)]: the map $(LX)^{-\tau} \to X^{-\tau}$ induced by loop evaluation is a morphism of A_∞-rings.

Our final result concerns Thom spectra of the form $(X \times^B X)^\xi$ where ξ is a spherical fibration on $X \times^B X$ and is induced from one on X via base change along the first factor projection $X \times^B X \to X$. As above, we assume that X is a connected Poincaré duality space.

Theorem D. *The Thom spectrum*

$$(X \times^B X)^{\xi}$$

is a left $(X \times^B X)^{-\tau}$*-module.*

Corollary E. $\Sigma^\infty LX_+$ *is a left* $(LX)^{-\tau}$*-module.*

A homotopy associative version of this corollary was observed by Cohen and Jones.

Outline. §2 is language. In §3 we prove Theorem B; the main idea is to use the *norm map* for equivariant spectra constructed by the author in [Kl]. In §4 we use Theorem B together with results of [E-K-M-M] to prove Theorem A. In §5 we prove Theorem C. In §6 we use ideas similar to §4 and §5 to prove Theorem D.

2. Language

This section is not intended to be complete. A more detailed exposition of this material appears in [Kl].

Spaces. All spaces will be compactly generated, and we make the convention that products are to be re-topologized using the compactly generated topology. Mapping spaces are to be given the compactly generated, compact open topology.

A *weak equivalence* of spaces denotes (a chain of) weak homotopy equivalence(s).

If $f\colon A \to C$ and $g\colon B \to C$ are maps of spaces, then the *homotopy fiber product* (or *homotopy pullback*) is the space $A \times^C B$ consisting of triples (a, λ, b) with $a \in A$, $b \in B$ and $\lambda\colon [0,1] \to C$ satisfying $f(a) = \lambda(0)$ and $g(b) = \lambda(1)$. If either f or g is a fibration, then the evident map from the fiber product into the homotopy fiber product is a weak equivalence.

Poincaré spaces. A finitely dominated space X is said to be an *orientable Poincaré duality space* of (formal) dimension n if there exists a fundamental class $[X] \in H_n(X;\mathbb{Z})$ such that the associated cap product homomorphism

$$\bigcap [X]\colon H^*(X;M) \to H_{n-*}(X;M)$$

is an isomorphism in all degrees and all local coefficient modules M. Similarly, one has the notion of Poincaré duality with respect to a multiplicative cohomology theory E, where a fundamental class is required to live in the abelian group $E_n(X) := \pi_n(E \wedge X_+)$.

Naive G-spectra. Let G be the geometric realization of a simplicial group. A *(naive) G-spectrum* consists of based (left) G-spaces E_i for $i \geq 0$, and equivariant based maps $\Sigma E_i \to E_{i+1}$ (where we let G act trivially on the suspension coordinate of ΣE_i). A *morphism* $E \to E'$ of G-spectra consists of maps of based spaces $E_i \to E'_i$ which are compatible with the structure maps. A *weak equivalence* of G-spectra is a map inducing an isomorphism on homotopy groups. E is an Ω*-spectrum* if the adjoint maps $E_i \to \Omega E_{i+1}$ are weak homotopy equivalences.

If X is a based G-space, then its *suspension spectrum* $\Sigma^\infty X$ is a G-spectrum with jth space $Q(S^j \wedge X)$, where $Q = \Omega^\infty\Sigma^\infty$ is the stable homotopy functor (here G acts trivially on the suspension coordinates). We use the notation $S[G]$

for the suspension spectrum of G_+ considered as a $(G{\times}G)$-spectrum (the action on G_+ is given left multiplication with respect to the first factor of $G{\times}G$ and right multiplication composed with the involution $g \mapsto g^{-1}$ on the second factor).

We now give some constructions on G-spectra. The extent to which each construction is homotopy invariant is indicated parenthetically.

If U is a based G-space and E is a G-spectrum, then

$$U \wedge E$$

denotes the G-spectrum which in degree j is the smash product $U \wedge E_j$ provided with the diagonal action (this has the correct homotopy type when U, considered unequivariantly, is a based CW complex). The associated *orbit spectrum*

$$U \wedge_G E$$

is given by taking G-orbits degreewise (it has the correct homotopy type when U is a based G-CW complex which is free away from the basepoint).

Similarly, we can form the function spectrum

$$\mathrm{map}(U, E)$$

which in degree j is given by $\mathrm{map}(U, E_j)$ = the function space of unequivariant based maps from U to E_j. The action of G on $\mathrm{map}(U, E)$ is given by conjugation, i.e., $(g * f)(u) = gf(g^{-1}u)$ for $g \in G$ and $f \in \mathrm{map}(U, E_j)$ (the spectrum $\mathrm{map}(U, E)$ has the correct homotopy type when E is an Ω-spectrum and U is a CW complex).

Let

$$\mathrm{map}_G(U, E)$$

denote the *fixed point spectrum* of G acting on $\mathrm{map}(U, E)$, i.e., the spectrum whose jth space consists of the equivariant functions from U to E_j (this has the correct homotopy type if E is an Ω-spectrum and U is the retract of a based G-CW complex which is free away from the basepoint).

If E is a G-spectrum, then the *homotopy orbit spectrum* E_{hG} is

$$E \wedge_G EG_+ \,,$$

where EG is the free contractible G-space (arising from the bar construction), and EG_+ is the effect of adding a basepoint to EG.

The *homotopy fixed point spectrum* E^{hG} is

$$\mathrm{map}_G(EG_+, E) \,.$$

In the above constructions, the hypotheses granting the correct homotopy type can always be achieved by changing the input spectra up to natural weak equivalence. This follows from a specific choice of Quillen model structure on the category of G-spectra (for details, see [Sc]).

A_∞-rings. Roughly, a *(strict) A_∞-ring* consists of a spectrum R, a product $R \wedge R \to R$ and a unit $S^0 \to E$ such that the axioms for a classical ring (associativity, etc.) are relaxed to hold up to homotopy and all higher homotopy coherences (i.e., R is an algebra over the associahedron operad). More generally, we use the term A_∞-ring for any spectrum having the weak homotopy type of a strict A_∞-ring.

Essentially, the only fact we use in this paper about A_∞-rings is that taking the (enriched) endomorphisms of an object in a suitably nice category of naive

G-spectra forms an A_∞-ring. This will follow from the existence of a good smash product construction ([E-K-M-M]).

3. The proof of Theorem B

Some technical simplifications. Once a basepoint for X is chosen, $X \to B$ becomes a based map. Let B_0 denote the connected component of B containing the basepoint. Using the assumption that X is connected, it is straightforward to check that the map of homotopy fiber products

$$X \times^{B_0} X \to X \times^B X$$

is a weak homotopy equivalence (we omit the argument). Hence the associated map of Thom spectra is also a weak equivalence. *We therefore assume henceforth that B is a connected space.*

Let $G = \Omega B$ denote the geometric realization of the total singular complex of the Kan loop group of B (see [Wa]; in what follows, we abbreviate terminology and call ΩB the *loop group* of B). Then we have a natural weak equivalence of based spaces $BG \simeq B$. Using this identification, we will assume that B has been replaced by BG. We therefore have a map $X \to BG$. The homotopy fiber of this map is then identified with the (strict) fiber product

$$F \ := \ EG \times^{BG} X\,.$$

This description of the homotopy fiber equips it with a preferred action of G (arising from the action of G on the first factor EG). The Borel construction $EG \times_G F \to BG$ is then identified with the map $X \to B$ up to fiberwise weak equivalence.

Let $H = \Omega X$ be the loop group of X. Then the Borel construction of H acting on F gives a fibration

$$EH \times_H F \to BH\,,$$

which, with respect to the identification $BH \simeq X$, is fiberwise weak homotopy equivalent to the first factor projection $X \times^B X \to X$.

We next give a Thom spectrum version of the above. Let $S^{-\tau}$ denote the fiber of the Spivak normal fibration of $X \simeq BH$ desuspended to down to degree $-d$. As above, we equip $S^{-\tau}$ with an H-action in such a way that the Borel construction of the action coincides with the Spivak fibration of X. Then we have a weak equivalence of spectra

$$S^{-\tau} \wedge_{hH} F_+ \ \simeq \ (X \times^B X)^{-\tau} \tag{1}$$

where the left-hand side denotes the homotopy orbit spectrum of H acting diagonally on $S^{-\tau} \wedge F_+$.

The norm map. Let E be a G-spectrum. In [Kl], the author constructed a (weak) map of spectra

$$\eta\colon D_G \wedge_{hG} E \to E^{hG}$$

which is a natural in E. Here $D_G = S[G]^{hG}$ is the *dualizing spectrum* of G which is given by taking the homotopy fixed points of G acting by left multiplication on $S[G] =$ the suspension spectrum of G_+. Right multiplication by G composed with the involution $g \mapsto g^{-1}$ gives D_G the structure of a G-spectrum.

The author also proved in [Kl] that η is a weak equivalence whenever BG is a finitely dominated space. Furthermore, when BG is finitely dominated it was shown that BG is a Poincaré duality space of formal dimension d if and only if D_G

is unequivariantly weak equivalent to S^{-d}. In this instance, it was also shown that D_G gives a model for the *Spivak fiber* of BG, i.e., the unreduced Borel construction of G acting on D_G gives a stable spherical fibration $EG \times_G D_G \to BG$ which is the Spivak normal fibration of the Poincaré space BG. In the sequel, we fix the notation

$$D_G = S^{-\tau}$$

whenever BG is a Poincaré duality space.

We apply the norm map η to (the suspension spectrum of) F_+ and the group H. Since $BH = X$ is a Poincaré duality space, we see that η takes the form of a weak equivalence

$$S^{-\tau} \wedge_{hH} F_+ \xrightarrow{\sim} (\Sigma^\infty F)^{hH} . \tag{2}$$

Change of groups. In the following, let $H \to G$ be the homomorphism of loop groups arising from a map of based spaces.

Lemma 3.1 ("Shapiro's Lemma")**.** *Let W be a G-spectrum which is also an Ω-spectrum. Let Y be a based H-CW complex whose action is free away from the basepoint. Then there is a natural weak equivalence of spectra*

$$\mathrm{map}_H(Y, W) \ \simeq \ \mathrm{map}_G(Y \wedge_H G_+, W) .$$

Proof. In fact, we claim the two spectra are isomorphic. Taking adjoints, we get

$$\mathrm{map}_G(Y \wedge_H G_+, W) \cong \mathrm{map}_H(Y, \mathrm{map}_G(G_+, W)).$$

On the other hand, $W \cong \mathrm{map}_G(G_+, W)$. □

We now apply 3.1 to the homomorphism arising from the map $X \to B$, with $Y = EH_+$ and $W = \Sigma^\infty F_+$. The result yields a weak equivalence of spectra

$$(\Sigma^\infty F_+)^{hH} \ \simeq \ \mathrm{map}_G(EH \wedge_H G_+, \Sigma^\infty F_+) .$$

Clearly, there is a weak equivalence of based G-spaces $EH \wedge_H G_+ \simeq F_+$ and thus the right side is identified with $\mathrm{map}_G(F_+, \Sigma^\infty F_+)$. Regarding the latter as the spectrum of equivariant stable self-maps of F_+, let us substitute the notation

$$\mathrm{end}_G(\Sigma^\infty F_+) \ := \ \mathrm{map}_G(F_+, \Sigma^\infty F_+) .$$

Then (1) and (2) above yield a weak equivalence of spectra

$$(X \times^B X)^{-\tau} \ \simeq \ \mathrm{end}_G(\Sigma^\infty F_+) .$$

This completes the proof of Theorem B.

4. The proof of Theorem A

The second part of the statement of Theorem A follows from the first part by the Thom isomorphism in E-homology.

As for the first part, by Theorem B, it is enough to check that $\mathrm{end}_G(\Sigma^\infty F_+)$ is an A_∞-ring. We will simply quote [E-K-M-M] to prove this.

Recall that $S[G] := \Sigma^\infty(G_+)$. Then $S[G]$ is an S-algebra and $\Sigma^\infty F_+$ is a left $S[G]$-module (see [E-K-M-M, IV. Th. 7.8]). Furthermore, there is an evident identification

$$\mathrm{end}_G(\Sigma^\infty F_+) \simeq \mathrm{end}_{S[G]}(\Sigma^\infty F_+) ,$$

where the right side denotes the function object of $S[G]$-module self-maps of $\Sigma^\infty F_+$. By [E-K-M-M, III. Prop. 6.12], $\mathrm{end}_{S[G]}(\Sigma^\infty F_+)$ is an S-algebra. Finally, by [E-K-M-M, II. Lem. 3.4], one knows that an S-algebra is an A_∞-ring.

Remark 4.1. A nuts and bolts argument can be given to prove the weaker statement that $E := \mathrm{map}_G(F_+, \Sigma^\infty F_+)$ is a ring spectrum (i.e., without verifying the A_∞ condition). We now sketch this argument.

The jth space of E is

$$E_j = \mathrm{map}_G(F_+, Q(\Sigma^j F_+))\,,$$

where $Q = \Omega^\infty\Sigma^\infty$ denotes the stable homotopy functor.

For nonnegative integers i and j, there is a map

$$E_i \wedge E_j \to E_{i+j}$$

which is given as follows: by taking adjoints, a point in E_i amounts to a map of spectra $\Sigma^\infty F_+ \to \Sigma^\infty\Sigma^i F_+$. Using in also the evident homeomorphism $Q(\Sigma^j F_+) \cong \Omega^i Q(\Sigma^{i+j} F_+)$, a point in E_j amounts to a map of spectra $\Sigma^\infty\Sigma^i F_+ \to \Sigma^\infty\Sigma^{i+j} F_+$. The composition of these maps of spectra is then a map $\Sigma^\infty F_+ \to \Sigma^\infty\Sigma^{i+j} F_+$, which is adjoint to a point of E_{i+j}. The maps $E_i \wedge E_j \to E_{i+j}$ assemble to a morphism of *bispectra* $E \otimes E \to E$, where $E \otimes E$ denotes the external smash product of E with itself, and the codomain E is regarded as a bispectrum in the obvious way. This map of bispectra induces a map of spectra

$$E \wedge E \to E\,,$$

which makes E into a ring spectrum.

5. The proof of Theorem C

We rename $F = F_B$ to indicate that it is an ΩB-equivariant model for the homotopy fiber of $X \to B$ and similarly, we have $F_C =$ an equivariant model for the homotopy fiber of $X \to C$. Then there is an ΩC-equivariant weak equivalence of spaces

$$F_C \;\simeq\; F_B \times_{\Omega B} \Omega C\,.$$

Using Theorem B, the map $(X \times^B X)^{-\tau} \to (X \times^C X)^{-\tau}$ corresponds to

$$\mathrm{end}_{\Omega B}(\Sigma^\infty (F_B)_+) \to \mathrm{end}_{\Omega C}(\Sigma^\infty (F_C)_+)\,. \tag{3}$$

A straightforward checking of the details of the proof of Theorem B which we omit shows that this map is described up to homotopy by *induction* along the homomorphism $\Omega B \to \Omega C$. Explicitly, a self-map $f\colon \Sigma^\infty (F_B)_+ \to \Sigma^\infty (F_B)_+$ induces a self-map

$$f \wedge \mathrm{id}_{(\Omega C)_+}\colon \Sigma^\infty (F_B)_+ \wedge_{\Omega B} (\Omega C)_+ \;\to\; \Sigma^\infty (F_B)_+ \wedge_{\Omega B} (\Omega C)_+\,,$$

and using the identification of F_C above, we get an ΩC-equivariant weak equivalence

$$\Sigma^\infty (F_C)_+ \;\simeq\; \Sigma^\infty (F_B)_+ \wedge_{\Omega B} (\Omega C)_+\,.$$

Hence, on the level of S-algebras, we can rewrite the map (3) as

$$\mathrm{end}_{S[\Omega B]}(\Sigma^\infty (F_B)_+) \to \mathrm{end}_{S[\Omega C]}(\Sigma^\infty (F_B)_+ \wedge_{S[\Omega B]} S[\Omega C])\,.$$

With $R = S[\Omega B]$ and $R' = S[\Omega C]$, the above amounts to the question of whether the extension of scalars functor $M \mapsto M \wedge_R R'$ is enriched over S-modules. As pointed out to me by M. Mandell, the latter is a formal consequence of the result that S-modules form a closed symmetric monoidal category [E-K-M-M, II. Th. 1.6].

6. The proof of Theorem D

We argue as in the proof of Theorem B, and use the notation of that proof.

Let $S^{\xi+\tau}$ denote the fiber of spherical fibration over $BH \simeq X$ which classifies $\xi+\tau$, where τ is the Spivak tangent bundle of BH (whose fiber is a sphere spectrum of dimension $d = \dim X$). Then $S^{\xi+\tau}$ is a sphere spectrum with H-action whose unreduced Borel construction represents $\xi + \tau$. Hence, we have

$$\begin{aligned}
(X \times_B X)^\xi &\simeq S^\xi \wedge_{hH} F_+, && \text{cf. §3,} \\
&\simeq S^{-\tau} \wedge_{hH} S^{\xi+\tau} \wedge F_+ && \text{by rewriting,} \\
&\simeq (S^{\xi+\tau} \wedge F_+)^{hH} && \text{by } \eta \text{ (cf. §3),} \\
&\simeq \mathrm{map}_G(F_+, S^{\xi+\tau} \wedge F_+) && \text{by change of groups.}
\end{aligned}$$

If we now translate the last term into a category of S-modules, it becomes identified with

$$\mathrm{map}_{S[G]}(\Sigma^\infty F_+, S^{\xi+\tau} \wedge F_+)\,.$$

The closed symmetric monoidal structure of S-modules then gives a composition pairing

$$\mathrm{end}_{S[G]}(\Sigma^\infty F_+) \;\wedge_S\; \mathrm{map}_{S[G]}(\Sigma^\infty F_+, S^{\xi+\tau} \wedge F_+) \;\to\; \mathrm{map}_{S[G]}(\Sigma^\infty F_+, S^{\xi+\tau} \wedge F_+)$$

and we infer that $\mathrm{map}_{S[G]}(\Sigma^\infty F_+, S^{\xi+\tau} \wedge F_+)$ is an $\mathrm{end}_{S[G]}(\Sigma^\infty F_+)$-module. It follows that $(X \times^B X)^\xi$ is a left $(X \times^B X)^{-\tau}$-module.

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