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POINCARÉ DUALITY ALGEBRAS AND RINGS OF COINVARIANTS

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ABSTRACT. Let $\varrho : G \hookrightarrow GL(n, \mathbb{F})$ be a faithful representation of a finite group G over the field $\mathbb{F}$. Via ϱ the group G acts on $V = \mathbb{F}^n$ and hence on the algebra $\mathbb{F}[V]$ of homogenous polynomial functions on the vector space V. R. Kane (1994) formulated the following result based on the work of R. Steinberg (1964): If the field $\mathbb{F}$ has characteristic 0, then $\mathbb{F}[V]_G$ is a Poincaré duality algebra if and only if G is a pseudoreflection group. The purpose of this note is to extend this result to the case $|G| \in \mathbb{F}^\times$ (i.e. the order $|G|$ of G is relatively prime to the characteristic of $\mathbb{F}$).

1. Introduction

Let G be a finite group, $\mathbb{F}$ a field and V an n-dimensional $\mathbb{F}$-vector space. For a representation $\varrho : G \hookrightarrow GL(n, \mathbb{F}) \cong GL(V)$ the group G acts on the algebra $\mathbb{F}[V]$ of homogeneous polynomial functions on $V = \mathbb{F}^n$. The **ring of invariants** is defined by

$$\mathbb{F}[V]^G := \{f \in \mathbb{F}[V] \mid \sigma f = f, \ \forall \ \sigma \in G\}.$$

We define the **ring of coinvariants** to be the graded quotient algebra

$$\mathbb{F}[V]_G := \mathbb{F}[V]/h(G),$$

where $h(G)$ is the ideal in $\mathbb{F}[V]$ generated by all G-invariant homogeneous polynomials of strictly positive degree. $h(G)$ is called the **Hilbert's ideal**. As a convenient reference for invariant theory see [2] or [14].

Note that the Hilbert's ideal $h(G)$ is a $\overline{\mathbb{F}[V]}$-primary ideal, where $\overline{\mathbb{F}[V]}$ denotes the augmentation ideal of $\mathbb{F}[V]$, i.e. $\bigoplus_{i>0} \mathbb{F}[V]_i$.

Definition 1.1. Let H be a graded connected commutative algebra over a field $\mathbb{F}$. We say that H is a **Poincaré duality algebra** of **formal dimension** d, if

(1) $H_i = 0$ for all $i > d$,
(2) $\dim_{\mathbb{F}}(H_d) = 1$,
(3) the bilinear form $H_i \otimes_{\mathbb{F}} H_{d-i} \longrightarrow H_d,\ a \otimes b \mapsto a \cdot b$, is nonsingular, i.e. an element $a \in H_i$ is 0 if and only if $a \cdot b = 0$ for all $b \in H_{d-i}$.

If H is a Poincaré duality algebra of formal dimension d, then there is an element $[H] \neq 0$ in H_d. Such an element is referred to as a **fundamental class** for H.

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The following theorem is well known.

Theorem 1.2 (G.C. Shephard, J.A. Todd and C. Chevalley). *If $\varrho : G \hookrightarrow GL(V)$ is a representation of a finite group G over a field $\mathbb{F}$ and $|G| \in \mathbb{F}^{\times}$, then $\mathbb{F}[V]^G$ is a polynomial algebra if and only if G is a pseudoreflection group.*

Recall that a linear automorphism $s : V \longrightarrow V$ is called a **pseudoreflection** if it is not the identity, has finite order, and $\dim_{\mathbb{F}}((1-s)V) = 1$. A **pseudoreflection group** is the group generated by pseudoreflections.

For a field $\mathbb{F}$ of characteristic 0 we have ([8], [16], or [9])

Theorem 1.3 (R. Steinberg, R. Kane). *If $\varrho : G \hookrightarrow GL(V)$ is a representation of a finite group G over a field $\mathbb{F}$ of characteristic 0, then G is a pseudoreflection group if and only if $\mathbb{F}[V]_G$ is a Poincaré duality algebra.*

In this note we shall prove the result of the Theorem (1.3) for the case $\mid G \mid \in \mathbb{F}^{\times}$, namely

Main Theorem 1.4. *If $\varrho : G \hookrightarrow GL(V)$ is a representation of a finite group G over a field $\mathbb{F}$ and $\mid G \mid \in \mathbb{F}^{\times}$, then G is a pseudoreflection group if and only if $\mathbb{F}[V]_G$ is a Poincaré duality algebra.*

To prove the Main Theorem we make use of a result of L. Smith ([13]).

Theorem 1.5 (L. Smith). *Let $\mathbb{F}$ be a field, let $V \cong \mathbb{F}^n$, and let $f_1, \cdots, f_n \in \mathbb{F}[V]$ be a regular sequence of maximal length. Then the quotient $F[V]/(f_1, \cdots, f_n)$ is a Poincaré duality algebra.*

Corollary 1.6. *If $\varrho : G \hookrightarrow GL(V)$ is a representation of a finite group G over a field $\mathbb{F}$ of arbitrary characteristic and $F[V]^G$ is a polynomial algebra, then $\mathbb{F}[V]_G$ is a Poincaré duality algebra.*

Therefore, we only need to prove that the converse of Corollary 1.6 is true in the case $|G| \in \mathbb{F}^{\times}$. In the next two sections, we explain the process.

2. Preliminaries

We assume that $\mathbb{F}$ is a perfect field of characteristic p. First of all, we recall the construction of the commutative rings, denoted by $W_m(\mathbb{F})$, for all $m \geq 1$, with the use of Witt's vector calculus. We call $W_m(\mathbb{F})$ the **ring of Witt vector** of **length** m over $\mathbb{F}$. The elements in $W_m(\mathbb{F})$ are m-tupels $(a_1, \cdots, a_m)$, $a_i \in \mathbb{F}$, $i = 1, \cdots, m$, with a particular ring structure; for more details see [5], [17]. These rings define the projective limit $W(\mathbb{F}) := \varprojlim_r W_r(\mathbb{F})$; the ring $W(\mathbb{F})$ is the ring of Witt vectors of infinite length over $\mathbb{F}$. We refer to $W(\mathbb{F})$ as the **Witt-ring** of $\mathbb{F}$. The Witt-ring $W(\mathbb{F})$ is a discrete valuation ring whose residue class field is just the field $\mathbb{F}$. The Witt-ring $W(\mathbb{F})$ is an integral domain of characteristic 0, so that its fraction field, denoted by $\mathbb{K}$, is also of characteristic 0. If $\mathbb{F}$ is a prime field $\mathbb{F}_p$, the Witt-ring $W(\mathbb{F}_p)$ is exactly the p-adic integers $\hat{\mathbb{Z}}_p$.

The next step is to assume $\varrho : G \hookrightarrow GL(n, \mathbb{F})$ is a representation of G over $\mathbb{F}$. In the sequel we shall define a representation $\tilde{\varrho} : G \hookrightarrow GL(n, W(\mathbb{F}))$ realized as a lifting of ϱ from $\mathbb{F}$ to $W(\mathbb{F})$. Since we have the inclusion $W(\mathbb{F}) \hookrightarrow \mathbb{K}$, the lifting from $\tilde{\varrho} : G \hookrightarrow GL(n, W(\mathbb{F}))$ to $\hat{\varrho} : G \hookrightarrow GL(n, \mathbb{K})$ is obvious. Furthermore, if the ring $\mathbb{F}[V]_G$ of coinvariants is a Poincaré duality algebra, so is $W(\mathbb{F})[M]_{\tilde{G}}$ and also $\mathbb{K}[T]_{\hat{G}}$. By Theorems 1.2 and 1.3 $\mathbb{K}[T]^{\hat{G}}$ is then a polynomial algebra. As a consequence we will show in the next section that $\mathbb{F}[V]^G$ is a polynomial algebra as well. In this section, we give the argument for the lifting of a representation $G \hookrightarrow GL(n, \mathbb{F})$ on $W(\mathbb{F})$ and $\mathbb{K}$.

Notation. $M :=$ $W(\mathbb{F})$-module of rank n, so that $V \cong \mathbb{F} \otimes_{W(\mathbb{F})} M \cong \mathbb{F}^n$,
$T := \mathbb{K} \otimes M \cong \mathbb{K}^n$,
$G := \varrho(G) \leq GL(n, \mathbb{F}) \cong GL(V)$,
$\tilde{G} := \tilde{\varrho}(G) \leq GL(n, W(\mathbb{F})) \cong GL(M)$,
$\hat{G} := \hat{\varrho}(G) \leq GL(n, \mathbb{K}) \cong GL(T)$,
Note that $\mid G \mid = \mid \tilde{G} \mid = \mid \hat{G} \mid$.

Theorem 2.1. *Suppose that G is a finite group with $\mid G \mid \in \mathbb{F}^\times$. Let $\rho : G \hookrightarrow GL(n, \mathbb{F})$ be a representation of G over the field $\mathbb{F}$. Then ϱ can be lifted to $\tilde{\varrho} : G \hookrightarrow GL(n, W(\mathbb{F}))$, and this lifting is unique, up to conjugation.*

Since $W(\mathbb{F}) = \varprojlim_r W_r(\mathbb{F})$, it follows that $GL(n, W(\mathbb{F})) = \varprojlim_r GL(n, W_r(\mathbb{F}))$. It is then enough to prove that:

Theorem 2.2. *Suppose that G is a finite group with $\mid G \mid \in \mathbb{F}^\times$. Let $\varrho_r : G \hookrightarrow GL(n, W_r(\mathbb{F}))$ be a representation of G over the ring $W_r(\mathbb{F})$. Then ϱ_r can be lifted to a representation $\varrho_{r+1} : G \hookrightarrow GL(n, W_{r+1}(\mathbb{F}))$ for all $r \geq 1, r \in \mathbb{N}$. This lifting is unique, up to conjugation.*

The sketch of the proof is as follows: We consider the exact sequence of groups

$$0 \longrightarrow \ker(\breve{\pi}_r) \hookrightarrow \breve{G} \xrightarrow{\breve{\pi}_r} \varrho_r(G) \longrightarrow 1 \quad (r \geq 1), \tag{1}$$

where $\breve{G} \leq GL(n, W_{r+1}(\mathbb{F}))$ is a subgroup and $\ker(\breve{\pi}_r)$ is an abelian p-group and a normal subgroup of $\breve{G}$.

Since $|G| \in \mathbb{F}^\times$, the cohomology group $H^i(\varrho_r(G), \ker(\breve{\pi}_r)) = 0$ for all $i \geq 1$ (see [6]). Since $H^2(\varrho_r(G), \ker(\breve{\pi}_r)) = 0$ it follows that this exact sequence (1) splits. Thus we can choose a splitting and define ϱ_{r+1}. The set of $\ker(\breve{\pi}_r)$ conjugation classes of splitting homomorphisms $\varrho_r(G) \longrightarrow \breve{G}$ is in 1-1 correspondence with the first cohomology group $H^1(\varrho_r(G), \ker(\breve{\pi}_r))$. Thus there is precisely one conjugation class.

We have therefore constructed a representation $\tilde{\varrho} : G \hookrightarrow GL(n, W(\mathbb{F}))$ from a given representation $\varrho : G \hookrightarrow GL(n, \mathbb{F})$. Hence, we can define a representation $\hat{\varrho} : G \hookrightarrow GL(n, \mathbb{K})$ via the inclusion $W(\mathbb{F}) \hookrightarrow \mathbb{K}$.

Remark 2.3. If $\mathbb{F}$ is a field of characteristic $p \neq 0$ which is not perfect, then we can always find a local ring R of characteristic 0 whose residue class field is $\mathbb{F}$. The ring R may be taken to be a subring of the Witt ring $W(\mathbb{F})$ of $\mathbb{F}$ (see [11]). Therefore, Theorem 2.1 is as well true for the ring R. In the next section we work with R instead of $W(\mathbb{F})$, i.e. we set $\mathbb{K} = FF(R)$ and $M \cong R^n$.

3. The main theorem

In this section we use the result of §2 to prove the following main theorem.

Theorem 3.1. *Let $\varrho : G \hookrightarrow GL(n, \mathbb{F}) \cong GL(V)$ be a representation of a finite group G over a field $\mathbb{F}$ of characteristic p. Suppose that $|G| \in \mathbb{F}^\times$. If $\mathbb{F}[V]_G$ is a Poincaré duality algebra, then G is a pseudoreflection group.*

Proposition 3.2 (see, e.g. [4]). *Suppose that* $| G | \in \mathbb{F}^\times$. *If* $\tilde{\varrho} : G \hookrightarrow GL(M)$ *is a lifting of the representation* $\varrho : G \hookrightarrow GL(V)$ *and* $\hat{\varrho} : G \hookrightarrow GL(T)$ *is an extension of* $\tilde{\varrho}$*, then there exist isomorphisms*

$$\mathbb{F} \otimes_R R[M]^{\tilde{G}} \cong \mathbb{F}[V]^G \quad \textit{and} \quad \mathbb{K} \otimes_R R[M]^{\tilde{G}} \cong \mathbb{K}[T]^{\hat{G}}.$$

Proof. We consider the mapping $\psi : \mathbb{F} \otimes_R R[M]^{\tilde{G}} \longrightarrow \mathbb{F}[V]^G$ given by $a \otimes \tilde{f} \mapsto a\tilde{f}$, for all $\tilde{f} \in R[M]^{\tilde{G}}$, $a \in \mathbb{F}$. It remains to show that ψ is bijective.

$\underline{\psi \text{ is surjective}}$: For an invariant polynomial $f \in \mathbb{F}[V]^G$ there is always a polynomial $\tilde{f} \in R[M]$. Since $| G | \in \mathbb{F}^\times$, the Reynolds operation $\pi^{\tilde{G}}(\tilde{f}) = \frac{1}{|\tilde{G}|} \sum_{\sigma \in \tilde{G}} \sigma \tilde{f}$, for all $\tilde{f} \in R[M]$, is surjective. This implies that ψ is surjective.

$\underline{\psi \text{ is injective}}$: We choose an invariant polynomial $\tilde{f} \in R[M]^{\tilde{G}}$ so that $\psi(1 \otimes \tilde{f}) = 0$, i.e., $1 \otimes \tilde{f} \equiv 0 \bmod p$. So, we may choose an invariant polynomial $\tilde{h} \in R[M]^{\tilde{G}}$ with $\tilde{f} = p\tilde{h}$. This immediately implies that $1 \otimes \tilde{f} \equiv 0 \bmod p$. The remaining case is analogous. □

The following corollary is an easy consequence of Proposition 3.2.

Corollary 3.3. *Suppose that* $| G | \in \mathbb{F}^\times$. *If* $\tilde{\varrho} : G \hookrightarrow GL(M)$ *is a lifting of the representation* $\varrho : G \hookrightarrow GL(V)$ *and* $\hat{\varrho} : G \hookrightarrow GL(T)$ *is an extension of* $\tilde{\varrho}$*, then there exist isomorphisms*

$$\mathbb{F} \otimes_R R[M]_{\tilde{G}} \cong \mathbb{F}[V]_G \quad \textit{and} \quad \mathbb{K} \otimes_R R[M]_{\tilde{G}} \cong \mathbb{K}[T]_{\hat{G}}.$$

Let $\mathfrak{R}$ be a graded connected commutative Noetherian ring. For a finitely generated graded $\mathfrak{R}$-module N we define a finitely generated graded $\mathfrak{R}_0$-module

$$Soc(N) := \{x \in N \mid \overline{\mathfrak{R}} \cdot x = 0\},$$

where $\overline{\mathfrak{R}}$, is denoted as $\bigoplus_{i \geq 1} \mathfrak{R}_i$, and call it the **socle** of N.

Theorem 3.4. *Let* $\mathfrak{R}$ *be a graded connected commutative Noetherian ring and let* $\mathfrak{q}$ *be a* $\overline{\mathfrak{R}}$*-primary ideal in* $\mathfrak{R}$*. Then the following conditions are equivalent:*

(1) $Soc(\mathfrak{R}/\mathfrak{q})$ *is a finitely generated free* $\mathfrak{R}/\overline{\mathfrak{R}}$*-module of rank* 1.
(2) *The quotient* $\mathfrak{R}/\mathfrak{q}$ *is a Poincaré duality algebra.*

Proof. (1) ⇒ (2): We put $\mathfrak{R}/\mathfrak{q} = \bigoplus_{i \geq 0} H_i$, where H_i is the component of $\mathfrak{R}/\mathfrak{q}$ of degree i. There is an integer $d \in \mathbb{N}$ so that $H_d \cong Soc(\mathfrak{R}/\mathfrak{q})$ and $H_i = 0$ for $i > d$. It remains to show that the pairing $H_i \otimes H_{d-i} \longrightarrow H_d$ is nonsingular. Since the module $H_d \cong Soc(\mathfrak{R}/\mathfrak{q})$ is of rank 1, we must have a homogeneous element $a \in \mathfrak{R}/\mathfrak{q}$, $a \neq 0$, satisfying $H_d \subseteq (a) \subset \mathfrak{R}/\mathfrak{q}$. Thus, for every homogeneous element $x \neq 0$ in H_d there exists a homogeneous element b in $\mathfrak{R}/\mathfrak{q}$ satisfying $x = ab \neq 0$.

(2) ⇒ (1): This is obvious. □

From Theorem 3.4 together with Corollary 3.3 we immediately obtain the following theorem.

Theorem 3.5. *The conditions are as in Proposition* 3.2. *If* $\mathbb{F}[V]_G$ *is a Poincaré duality algebra of formal dimension* d*, then* $R[M]_{\tilde{G}}$ *is a Poincaré duality algebra of formal dimension* d*, and therefore,* $\mathbb{K}[T]_{\hat{G}}$ *is a Poincaré duality algebra of formal dimension* d *as well.*

We recall again that if the field $\mathbb{K}$ has characteristic 0, then $\mathbb{K}[T]_{\hat{G}}$ is a Poincaré duality algebra if and only if $\hat{G}$ is a pseudoreflection group, equivalently, $\mathbb{K}[T]^{\hat{G}}$ is a polynomial algebra.

Theorem 3.6. *The conditions are as in Proposition* 3.2. *If* $\mathbb{K}[T]^{\hat{G}}$ *is a polynomial algebra, then* $R[M]^{\tilde{G}}$ *is a polynomial algebra, and therefore,* $\mathbb{F}[V]^G$ *is a polynomial algebra as well.*

Proof. Let $\mathbb{K}[T]^{\hat{G}}$ be a polynomial algebra generated by invariant polynomials $f_1, \cdots, f_n$. From the result of Proposition 3.2 we know that for each generator f_i, $i = 1, \cdots, n$, for $\mathbb{K}[T]^{\hat{G}}$ there is an invariant polynomial, denoted by $\tilde{f}_i$ in $R[M]^{\tilde{G}}$. Therefore, $R[\tilde{f}_1, \cdots, \tilde{f}_n] \longrightarrow R[M]^{\tilde{G}}$ is a monomorphism. We set $d_i = \deg(f_i)$ for each $i = 1, \cdots, n$. So we have the Poincaré series

$$P(\mathbb{K}[T]^{\hat{G}}, t) = P(R[M]^{\tilde{G}}, t) = P(R[\tilde{f}_1, \cdots, \tilde{f}_n], t)$$

and

$$\begin{aligned}\deg(\mathbb{K}[T]^{\hat{G}}) = \frac{1}{\mid \hat{G} \mid} = \frac{1}{d_1 \cdots d_n} &= (1-t)^n P(R[\tilde{f}_1, \cdots, \tilde{f}_n], t) \mid_{t=1} \\ &= (1-t)^n P(R[M]^{\tilde{G}}, t) \mid_{t=1} = \deg(R[M]^{\tilde{G}}) = \frac{1}{\mid \tilde{G} \mid}.\end{aligned}$$

Hence, we have $\mid \hat{G} \mid = d_1 \cdots d_n = \mid \tilde{G} \mid$. In addition, since $\tilde{f}_1, \cdots, \tilde{f}_n$ are algebraically independent over R, we obtain $R[M]^{\tilde{G}} \cong R[\tilde{f}_1, \cdots, \tilde{f}_n]$. The proof for the last result is analogous. □

From Theorem 3.5 and Theorem 3.6 together with Theorem 1.2 we complete the proof of the main theorem.

Remark 3.7. This result may fail for p-groups. There is a counterexample when $p = 2$ and $\dim_{\mathbb{F}_p} V = 4$ (see [15]).

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