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ON ROURKE'S EXTENSION OF GROUP PRESENTATIONS AND A CYCLIC VERSION OF THE ANDREWS–CURTIS CONJECTURE

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Abstract. In 1979, Rourke proposed to extend the set of cyclically reduced defining words of a group presentation $\mathcal{P}$ by using operations of cyclic permutation, inversion and taking double products. He proved that iterations of these operations yield all cyclically reduced words of the normal closure of defining words of $\mathcal{P}$ if the group, defined by the presentation $\mathcal{P}$, is trivial. We generalize this result by proving it for every group presentation $\mathcal{P}$ with an obvious exception. We also introduce a new, "cyclic", version of the Andrews–Curtis conjecture and show that the original Andrews–Curtis conjecture with stabilizations is equivalent to its cyclic version.

1. Two theorems

Let a group $\mathcal{G}$ be defined by a presentation $\mathcal{P}$ in terms of generators and defining relations

$$\mathcal{P} = \langle a_1, \dots, a_m \parallel \ R = 1,\ R \in \mathcal{R} \rangle, \tag{1}$$

where $\mathcal{R}$ is a set of words in the alphabet $\mathcal{A}^{\pm 1} = \{a_1^{\pm 1}, \dots, a_m^{\pm 1}\}$, whose elements $R \in \mathcal{R}$ are assumed to be cyclically reduced words and are called *defining words* of $\mathcal{G}$. Recall that this means that $\mathcal{G} = F(\mathcal{A})/N(\mathcal{R})$ is the quotient group of the free group $F(\mathcal{A})$ in $\mathcal{A}$ by the normal closure $N(\mathcal{R})$ of $\mathcal{R}$ in $F(\mathcal{A})$.

In [7], Rourke introduced the two operations $^+, ^\sim$ over a set $\mathcal{R} \subset F(\mathcal{A})$ of cyclically reduced words so that $\mathcal{R}^+$ denotes the set of all cyclic permutations of words in $\mathcal{R}$ and $\mathcal{R}^{-1}$, and $\mathcal{R}^\sim$stands for the union of $\mathcal{R}$ and the set of all cyclically reduced words obtained from all possible products $R_1 R_2$, where $R_1, R_2 \in \mathcal{R}$.

Let $\mathcal{R}_\infty$ be the $(^+, ^\sim)$-closure of $\mathcal{R}$, that is, $\mathcal{R}_\infty$ is the minimal set such that $\mathcal{R} \subseteq \mathcal{R}_\infty$ and $\mathcal{R}_\infty^+ = \mathcal{R}_\infty$, $\mathcal{R}_\infty^\sim = \mathcal{R}_\infty$. Clearly, $\mathcal{R}_\infty$ can be obtained from $\mathcal{R}$ by iterations of operations $^+, ^\sim$.

The main result of Rourke's article [7] states that if the group $\mathcal{G}$ is trivial, then $\mathcal{A} \subset \mathcal{R}_\infty$, that is, $\mathcal{R}_\infty$ is the set of all cyclically reduced words of $F(\mathcal{A})$. This result was reproved by Scarabotti [8]. It is also observed in [7] that, in general, the set $\mathcal{R}_\infty$ is "distinctly smaller" than the set $\widehat{N}(\mathcal{R})$ of all cyclically reduced words of the normal closure $N(\mathcal{R})$ of $\mathcal{R}$. For example, if $\mathcal{A} = \{a, b\}$ and $\mathcal{R} = \{a\}$, then

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$\mathcal{R}_\infty = \{a^k \mid k \in \mathbb{Z}\} \neq \widehat{N}(\mathcal{R})$. More generally, if there is a letter $a \in \mathcal{A}$ such that a, a^{-1} do not occur in words of $\mathcal{R}$, then, obviously, $\mathcal{R}_\infty \neq \widehat{N}(\mathcal{R})$. The first result of this article claims that the existence of such a letter is the only obstruction to the equality $\mathcal{R}_\infty = \widehat{N}(\mathcal{R})$.

Theorem 1. *Let $F(\mathcal{A})$ be the free group in an alphabet $\mathcal{A}$ and let $\mathcal{R} \subset F(\mathcal{A})$ be a set of cyclically reduced words. Then the Rourke closure $\mathcal{R}_\infty$ of $\mathcal{R}$ is the set $\widehat{N}(\mathcal{R})$ of all cyclically reduced words of the normal closure $N(\mathcal{R})$ of $\mathcal{R}$ in $F(\mathcal{A})$ unless there exists a letter $a \in \mathcal{A}$ such that both a, a^{-1} do not occur in words of $\mathcal{R}$.*

It is rather interesting to apply the Rourke idea of using only cyclic permutations (in place of arbitrary conjugations) in the context of the Andrews–Curtis conjecture; see [1], [2], [3], [5].

Recall that if $\mathcal{W} = (W_1, \dots, W_p)$ is a p-tuple of reduced words in $\mathcal{A}^{\pm 1}$, then Nielsen operations over $\mathcal{W}$ of types (T1)–(T2) are defined as follows.

(T1) For some i, $W_i \to W_i^{-1}$ and $W_k \to W_k$ for all k, $k \neq i$.
(T2) For some i, $W_i \to W_i W_j$, $j \neq i$, and $W_k \to W_k$ for all k, $k \neq i$.

We also consider an operation of the third type (T3) so that

(T3) For some i, $W_i \to S W_i S^{-1}$, where S is a word in $\mathcal{A}^{\pm 1}$, and $W_k \to W_k$ for all k, $k \neq i$.

Recall that extended Nielsen operations, termed *EN-operations*, are those of types (T1)–(T3); see [2].

The Andrews–Curtis conjecture, or *AC-conjecture*, states that *if a balanced presentation*

$$\mathcal{P} = \langle a_1, \dots, a_m \parallel \ R_1 = 1, \dots, R_m = 1 \rangle \tag{2}$$

defines the trivial group, then the m-tuple $\mathcal{R} = (R_1, \dots, R_m)$ can be brought to $\mathcal{A} = (a_1, \dots, a_m)$ by a finite sequence of EN-operations.

Consider a new, "cyclic", version of an EN-operation of type (T3) so that

(CT3) for some i, $W_i \to \overline{W}_i$, where $\overline{W}_i$ is a cyclic permutation of W_i, and $W_k \to W_k$ for all k, $k \neq i$.

Thus, in place of an arbitrary conjugation of W_i, we can now use only a cyclic permutation of W_i. Such redefined operations will be called cyclically extended Nielsen operations, or, briefly, *CEN-operations*, of types (CT1)–(CT3) (operations of types (CT1)–(CT2) are identical to those of types (T1)–(T2)).

Recall that EN-operations apply to tuples of (reduced) words and, for the cyclic version (as in the definition of Rourke's closure), it is more natural to require that CEN-operations apply to tuples of cyclically reduced words. Hence, when making a CEN-operation of type (CT2), we always cyclically reduce the resulting new word $W_i W_j$.

Now the cyclic version of the Andrews–Curtis conjecture, or *CAC-conjecture*, states that *if the presentation* (2) *defines the trivial group, then the m-tuple $\mathcal{R} = (R_1, \dots, R_m)$ can be brought to $\mathcal{A} = (a_1, \dots, a_m)$ by a finite sequence of CEN-operations.*

Recall that there is another, more general, version of the AC-conjecture, or *AC-conjecture with stabilizations*, in which a fourth type of operations, called *stabilizations*, is allowed.

(T4) Add (or remove) a new letter b both to the alphabet $\mathcal{A}$ and to the set $\mathcal{R}$ of defining words (when removing, b and b^{-1} must not occur in all other words of $\mathcal{R}$).

A stabilization that increases the number of letters in $\mathcal{A}$ is termed *positive*, otherwise, it is *negative*.

Quite analogously, we consider the cyclic version of the AC-conjecture with stabilizations, or *CAC-conjecture with stabilizations*, in which we use only cyclic permutations (versus arbitrary conjugations) and cyclically reduce the resulting words after each CEN-operation. Observe that, in general, the CAC-conjecture (with stabilizations) is stronger than the original AC-conjecture (with stabilizations, respectively) in the sense that if the CAC-conjecture (with stabilizations) holds for a presentation $\mathcal{P}$, then the AC-conjecture (with stabilizations, respectively) also holds for this $\mathcal{P}$.

Theorem 2. *The original Andrews–Curtis conjecture with stabilizations is equivalent to its cyclic version with stabilizations. More specifically, if* (2) *is a balanced presentation of the trivial group and the defining words $R_1, \dots, R_m$ are cyclically reduced, then the m-tuple $\mathcal{R} = (R_1, \dots, R_m)$ can be turned into $\mathcal{A} = (a_1, \dots, a_m)$ by a finite sequence of operations* (T1)–(T4) *if and only if $\mathcal{R}$ can be turned into $\mathcal{A}$ by a finite sequence of operations* (CT1)–(CT3), (T4). *In addition, if $\mathcal{R}$ can be transformed into $\mathcal{A}$ by operations* (T1)–(T3) *and $2s$ stabilizations, then $\mathcal{R}$ can be transformed into $\mathcal{A}$ by operations* (CT1)–(CT3) *and $2(m + 2s + 1)$ stabilizations.*

2. Proof of Theorem 1

Since one direction in Theorem 1 is obvious, we will be proving that if every letter $a \in \mathcal{A}$ occurs in words of $\mathcal{R}^{\pm 1}$, then $\mathcal{R}_\infty = \widehat{N}(\mathcal{R})$. If $\mathcal{A}$ has a single letter, then Theorem 1 is obvious and we can assume that $|\mathcal{A}| > 1$.

Let $X = Y$ denote the equality of elements of the free group $F(\mathcal{A})$ in $\mathcal{A}$ (also called $\mathcal{A}$-*words* or just *words*) and let $X \equiv Y$ stand for the graphical (= letter-by-letter) equality of words.

Lemma. *Let B be a reduced $\mathcal{A}$-word. Then there exists a word $R \in \mathcal{R}_\infty$ such that $R \equiv BC$ with some C.*

Proof. Pick a letter $a \in \mathcal{A}^{\pm 1}$ and let R be a word in $\mathcal{R}_\infty$ so that $R \equiv aD$ (such a word R can already be found in $\mathcal{R}^+$). Note that if E is a reduced word such that $E = D^3$, then $a^3E \in \mathcal{R}_\infty$, in particular, a^3E is cyclically reduced.

Now, by induction on the length $|B|$ of a word B, we will prove the following modified claim that obviously implies the Lemma: Suppose B is a reduced $\mathcal{A}$-word and $B \equiv b_1B_1 \equiv B_2b_2$, where $b_1, b_2 \in \mathcal{A}^{\pm 1}$. Then there exists a word $R \in \mathcal{R}_\infty$ such that $R \equiv b_1Bb_2C$. Note that this claim is actually proved above when $|B| = 1$. To make the induction step, we assume $|B| > 1$ and let $B \equiv b_1B_1 \equiv B_2b_2 \equiv B_3b_3b_2$, where $b_1, b_2, b_3 \in \mathcal{A}^{\pm 1}$. By the induction assumption, there are words R and S in $\mathcal{R}_\infty$ for B_2 and b_2, respectively, such that $R \equiv b_1B_2b_3C$ and $S \equiv b_2^3D$. Multiplying cyclic permutations $\bar{R} \equiv b_3Cb_1B_2$, $\bar{S} \equiv b_2^2Db_2$ of R, S, we get the word

$$\bar{R}\bar{S} \equiv b_3Cb_1B_2b_2^2Db_2 \equiv b_3Cb_1Bb_2Db_2$$

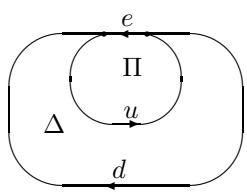
which is cyclically reduced and hence is in $\mathcal{R}_\infty$. Its cyclic permutation $b_1Bb_2Db_2b_3C$ is a word of the form required for B and the Lemma is proved. $\square$

Recall that a *van Kampen* (or *disk*) *diagram* Δ over presentation (1) is a planar, connected and simply connected 2-complex that is equipped with a labelling function φ from the set of oriented edges of Δ to $\mathcal{A}^{\pm 1}$ and has the following two properties (for more details see [4, 6]):

(L1) If e is an oriented edge of Δ, then $\varphi(e^{-1}) = \varphi(e)^{-1}$.
(L2) If Π is a face of Δ, then $\varphi(\partial\Pi)$ is a cyclic permutation of a word in $\mathcal{R} \cup \mathcal{R}^{-1}$.

Now suppose that W is a cyclically reduced nonempty word and $W = 1$ in the group $\mathcal{G}$ defined by presentation (1), that is, $W \in \widehat{N}(\mathcal{R})$. Then there is a van Kampen diagram $\Delta = \Delta(W)$ (see [4, 6]) over (1) such that $\varphi(\partial\Delta) \equiv W$, where the boundary $\partial\Delta$ of Δ is negatively (clockwise) oriented. We will prove that $W \in \mathcal{R}_\infty$ by induction on the number $|\Delta(2)|$ of faces in Δ.

As in [4, 6], assume that the boundary $\partial\Pi$ of a face Π of Δ is positively (counterclockwise) oriented. Pick a face Π in Δ so that there is an edge $e \in \partial\Pi$ with $e^{-1} \in \partial\Delta$. Denote $\partial\Pi = eu$, $\partial\Delta = e^{-1}d$; see Figure 1.



$\varphi(e) = a$
$\partial\Pi = eu$
$\partial\Delta = e^{-1}d$

FIGURE 1

Taking e and Π out of Δ, we will get a diagram Δ_1 such that $\partial\Delta_1 = du$ and $|\Delta_1(2)| = |\Delta(2)| - 1$.

If $\varphi(\partial\Delta_1) \equiv \varphi(d)\varphi(u) = 1$ in $F(\mathcal{A})$, then $\varphi(\partial\Delta) \equiv \varphi(\partial\Pi)^{-1} \equiv W$. Hence, $W \in \mathcal{R}_\infty$, as required.

Now assume that $\varphi(\partial\Delta_1) \neq 1$ in $F(\mathcal{A})$. Note that the words $\varphi(d)$, $\varphi(u)$ are subwords of cyclically reduced words W, $\varphi(\partial\Pi)$, respectively, and hence $\varphi(d)$, $\varphi(u)$ are reduced. Since the product $\varphi(d)\varphi(u)$ need not be (cyclically) reduced, it follows that the words $D \equiv \varphi(d)^{-1}$ and $U \equiv \varphi(u)$ can be represented in one of the following three forms (F1)–(F3).

(F1) $D \equiv BD_1E$ and $U \equiv BU_1E$, where $D_1U_1^{-1}$ is a nonempty cyclically reduced word (some of B, D_1, U_1, E may be empty).
(F2) $D \equiv BE$ and $U \equiv BSU_0S^{-1}E$, where U_0 is a nonempty cyclically reduced word (some of B, S, E may be empty).
(F3) $D \equiv BSD_0S^{-1}E$ and $U \equiv BE$, where D_0 is a nonempty cyclically reduced word (some of B, S, E may be empty).

Observe that there is a diagram Δ_0 such that $|\Delta_0(2)| \leq |\Delta(2)| - 1$ and $\varphi(\partial\Delta_0) \equiv D_1U_1^{-1}$ in Case (F1) or $\varphi(\partial\Delta_0) \equiv U_0$ in Case (F2) or $\varphi(\partial\Delta_0) \equiv D_0$ in Case (F3).

Since $\varphi(\partial\Delta_0)$ is cyclically reduced, it follows from the induction assumption that $\phi(\partial\Delta_0) \in \mathcal{R}_\infty$. Denote $\varphi(e) = a$, where $a \in \mathcal{A}^{\pm 1}$. In Cases (F1)–(F2), we can see that W is a cyclic permutation of the product of some cyclic permutations of $\phi(\partial\Delta_0)$ and $\varphi(\partial\Pi)^{-1}$; see Figure 2(a) in Case (F1) and Figure 2(b) in Case (F2).

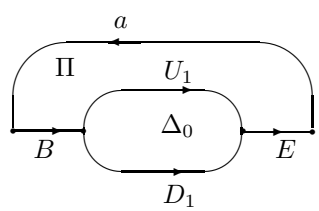


FIGURE 2(a)

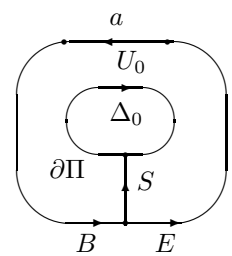


FIGURE 2(b)

In Case (F3), W is a cyclic permutation of the cyclically reduced product of words $\varphi(\partial\Pi)^{-1} \equiv B^{-1}a^{-1}E^{-1}$ and $SD_0^{-1}S^{-1}$; see Figure 3(a).

It follows from the Lemma, applied to a word of the form xSy, where $x, y \in \mathcal{A}^{\pm 1}$ are letters such that xE, D_0y are reduced (recall $|\mathcal{A}| > 1$), that there is a word $R \in \mathcal{R}_\infty$ such that $R \equiv ST$ for some T of the form $T \equiv yT_0x$ and the word $EaBSD_0T$ is cyclically reduced. Since the word $EaBSD_0T$ is the product of cyclic permutations of $\varphi(\partial\Pi)$ and $TS\varphi(\partial\Delta_0)^{-1}$, and $TS\varphi(\partial\Delta_0)^{-1}$ in its turn is the product of cyclic permutations of $R \equiv ST$ and $D_0 \equiv \varphi(\partial\Delta_0)^{-1}$ (see Figure 3(b)), it follows that $EaBSD_0T \in \mathcal{R}_\infty$. Now we observe that W^{-1} is a cyclic permutation of the product of some cyclic permutations of $EaBSD_0T$ and $R^{-1} \equiv T^{-1}S^{-1}$. Thus, $W \in \mathcal{R}_\infty$ and Theorem 1 is proved. □

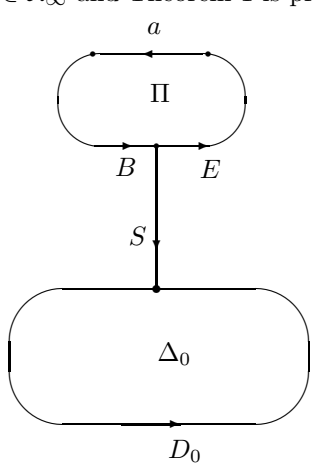


FIGURE 3(a)

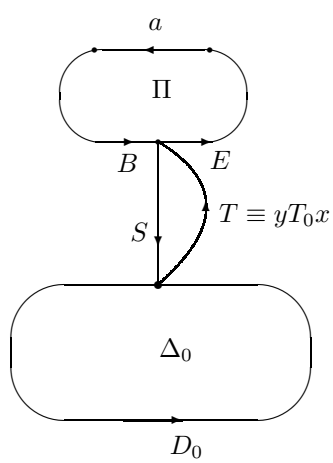


FIGURE 3(b)

3. PROOF OF THEOREM 2

Recall that the input and output of an EN-operation are tuples of (reduced) words while those of a CEN-operation are tuples of cyclically reduced words. Since CEN-operations are more restrictive and a cyclic permutation of a word can be realized by an EN-operation of type (T3), it follows that we only need to show that if $\mathcal{R} = (R_1, \ldots, R_m)$ can be turned into an $\mathcal{A} = (a_1, \ldots, a_m)$ by a sequence of EN-operations and $2s$ stabilizations, then $\mathcal{R}$ can also be turned into $\mathcal{A}$ by a sequence of CEN-operations and at most $2(m + 2s + 1)$ stabilizations.

Furthermore, it is clear that, in the process of getting $\mathcal{A}$ from $\mathcal{R}$ by EN-operations and $2s$ stabilizations, we can do all s positive stabilizations in the very beginning and all s negative stabilizations in the very end. Therefore, we can avoid stabilizations altogether and assume that the $(m+s)$-tuple $(\mathcal{R},\mathcal{B}) = (R_1,\dots,\ R_m, b_1,\dots,\ b_s)$, where $b_1,\dots,b_s$ are all new letters that were introduced by s positive stabilizations, can be turned into the $(m+s)$-tuple $(\mathcal{A},\mathcal{B}) = (a_1,\dots,a_m,b_1,\dots,b_s)$ by a sequence of EN-operations $\sigma_1,\sigma_2,\dots,\sigma_\ell$ of types (T1)–(T3).

Now, with the original $(m+s)$-tuple $\mathcal{W}(0) = (\mathcal{R},\mathcal{B})$, we associate a $(2m+2s+1)$-tuple

$$\mathcal{U}(0) \;=\; (R_1,\dots,R_m,\ b_1,\dots,b_s,\ x_1R_1,\dots,x_mR_m,\ x_{m+1}b_1,\dots,x_{m+s}b_s,\ y). \tag{3}$$

Clearly, $\mathcal{U}(0)$ can be obtained from $\mathcal{R}$ by $m+2s+1$ positive stabilizations and by CEN-operations (CT1)–(CT2).

Let

$$\mathcal{W}(k) = (W_1,\dots,W_{m+s})$$

be the $(m+s)$-tuple obtained from $\mathcal{W}(0)$ by the first $k \geq 0$ EN-operations $\sigma_1,\dots,\sigma_k$. Now, by induction on $k \geq 0$ (the base $k=0$ is trivial), assume that a $(2m+2s+1)$-tuple $\mathcal{U}(k) = (U_1,\dots,U_{2m+2s+1})$ of cyclically reduced words is already constructed so that U_i is freely conjugate (that is, conjugate in a free group) to W_i for $1 \leq i \leq m+s$ and, for $i > m+s$, we have

$$U_{m+s+1} \equiv x_1R_1,\dots,U_{2m+s} \equiv x_mR_m,\ U_{2m+s+1} \equiv x_{m+1}b_1,\dots,\ U_{2m+2s} \equiv x_{m+s}b_s, U_{2m+2s+1} \equiv y. \tag{4}$$

Let $\mathcal{W}(k+1)$ be obtained from $\mathcal{W}(k)$ by the next EN-operation σ_{k+1}. Our goal is to find a sequence of CEN-operations that would turn $\mathcal{U}(k)$ into $\mathcal{U}(k+1)$, where $\mathcal{U}(k+1)$ corresponds to $\mathcal{W}(k+1)$ in the same fashion as $\mathcal{U}(k)$ corresponds to $\mathcal{W}(k)$.

If σ_{k+1} has type (T1), that is, $W_i \to W_i^{-1}$, then we make a similar CEN-operation over $\mathcal{U}(k)$ which is $U_i \to U_i^{-1}$. If σ_{k+1} has type (T3), that is, $W_i \to SW_iS^{-1}$, then no change is needed and $\mathcal{U}(k+1) = \mathcal{U}(k)$.

Now consider the main case when σ_{k+1} has type (T2) and $W_i \to W_iW_j$, $j \neq i$, $1 \leq i,j \leq m+s$. Note that for some reduced word C in the alphabet $(\mathcal{A}\cup\mathcal{B})^{\pm1} = \{a_1^{\pm1},\dots,a_m^{\pm1},b_1^{\pm1},\dots,b_s^{\pm1}\}$, the word $U_iCU_jC^{-1}$ is freely conjugate to W_iW_j.

Let P_i denote a cyclically reduced word which is freely conjugate to $U_iCU_jC^{-1}$. We will construct a sequence of CEN-operations that turn the word U_i into P_i and preserve all other components of $\mathcal{U}(k)$. This new $(2m+2s+1)$-tuple will be the desired $\mathcal{U}(k+1)$.

Let $C \equiv c_1\dots c_d$, where $c_1,\dots,c_d$ are letters in $(\mathcal{A}\cup\mathcal{B})^{\pm1}$, and let $Q_1,\dots,Q_d$ be cyclic permutations of some words among $U_{m+s+1}^{\pm1},\dots,U_{2m+2s}^{\pm1}$ (see (4)) such that $Q_1 \equiv c_1T_1,\dots,\ Q_d \equiv c_dT_d$.

Consider a sequence of CEN-operations (CT1)–(CT3) so that

$$\begin{aligned} U_i &\to yU_iy \to yU_iyc_1T_1 \to T_1yU_iyc_1 \to yT_1yU_iyc_1c_2T_2 \\ &\to yT_2yT_1yU_iyc_1c_2 \to \dots \to yT_dyT_{d-1}\dots yT_1yU_iyc_1\dots c_dyU_j \\ &\to U_iyc_1\dots c_dyU_jyT_dyT_{d-1}\dots yT_1y \equiv U_{i,1} \end{aligned}$$

and all other components of $\mathcal{U}(k)$ are preserved. Now we will use more CEN-operations to turn the subwords $T_d,\dots,T_1$ of $U_{i,1}$ into $c_d^{-1},\dots,c_1^{-1}$, respectively.

Doing this transforms the word $U_{i,1}$ into

$$U_{i,2} \equiv U_i y c_1 \dots c_d y U_j y c_d^{-1} y c_{d-1}^{-1} \dots y c_1^{-1} y.$$

Using more CEN-operations of types (CT1)–(CT3) (the j index in CEN-operations of type (CT2) is now $2m+2s+1$), we delete all occurrences of letters $y^{\pm 1}$ in $U_{i,2}$. Doing this turns the word $U_{i,2}$ into a cyclic permutation of the word P_i and an application of (CT3) finally yields the desired $(2m+2s+1)$-tuple $\mathcal{U}(k+1)$ whose ith component is P_i and all other components are those of $\mathcal{U}(k)$.

Thus, by induction, it is proved that for every $k \geq 0$ there is a sequence of CEN-operations that turn the tuple $\mathcal{U}(0)$ (see (3)) into a $(2m+2s+1)$-tuple $\mathcal{U}(k) = (U_1, \dots, U_{2m+2s+1})$ such that $U_{m+s+1}, \dots, U_{2m+2s+1}$ are given by formulas (4) and $U_1, \dots, U_{m+s}$ are cyclically reduced words freely conjugate to $W_1, \dots, W_{m+s}$, respectively, where $\mathcal{W}(k) = (W_1, \dots, W_{m+s})$ is obtained from $\mathcal{W}(0)$ by the first k EN-operations $\sigma_1, \dots, \sigma_k$. Since $\mathcal{W}(\ell) = (a_1, \dots, a_m, b_1, \dots, b_s)$, it follows that

$$\mathcal{U}(\ell) = (a_1, \dots, a_m, b_1, \dots, b_s, U_{m+s+1}, \dots, U_{2m+2s+1}).$$

Now we can use obvious CEN-operations and $m+2s+1$ negative stabilizations to change $\mathcal{U}(\ell)$ into $(a_1, \dots, a_m)$. Theorem 2 is proved. $\square$

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