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SELF DELTA-EQUIVALENCE OF COBORDANT LINKS

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Abstract. Self Δ-equivalence is an equivalence relation for links, which is stronger than the link-homotopy defined by J. Milnor. It is known that cobordant links are link-homotopic and that they are not necessarily self Δ-equivalent. In this paper, we will give a sufficient condition for cobordant links to be self Δ-equivalent.

1. Introduction

In this paper, all links will be assumed to be ordered and oriented, and they will be considered up to ambient isotopy.

A Δ-*move* [6] is a local move on links as illustrated in Figure 1. If the three strands in Figure 1 belong to the same component of a link, we call it a *self* Δ-*move* [11]. Two links are said to be *self* Δ-*equivalent* if one can be deformed into the other by a finite sequence of self Δ-moves.

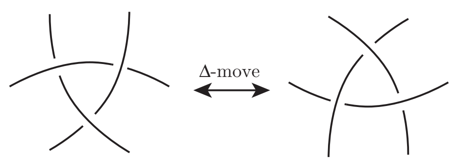

Figure 1.

Two links are said to be *link-homotopic* if one can be deformed into the other by a finite sequence of self-crossing changes [5]. Note that self Δ-equivalence implies link-homotopy, i.e., if two links are self Δ-equivalent, then they are link-homotopic.

Let $L_i = K_{i1}\cup\cdots\cup K_{in}$ $(i = 0, 1)$ be n-component links. Two links L_0 and L_1 are *cobordant* if there is a disjoint union $\mathcal{A} = A_1\cup\cdots\cup A_n$ of n annuli in $S^3\times[0,1]$ with $(\partial(S^3\times[0,1]), \partial A_j) = (S^3\times\{0\}, K_{0j})\cup(-S^3\times\{1\}, -K_{1j})$ $(j = 1, ..., n)$, where $-X$ denotes X with the opposite orientation. Then $\mathcal{A}$ is called a *concordance* between L_0 and L_1.

It is known that cobordism implies link-homotopy [1], [3]. In [11], it is shown that every ribbon link is self Δ-equivalent to a trivial link, and it is conjectured that if two links are cobordant, then they are self Δ-equivalent. Nakanishi and Shibuya [9] give a counterexample for this conjecture. After that Nakanishi and Ohyama

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give a classification for 2-component links up to self Δ-equivalence [7]. By using their classification theorem, we have the following: (1) If 2-component links with linking numbers zero are cobordant, then they are self Δ-equivalent. (2) The links illustrated in Figure 2 are cobordant and are not self Δ-equivalent if $|p| \neq 0$. This implies that for any integer $p \neq 0$, there are two links with linking number p such that they are cobordant and not self Δ-equivalent. The case that $|p| = 1$ is due to Nakanishi and Shibuya [9]. The linking number is an obstruction for links which are cobordant to be self Δ-equivalent. Our purpose is to find a sufficient condition for links which are cobordant to be self Δ-equivalent.

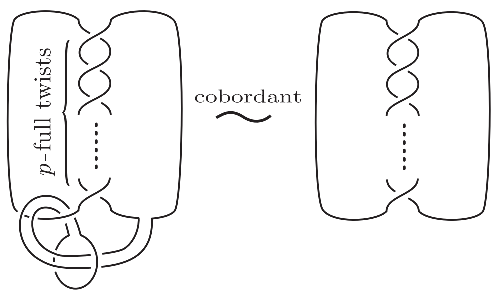

FIGURE 2.

Since any knot is (self) Δ-equivalent to a trivial knot [6], any link is self Δ-equivalent to a link each component of which is a trivial knot. For a 2-component link with trivial components, the following conditions are equivalent:

(1) The linking number is 0.

(2) It is link-homotopic to a trivial link.

(3) Each component is null-homotopic in the complement of the other component.

For links with n trivial components ($n \geq 3$), the conditions above are not equivalent. Note that (3) $\Rightarrow$ (2) $\Rightarrow$ (1). So we have the following questions.

Questions. Let L_0 and L_1 be cobordant and their components all trivial.

(1) If the linking numbers of all 2-component sublinks of L_i ($i = 0, 1$) are 0, then are they self Δ-equivalent?

(2) If L_0 and L_1 are link-homotopic to a trivial link, then are they self Δ-equivalent?

(3) If each component K_i of L_i is null-homotopic in $S^3 \setminus (L_i - K_i)$ ($i = 0, 1$), then are they self Δ-equivalent?

In this paper, we give a negative answer to Question (1) and an affirmative answer to Question (3). Question (2) is still open (likely negative).

The following theorem gives us an affirmative answer to Question (3).

Theorem 1. *Let L_0 and L_1 be links which are cobordant. If each component K_i of L_i is null-homotopic in $S^3 \setminus (L_i - K_i)$ ($i = 0, 1$), then L_0 and L_1 are self Δ-equivalent.*

The following propositon gives a negative answer to Question (1).

Proposition 1. *There exists a 3-component link with trivial components such that it is cobordant to Borromean rings and it is not self Δ-equivalent to Borromean rings.*

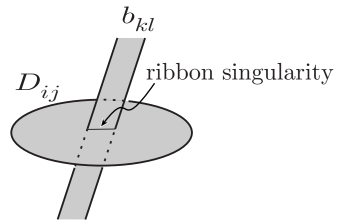

FIGURE 3.

2. PROOF OF THEOREM 1

A concordance $\mathcal{A}$ in $S^3 \times [0,1]$ between $L_0 \subset S^3 \times \{0\}$ and $L_1 \subset S^3 \times \{1\}$ is called a *ribbon concordance* [2] if the restriction to $\mathcal{A}$ of the projection $S^3 \times [0,1] \to [0,1]$ is a Morse function with no minimal points. In this case we write $L_0 \geq L_1$.

For a concordance $\mathcal{A}$ betwen L_0 and L_1, there is a concordance $\mathcal{A}'$ between L_0 and L_1 such that $\mathcal{A}'$ is ambient isotopic to $\mathcal{A}$ in $S^3 \times [0,1]$ and $\mathcal{A}' \cap (S^3 \times [0,1/2])$ (resp. $\mathcal{A}' \cap (S^3 \times [1/2,1])$) is a ribbon concordance for $\mathcal{A}' \cap (S^3 \times \{1/2\}) \geq K_0$ (resp. $\mathcal{A}' \cap (S^3 \times \{1/2\}) \geq K_1$). Therefore, Theorem 1 follows directly the following theorem.

Theorem 2. *Let L_0 and L_1 be links. If $L_0 \geq L_1$ and each component K of L_1 is null-homotopic in $S^3 \setminus (L_1 - K)$, then L_0 and L_1 are self Δ-equivalent.*

Proof. Suppose $L_0 = K_{01} \cup K_{02} \cup \cdots \cup K_{0n}$ and $L_1 = K_{11} \cup K_{12} \cup \cdots \cup K_{1n}$ are not self Δ-equivalent. We choose L_0 up to self Δ-equivalence so that the number of the maximal points for ribbon concordance $A_1 \cup A_2 \cup \cdots \cup A_n$ between L_0 and L_1 is minimum. It is known in [4] that $L_0 \geq L_1$ if and only if L_0 is a *band sum* of L_1 and a trivial link. Hence, there exist a disjoint union $\bigcup_{i,j} D_{ij}$ of 2-disks $D_{11}, ..., D_{1m_1}, ..., D_{n1}, ..., D_{nm_n}$ and a disjoint union $\bigcup_{i,j} b_{ij}$ of 2-disks, *bands*, $b_{11}, ..., b_{1m_1}, ..., b_{n1}, ..., b_{nm_n}$ such that

(1) $L_1 \cap (\bigcup_{ij} D_{ij}) = \emptyset$,

(2) $b_{kl} \cap (L_1 \cup \bigcup_{ij} \partial D_{ij}) = b_{kl} \cap (K_{1k} \cup \bigcup_j \partial D_{kj})$ consists of disjoint two-arcs in ∂b_{kl},

(3) $b_{kl} \cap (\bigcup_{ij} \mathrm{int} D_{ij})$ consists of proper arcs in b_{kl}, which are called *ribbon singularities* (see Figure 3),

(4) $L_0 = L_1 \cup \bigcup_{ij}(\partial D_{ij} \cup \partial b_{ij}) - \mathrm{int}((\bigcup_{ij} \partial b_{ij}) \cap (L_1 \cup \bigcup_{ij} \partial D_{ij}))$, and

(5) $K_{0k} = K_{1k} \cup \bigcup_j(\partial D_{kj} \cup \partial b_{kj}) - \mathrm{int}((\bigcup_j \partial b_{kj}) \cap (K_{1k} \cup \bigcup_j \partial D_{kj}))$.

Note that the 2-disks D_{ij}'s correspond to the maximal points for the ribbon concordance. Set $\mathcal{D}_k = \bigcup_j D_{kj},\ \mathcal{D} = \bigcup_k \mathcal{D}_k,\ \mathcal{B}_k = \bigcup_j b_{kj},\ \mathcal{B} = \bigcup_k \mathcal{B}_k$. We may suppose that $\mathcal{D}_1 \neq \emptyset$. Since K_{11} is null-homotopic in $S^3 \setminus (L_1 - K_{11})$, K_{11} bounds a singular 2-disk D_0 in $L_1 - K_{11}$ each singularity of which is a *clasp* [10] (see Figure 4). We may assume that $D_0 \cap \mathcal{D} = \emptyset$, $\mathcal{B}$ is disjoint from the clasp singularities of D_0, and $(D_0 - K_{11}) \cap \mathcal{B}$ consists of ribbon singularities.

Moreover, by *sliding bands* in $\mathcal{B}_1$ suitably [4], we may assume that b_{11} connects ∂D_0 and ∂D_{11}, and that b_{1j} connects $\partial D_{1(j-1)}$ and ∂D_{1j}, without changing the number of 2-disks in $\mathcal{D}$.

We deform $D_0 \cup \mathcal{D}_1 \cup \mathcal{B}_1$ into $D_0' \cup \mathcal{D}_1' \cup \mathcal{B}_1'$ as illustrated in Figure 5 (a), (b) so that $D_0' \cap \mathcal{B}_1' = \emptyset$ and that each 2-disk in $\mathcal{D}_1'$ contains at most a single ribbon singularity of $\mathcal{D}_1' \cap \mathcal{B}_1'$. Here we temporarily ignore $\mathcal{D}_1' \cap \mathcal{B}_k$ ($k \geq 2$). Set $\mathcal{D}_1' =$

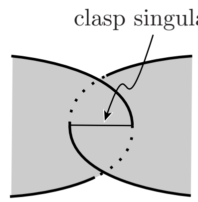

FIGURE 4.

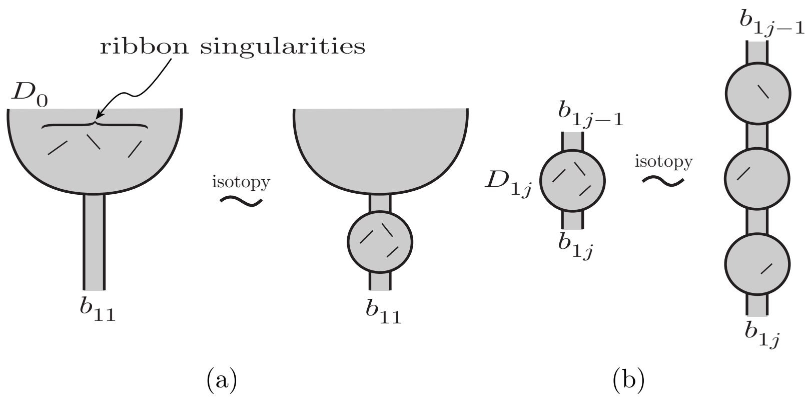

FIGURE 5.

$D_1 \cup D_2 \cup \cdots \cup D_m$, $\mathcal{B}'_1 = b_1 \cup b_2 \cup \cdots \cup b_m$, and assume that b_1 connects D'_0 and D_1, and that b_j connects D_{j-1} and D_j $(j \geq 2)$. □

Claim. The deformations as illustrated in Figure 6 (a), (b) and (c) are realized by Δ-moves.

Proof of Claim. Since the local move as illustrated in Figure 7 is realized by a single Δ-move and ambient isotopies (for example, see [12]), the claim above follows Figure 8 (a), (b) and (c). □

By combining the deformations as illustrated in Figure 6 (a), (b) and (c), we can change $D_m \cup b_m$ so that $D_m \cap \mathcal{B}'_1 = \emptyset$. Then we shrink $D_m \cup b_m$ into a part of D_{m-1}. In these deformations, the ambient isotopy class of $(L_1 - K_{11}) \cup \bigcup_{k\geq 2}(\mathcal{D}_k \cup \mathcal{B}_k)$ is preserved, although $\mathcal{B}_k$ might be trailed by D_m. Thus we have a new band sum of L_1 and $\partial(\mathcal{D}'_1 - D_m) \cup \bigcup_{k\geq 2} \partial\mathcal{D}_k$ with bands $(\mathcal{B}'_1 - b_m) \cup \bigcup_{k\geq 2} \mathcal{B}'_k$. Repeating these deformations, we have a band sum $L'_0 = K_{11} \cup K'_{02} \cup \cdots \cup K'_{0n}$ of L_1 and $\bigcup_{k\geq 2} \partial\mathcal{D}_k$ with bands $\bigcup_{k\geq 2} \mathcal{B}''_k$ such that $(L_1 - K_{11}) \cup \bigcup_{k\geq 2}(\mathcal{D}_k \cup \mathcal{B}''_k)$ is ambient isotopic to $(L_1 - K_{11}) \cup \bigcup_{k\geq 2}(\mathcal{D}_k \cup \mathcal{B}_k)$. We note that L'_0 is self Δ-equivalent to L_0, and that L'_0 and L_1 bound a ribbon concordance $(K_{11} \times I) \cup A_2 \cup \cdots \cup A_n$. This contradicts the minimality of the number of maximal points for the ribbon concordance. □

3. PROOF OF PROPOSITION 1

Let L_0 be Borromean rings, and let L_1 be a link as illustrated in Figure 9. Since L_0 and L_1 are cobordant, we will show that L_0 and L_1 are not self Δ-equivalent.

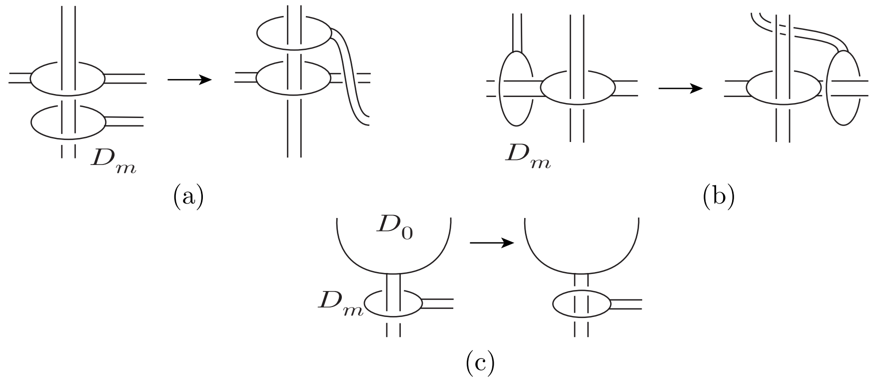

FIGURE 6.

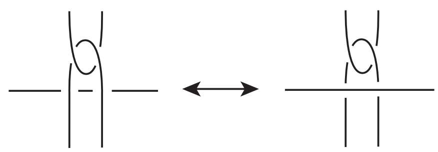

FIGURE 7.

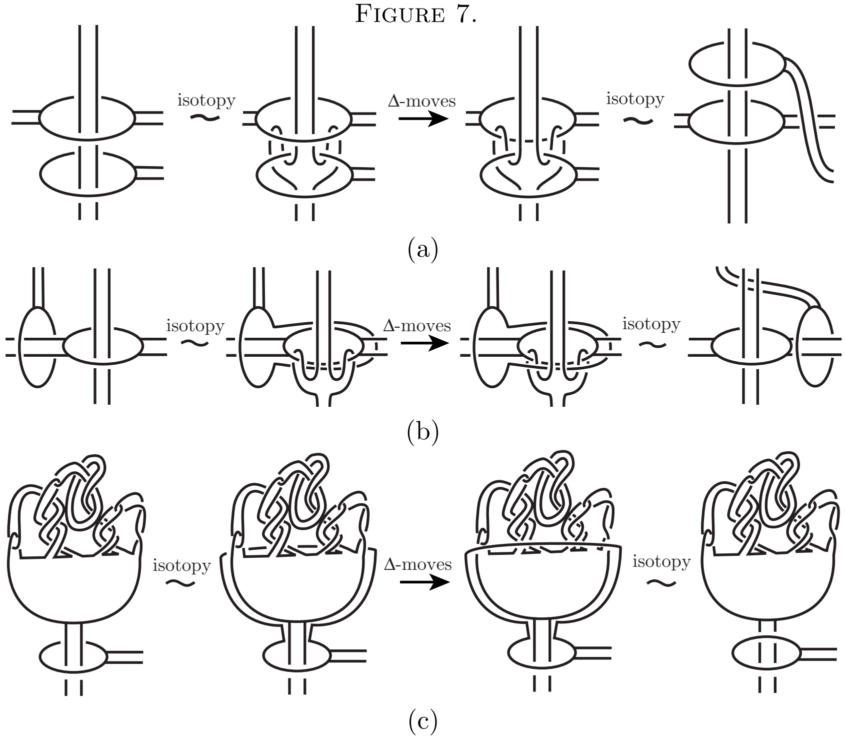

FIGURE 8.

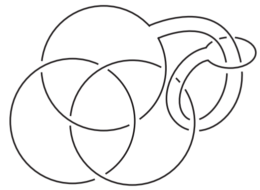

FIGURE 9.

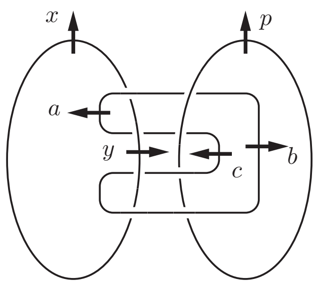

FIGURE 10

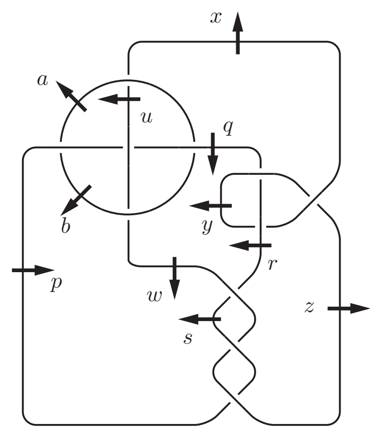

FIGURE 11

In order to prove this, we need the following proposition given in [8].

Proposition 2 ([8, Lemma 3.1])**.** *If two n-component links are self Δ-equivalent, then they have the same Alexander matrices modulo $((1-t_1)^2, ..., (1-t_n)^2)$. Further, we have similar statements for the elementary ideals of deficiency greater than* 0.

Proof of Proposition 1. We take a diagram of L_0 as illustrated in Figure 10. Then we have

$$\pi_1(S^3 \setminus L_0) = \left\langle a, b, c, p, x, y \;\middle|\; \begin{array}{l} y = axa^{-1}, x = b^{-1}yb, p = b^{-1}cpc^{-1}b, \\ b = pap^{-1}, c = y^{-1}by, a = yp^{-1}cpy^{-1} \end{array} \right\rangle.$$

This can be shown to be isomorphic to $\langle a, p, x | S_1, S_2 \rangle$, where

$$\begin{array}{l} S_1 = pa^{-1}p^{-1}axa^{-1}pap^{-1}x^{-1}, \\ S_2 = a^{-1}p^{-1}ax^{-1}a^{-1}pap^{-1}axa^{-1}pax^{-1}a^{-1}pa^{-1}p^{-1}axa^{-1}pap^{-1}. \end{array}$$

Hence the Alexander matrix is equivalent to the following matrix modulo $((a-1)^2, (p-1)^2, (x-1)^2)$:

$$\begin{pmatrix} (p-1)(x-1) & -(a-1)(x-1) & 0 \\ (p-1)(x-1) & -a^{-1}(a-1)(p-1)(x-1) & (a-1)(p-1) \end{pmatrix}.$$

So we have

$$E_1 \equiv (0) \mod ((a-1)^2, (p-1)^2, (x-1)^2).$$

On the other hand, we take a diagram of L_1 as illustrated in Figure 11. Then we have

$$\pi_1(S^3 \setminus L_1) = \left\langle \begin{array}{c} x, y, z, w, u, \\ a, b, p, q, r, s \end{array} \;\middle|\; \begin{array}{l} y = r^{-1}ar, z = x^{-1}yx, w = szs^{-1}, u = bwb^{-1}, \\ x = a^{-1}ua, b = q^{-1}aq, a = pbp^{-1}, q = upu^{-1}, \\ r = yqy^{-1}, s = w^{-1}rw, p = z^{-1}sz \end{array} \right\rangle.$$

This can be shown to be isomorphic to $\langle y, w, a, q | R_1, R_2, R_3\rangle$, where

$$\begin{aligned} R_1 &= q^{-1}y^{-1}ayqy^{-1}, \\ R_2 &= yqy^{-1}wa^{-1}q^{-1}aqw^{-1}q^{-1}a^{-1}qaya^{-1}q^{-1}aqwq^{-1}a^{-1}qaw^{-1}yq^{-1}y^{-1}w^{-1}, \\ R_3 &= aqwq^{-1}a^{-1}qa^{-1}q^{-1}aqw^{-1}q^{-1}a^{-1}qay^{-1}a^{-1}q^{-1}aqwq^{-1}a^{-1}qaw^{-1}yq. \end{aligned}$$

The Alexander matrix is modulo $((a-1)^2, (q-1)^2, (y-1)^2)$ equivalent to

$$\begin{pmatrix} -a(q-1)-1 & 0 & 1 & y(a-1) \\ y(q-1)+1 & -1 & 0 & -y(y-1) \\ 0 & -yaq(q-1)(a-1) & ya(q-1)(y-1) & (y-1)(a-1)(q-1) \end{pmatrix}$$

By fundamental deformations of presentation matrices up to modulo $((a-1)^2$, $(q-1)^2, (y-1)^2)$, we have

$$\begin{pmatrix} (q-1)(y-a) & (y-1)(a-1)(q-1) \end{pmatrix}.$$

Hence we have

$$E_1 \equiv ((q-1)(y-a)) \mod ((a-1)^2, (q-1)^2, (y-1)^2).$$

By Proposition 2, we have the conclusion. □

References

1. C. H. Giffen, *Link concordance implies link homotopy*, Math. Scand. **45**(1979), 243–254. MR0580602 (82a:57005)
2. C. McA. Gordon, *Ribbon concordance of knots in the* 3*-sphere*, Math. Ann. **257**(1981), 157–170. MR0634459 (83a:57007)
3. D. L. Goldsmith, *Concordance implies homotopy for classical links in M^3*, Comment. Math. Helv. **54**(1979),347–355. MR0543335 (80h:57006)
4. A. Kawauchi, T. Shibuya and S. Suzuki, *Descriptions on surfaces in four-space, I. Normal forms*, Math. Sem. Notes, Kobe Univ., **10** (1982), 75–125. MR0672939 (84d:57017)
5. J. Milnor, *Link groups*, Ann. Math., **59** (1954) 177–195. MR0071020 (17:70e)
6. H. Murakami and Y. Nakanishi, *On a certain move generating link-homology*, Math. Ann., **284**(1989), 75–89. MR0995383 (90f:57007)
7. Y. Nakanishi and Y. Ohyama, *Delta link homotopy for two component links. III*, J. Math. Soc. Japan **55**(2003), 641–654. MR1978214 (2004a:57009)
8. Y. Nakanishi and T. Shibuya, *Relations among self delta-equivalence and self sharp-equivalences for links*, Knots in Hellas '98 (Delphi), 353–360, Ser. Knots Everything, **24**, World Sci. Publishing, River Edge, NJ, 2000. MR1865717 (2002h:57014)
9. Y. Nakanishi and T. Shibuya, *Link homotopy and quasi self delta-equivalence for links*, J. Knot Theory Ramifications **9**(2000), 683–691. MR1762762 (2001d:57011)
10. T. Shibuya, *Some relations among various numerical invariants for links*, Osaka J. Math. **11**(1974), 313–322. MR0353295 (50:5779)

11. T. Shibuya, *Self Δ-equivalence of ribbon links*, Osaka J. Math, **33**(1996), 751–760. MR1424684 (97k:57012)
12. K. Taniyama and A. Yasuhara, *Clasp-pass moves on knots, links and spatial graphs*, Topology Appl. **122**(2002), 501–529. MR1911697 (2003g:57012)

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