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CHARACTERS OF p'-DEGREE WITH CYCLOTOMIC FIELD OF VALUES

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ABSTRACT. If p is a prime number and G is a finite group, we show that G has an irreducible complex character of degree not divisible by p with values in the cyclotomic field $\mathbb{Q}_p$.

1. INTRODUCTION

R. Gow conjectured that every finite group of even order has a nontrivial irreducible complex character with odd degree and rational values. This conjecture was finally proven in [7]. In this note we seek an analog of this result which works for every prime p. If G is a finite group and $\chi \in \mathrm{Irr}(G)$ is an irreducible complex character of G, we denote by $\mathbb{Q}(\chi)$ the field of values of χ. Also, we let $\mathbb{Q}_n$ be the cyclotomic field generated by a primitive nth root of unity.

Theorem A. *Let p be a prime and let G be a finite group of order divisible by p. Then there exists a nontrivial $\chi \in \mathrm{Irr}(G)$ of p'-degree such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_p$.*

Of course, the $p = 2$ case of Theorem B is Gow's conjecture. We are able, in fact, to prove the following.

Theorem B. *Let p be a prime and let G be a finite group. Then the trivial character is the only p'-degree irreducible character of G with values in $\mathbb{Q}_p$ if and only if G is a group of odd order not divisible by p.*

Once we have that groups of order divisible by p have nontrivial irreducible p'-degree characters with values in $\mathbb{Q}_p$, it is natural to ask if we can even obtain them having rational values. Of course, some conditions are necessary (since odd p-groups certainly do not possess these characters, nor does $L_2(3^{2a+1})$ for $p = 3$). The following improves on some of the results in [5].

Theorem C. *Let G be a finite group and let p be a prime.*

(i) *Let G be nonsolvable. Assume that either $p \neq 3$, or that $p = 3$ and G has no composition factor isomorphic to $L_2(3^{2a+1})$ for any $a \geq 1$. Then G has a nontrivial, rational, irreducible character of p'-degree.*

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(ii) *Let G be solvable. Then G has a nontrivial, rational, irreducible character of p'-degree if and only if $|\mathbf{N}_G(P)|$ is even, where $P \in \mathrm{Syl}_p(G)$.*

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2. Simple groups

The proof of our main results depends on the Classification of Finite Simple Groups.

(2.1) Theorem. *Let p be an odd prime and let G be a finite nonabelian simple group.*

(i) *There exists a nontrivial $\chi \in \mathrm{Irr}(G)$ of p'-degree such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_p$.*
(ii) *If $p = 3$, assume in addition that G is not isomorphic to $L_2(3^{2a+1})$ for any $a \geq 1$*

Then there exists a nontrivial rational-valued $\chi \in \mathrm{Irr}(G)$ of p'-degree. Furthermore, if a p-group Q acts on G, then χ may be chosen to be Q-invariant.

Proof. If G is an sporadic simple group, then a check of the ATLAS [2] gives us a desired χ, as Q must act on G only via inner automorphisms. Similarly, if $G = A_n$ is an alternating group with $n \geq 5$, as pointed out in Lemma (6.3) of [7], we have that G has two irreducible rational-valued characters of degree $d > 1$ and $d + 1$. One of these two characters has degree not divisible by p and is Q-invariant, by the same reason as before, and this case follows. Furthermore, if G is a finite group of Lie type in characteristic $\ell \neq p$, then we can take χ to be the Steinberg character. So we may assume that G is a finite group of Lie type in characteristic p.

Assume $G = L_2(q)$. If $q \neq 3^{2a+1}$, then the proof of Lemma (6.4) of [7] shows that G has a rational irreducible character of degree $q \pm 1$ which is stable under field automorphisms of G, and so we are done. On the other hand, if $q = 3^{2a+1}$, then G has two so-called Weil characters of degree $(q-1)/2$ with character value field equal to $\mathbb{Q}_3$. Thus we may assume G is not of type A_1.

There is a simple simply connected algebraic group $\mathcal{G}$ over the algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$ and a Frobenius map F on $\mathcal{G}$ such that $G = \mathcal{G}^F/\mathbf{Z}(\mathcal{G}^F)$. Let $(\mathcal{G}^*, F^*)$ be dual to $(\mathcal{G}, F)$. For every conjugacy class $[s]$ of a semisimple element $s \in \mathcal{G}^{F^*}$ such that $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected, there is a so-called semisimple irreducible character $\chi_s \in \mathrm{Irr}(\mathcal{G}^F)$ of degree coprime to p [1], [3], [6]. Furthermore, χ_s belongs to the Lusztig series corresponding to $[s]$. Clearly, $\mathcal{G}^F$ has a unique irreducible character of degree 1 — the trivial character — and it belongs to the Lusztig series corresponding to [1]. Since Lusztig series are disjoint, we see that $\chi_s(1) > 1$ if $s \neq 1$. If s is contained in the derived subgroup of $\mathcal{G}^{*F^*}$, then $\mathbf{Z}(\mathcal{G}^F) \leq \ker \chi_s$, and so χ_s defines an irreducible character of G. Moreover, by [6], this character is invariant under $\mathrm{Aut}(G) = \mathrm{Aut}(\mathcal{G}^F)$ provided that the class $[s]$ is invariant under $\mathrm{Aut}(\mathcal{G}^{*F^*})$ (and $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected as we assumed above). Finally, by Lemma (6.1) of [7], χ_s is rational-valued if s is rational in $\mathcal{G}^{*F^*}$. Hence it suffices to find a nontrivial rational semisimple element s in $\mathcal{G}^{*F^*}$, lying in the derived subgroup of $\mathcal{G}^{*F^*}$, having connected centralizer in $\mathcal{G}^*$, and such that its conjugacy class is invariant under $\mathrm{Aut}(\mathcal{G}^{*F^*})$. This element s has indeed been exhibited in the proof of Theorem (4.1) of [5]. $\square$

3. Proof of Theorem B

If $N \triangleleft G$ and $\theta \in \mathrm{Irr}(N)$, we write $\mathrm{Irr}(G|\theta)$ for the irreducible characters χ of G having θ as an irreducible constituent of χ_N. Recall that if $\chi \in \mathrm{Irr}(G)$ and $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$, then $\chi^\sigma \in \mathrm{Irr}(G)$ and $\mathbb{Q}(\chi) = \mathbb{Q}(\chi^\sigma)$. We need the following lemma.

(3.1) Lemma. *Suppose that $N \triangleleft G$ and that $\theta \in \mathrm{Irr}(N)$ is G-invariant with $\mathbb{Q}(\theta) \subseteq \mathbb{Q}_{p^e}$, where $p > 2$ is prime and $e \geq 1$ is some integer. If G/N is an odd p'-group, then there exists a unique $\tau \in \mathrm{Irr}(G|\theta)$ such that $\mathbb{Q}(\tau) \subseteq \mathbb{Q}_{p^f}$, where $f \geq 1$ is some integer. In fact, $\tau_N = \theta$.*

Proof. We argue by induction on $|G : N|$. By using the fact that G/N is solvable, we let $N \subseteq M \triangleleft G$ with $|G : M| = q$, where $2 \neq q \neq p$ is a prime. By induction, there exists a unique $\psi \in \mathrm{Irr}(M)$ over θ with values in some cyclotomic field $\mathbb{Q}_{p^f}$. In fact, $\psi_M = \theta$. Now, if $g \in G$, then ψ^g lies over θ and has values in $\mathbb{Q}_{p^f}$. By uniqueness, we have that $\psi = \psi^g$. Now, ψ is q-rational, and by Theorem (6.30) of [4] there exists a unique q-rational extension $\tau \in \mathrm{Irr}(G)$ of ψ to G. Now, $\mathbb{Q}(\psi) \subseteq \mathbb{Q}(\tau)$. If $\sigma \in \mathrm{Gal}(\mathbb{Q}(\tau)/\mathbb{Q}(\psi))$, it follows that $\tau^\sigma \in \mathrm{Irr}(G)$ is a q-rational character extending ψ. By uniqueness, $\tau^\sigma = \tau$, and we deduce that $\mathbb{Q}(\tau) = \mathbb{Q}(\psi) \subseteq \mathbb{Q}_{p^f}$. Suppose now that $\rho \in \mathrm{Irr}(G|\theta)$ has its values in $\mathbb{Q}_{p^r}$ for some integer $r \geq 1$. By Gallagher's Corollary (6.17) of [4], we have that $\rho = \xi\tau$, where $\xi \in \mathrm{Irr}(G/N)$. We claim that $\xi = 1$. Since G/N is odd, by Burnside's theorem it suffices to show that ξ is rational valued. Now, let $\alpha \in \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$. Since G/N is a p'-group, we may extend α to some $\beta \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$ that fixes all p-power roots of unity. In particular, θ, ρ and τ are β-fixed. By the uniqueness in Corollary (6.17) of [4], we have that ξ is β-fixed, and therefore α-fixed. This proves the lemma. □

If $K \subseteq \mathbb{C}$ is a field, let us denote by $\mathrm{Irr}_{p',K}(G)$ the set of all irreducible complex characters of G with degree not divisible by p and with values in K. The following is Theorem B of the Introduction.

(3.2) Theorem. *Let p be a prime and let G be a finite group. Then $|\mathrm{Irr}_{p',\mathbb{Q}_p}(G)| = 1$ if and only if G has odd order not divisible by p.*

Proof. Suppose that G has odd order not divisible by p, and let $\chi \in \mathrm{Irr}(G)$ with values in $\mathbb{Q}_p$. Then χ has its values in $\mathbb{Q}_{|G|} \cap \mathbb{Q}_p = \mathbb{Q}$. Since G has odd order, we conclude that $\chi = 1$ by Burnside's theorem.

Now, assume that $|\mathrm{Irr}_{p',\mathbb{Q}_p}(G)| = 1$. We prove that G is of odd order not divisible by p, by arguing by induction on $|G|$. By Theorem B of [7], we may assume that p is odd and that $|G|$ is divisible by p. Now, let $1 < N$ be a minimal normal subgroup of G. Hence, by induction, we have that G/N is of odd order not divisible by p. Suppose first that N is abelian. If N is of odd order not divisible by p, then we are done. So we may assume that N is an elementary abelian 2-group or p-group. In any case, $(|G : N|, |N|) = 1$. Now, let $1 \neq \lambda \in \mathrm{Irr}(N)$ and let T be the stabilizer of λ in G. By Corollary (8.16) of [4], there exists a unique extension $\hat{\lambda} \in \mathrm{Irr}(T)$ of λ with determinantal order $o(\hat{\lambda}) = o(\lambda)$. Hence $\mathbb{Q}(\hat{\lambda}) = \mathbb{Q}(\lambda) \subseteq \mathbb{Q}_p$ (in both cases). Now, by using the Clifford correspondence, we have that $1 \neq \chi = \hat{\lambda}^G \in \mathrm{Irr}(G)$ has its values in $\mathbb{Q}_p$ and has p'-degree.

So we may assume that N is a direct product of nonabelian isomorphic simple groups of order divisible by p. Now, by Theorem (2.1), there exists $1 \neq \theta \in \mathrm{Irr}(N)$

of p'-degree with values in $\mathbb{Q}_p$. Let I be the stabilizer of θ in G. By Lemma (3.1), there exists $\rho \in \mathrm{Irr}(I)$ over θ with values in $\mathbb{Q}_p$. Now $1 \neq \rho^G \in \mathrm{Irr}(G)$ has p'-degree and has its values in $\mathbb{Q}_p$. □

4. Rational characters of p'-degree

We need the following lemma.

(4.1) Lemma. *Suppose that $N \triangleleft G$, where G/N is solvable, $p > 2$ is prime, $P/N \in \mathrm{Syl}_p(G/N)$ and $|\mathbf{N}_{G/N}(P/N)|$ is odd. If $\theta \in \mathrm{Irr}(N)$ is rational, P-invariant of p'-degree, then there exists $\chi \in \mathrm{Irr}(G|\theta)$ rational of p'-degree.*

Proof. We argue by induction on $|G : N|$. Let M/N be a chief factor of G. If M/N is a p-group, then θ is M-invariant and has a unique real extension $\eta \in \mathrm{Irr}(M)$ by Lemma (2.1) of [7]. By uniqueness, η is rational and P-invariant. Since $|G : M| < |G : N|$, then the lemma follows by induction. Assume now that M/N is an odd p'-group. By Corollary (2.2) of [7], there is a unique rational $\eta \in \mathrm{Irr}(M|\theta)$. Again by uniqueness, η is P-invariant, and we note that has p'-degree since M/N is a p'-group (by Corollary (11.29) of [4]). Hence, the lemma again follows by induction. Finally, assume that M/N is a 2-group. By hypothesis, we have that $\mathbf{C}_{M/N}(P) = 1$. By Problem (13.10) of [4], we have that there exists a unique P-invariant $\eta \in \mathrm{Irr}(M|\theta)$. By uniqueness, η is P-invariant, rational, of p'-degree, and we may apply induction once again. □

Suppose that G is a p-solvable group, and let P be a Sylow p-subgroup of G. In [5], it was proven that $|\mathrm{Irr}_{p',\mathbb{R}}(G)| = 1$ if and only if $|\mathbf{N}_G(P)|$ is odd. Combined with this result, the following theorem yields Theorem C.

(4.2) Theorem. *Let G be a finite group and let $P \in \mathrm{Syl}_p(G)$, where p is prime. If $p = 3$, assume in addition that G has no composition factor isomorphic to $L_2(3^{2a+1})$ for any $a \geq 1$. Suppose that $|\mathrm{Irr}_{p',\mathbb{Q}}(G)| = 1$. Then $|\mathbf{N}_G(P)|$ is odd and G is solvable.*

Proof. If $p = 2$, then by Theorem (5.2) of [7], we see that $|G|$ is odd, and so we are done. We assume, therefore, that $p \neq 2$. We argue by induction on $|G|$.

Let N be minimal normal in G. Then $|\mathrm{Irr}_{p',\mathbb{Q}}(G/N)| = 1$ and by the induction hypothesis, G/N is solvable and $|\mathbf{N}_{G/N}(PN/N)|$ is odd. Since $\mathbf{N}_{G/N}(PN/N) = \mathbf{N}_G(P)N/N$, it follows that $|\mathbf{N}_G(P) : \mathbf{N}_G(P) \cap N|$ is odd. Hence, it suffices to show that $|\mathbf{N}_G(P) \cap N|$ is odd and that N is solvable. In particular, since $p \neq 2$, we may assume that N is not a p-group.

Assume that N is a p'-group. We claim that $|\mathbf{C}_N(P)|$ is odd. Otherwise, $|\mathbf{C}_N(P)|$ is even, and by Theorem (5.2) of [7] $\mathbf{C}_N(P)$ has a nontrivial rational $\eta \in \mathrm{Irr}(\mathbf{C}_N(P))$. Now, let $1 \neq \theta \in \mathrm{Irr}(N)$ be P-invariant such that the Glauberman correspondent of θ is η. Since the Glauberman correspondence commutes with Galois action, it follows that θ is rational, of p'-degree and P-invariant. By Lemma (4.1), there exists $\chi \in \mathrm{Irr}(G|\theta)$ rational of p'-degree, a contradiction. Since $|\mathbf{C}_N(P)|$ is odd, N is solvable by Theorem (3.4) of [5]. Furthermore, $\mathbf{N}_G(P) \cap N = \mathbf{C}_N(P)$, so we are done in this case.

Finally, assume that the minimal normal subgroup N of G is neither a p-group nor a p'-group. Then $N = K_1 \times \cdots \times K_k$, where each K_i is a product of nonabelian simple groups transitively permuted by P. Write $K_1 = T_1 \times \cdots \times T_r$, where the simple groups T_i are transitively permuted by P. Suppose that $Q = \mathbf{N}_P(T_1)$. By our assumption, $T_1 \not\cong L_2(3^{2a+1})$ for any $a \geq 1$. By Theorem (2.1(ii)), there exists $1 \neq \phi \in \mathrm{Irr}(T_1)$ rational of p'-degree and Q-invariant. For each subscript i with $1 \leq i \leq r$, choose $x_i \in P$ so that $(T_1)^{x_i} = T_i$, and let $\phi_i = \phi^{x_i} \in \mathrm{Irr}(T_i)$. It is straightforward to check that $1 \neq \eta = \phi_1 \times \cdots \times \phi_r$ is P-invariant, rational of p'-degree. Now, consider η as an irreducible character of N (with $K_2 \cdots K_k$ in its kernel) and apply Lemma (4.1) to get a nontrivial rational p'-degree character of G. This final contradiction completes the proof of the theorem. □

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