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# AN ELEMENTARY PROOF OF THE CHARACTERIZATION OF ISOMORPHISMS OF STANDARD OPERATOR ALGEBRAS

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Abstract. This study provides an elementary proof of the well-known fact that any isomorphism $\pi : \mathcal{A} \to \mathcal{B}$ of standard operator algebras on normed spaces $X, Y$, respectively, is spatial; i.e., there exists a topological isomorphism $T : X \to Y$ such that $\pi(A) = TAT^{-1}$ for any $A \in \mathcal{A}$. In particular, $\pi$ is continuous.

Paul R. Chernoff has shown that isomorphisms of standard operator algebras are spatial in the sense explained below. An elementary proof of this theorem is given in this study by an adaptation of a remark by P. R. Chernoff [1].

Let $X, Y$ be normed spaces. Denote by $B(X)$ the algebra of all bounded linear operators on $X$. A subalgebra of $B(X)$ which contains $F(X)$ (the ideal of all finite rank operators in $B(X)$) is called a standard operator algebra on $X$.

The dual space of $X$ is denoted by $X^*$. If $x \in X$ and $f \in X^*$, then $x \otimes f$ denotes the operator defined by

$$(x \otimes f)(z) = f(z)x, \text{ for any } z \in X.$$

If $A : X \to Y$ is a bounded linear operator, then $A^* : Y^* \to X^*$, the adjoint of $A$, is defined by $A^*(f) = f \circ A$, for any $f \in Y^*$.

**Theorem.** *Let $X$ and $Y$ be normed spaces, and let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on $X$ and $Y$, respectively. Then every algebraic isomorphism $\pi : \mathcal{A} \to \mathcal{B}$ is spatial; i.e., there is a linear topological isomorphism $T : X \to Y$ such that for any $A \in \mathcal{A}$, $\pi(A) = TAT^{-1}$. In particular, $\pi$ is continuous.*

*Proof.* Let $P \in \mathcal{A}$ be a rank one idempotent; then $\pi(P)$ is rank one idempotent, too. Then there exists $x_0 \in X$, $y_0 \in Y$ such that $P(x_0) = x_0$ and $\pi(P)(y_0) = y_0$. Also, let $S : X \to Y$ be a linear operator such that $S(x_0) = y_0$. It is easy to see that there are linear isomorphisms $\psi_1 : \mathcal{A}P \to X$ and $\psi_2 : \mathcal{B}\pi(P) \to Y$ such that $\psi_1(AP) = APx_0$ and $\psi_2(B\pi(P)) = B\pi(P)y_0$ for any $A \in \mathcal{A}$, $B \in \mathcal{B}$.
Hence, there exists an isomorphism $T : X \to Y$ such that for any $A \in \mathcal{A}$,

$$TAPx_0 = \pi(A)\pi(P)y_0 = \pi(A)\pi(P)Sx_0.$$

Therefore $TAP = \pi(A)\pi(P)SP$, and so for any $A' \in \mathcal{A}$,

$$TAA'P = \pi(A)\pi(A')\pi(P)SP = \pi(A)TA'P.$$

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It follows that

$$(*) \qquad TA = \pi(A)T.$$

To see that $T$ is bounded, let $0 \neq f \in Y^*$ and $0 \neq y \in Y$. Since $\pi$ is surjective, there exists an $A \in \mathcal{A}$ such that $y \otimes f = \pi(A) = TAT^{-1}$. This implies $A = T^{-1}(y \otimes f)T = T^{-1}(y) \otimes (f \circ T) \in \mathcal{A}$. Therefore $T^{-1}(y) \otimes (f \circ T)$ is bounded and hence $f \circ T$ is bounded.

Then, $f \to f \circ T$ (which is the same as $T^*$) is a linear transformation from $Y^*$ into $X^*$.

Suppose $f_n \to 0$ in $Y^*$ and $T^* f_n \to g$ in $X^*$. Clearly, $\pi(x \otimes g)^* \in F(Y^*)$, for any $x \in X$. Therefore, $(x \otimes g)^* T^* f_n \to 0$ by $(*)$. But $(x \otimes g)^*(T^* f_n) \to (x \otimes g)^* g$. This implies $g = 0$. By the closed graph theorem $T^*$ is bounded, and this implies that $T$ is bounded. $\square$

*Remark.* We note that in our proof, we did not use the completeness of $X$ and $Y$; so, many similar results in the literature (e.g., [2], [4], [5]), which use Banach spaces, can be generalized to normed spaces, using the above method.

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