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KHOVANOV-JACOBSSON NUMBERS AND INVARIANTS OF SURFACE-KNOTS DERIVED FROM BAR-NATAN'S THEORY

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Abstract. Khovanov introduced a cohomology theory for oriented classical links whose graded Euler characteristic is the Jones polynomial. Since Khovanov's theory is functorial for link cobordisms between classical links, we obtain an invariant of a surface-knot, called the *Khovanov-Jacobsson number*, by considering the surface-knot as a link cobordism between empty links. In this paper, we study an extension of the Khovanov-Jacobsson number derived from Bar-Natan's theory, and prove that any T^2-knot has trivial Khovanov-Jacobsson number.

1. Introduction

Khovanov [8] introduced a cohomology theory for oriented classical links which takes values in graded $\mathbb{Z}$-modules and whose graded Euler characteristic is the Jones polynomial. Khovanov's cohomology theory is based on a (1+1)-dimensional TQFT $\mathcal{F}$ associated to a Frobenius algebra V (cf. Section 2). We denote the Khovanov's cohomology group of an oriented link L by $H(L,\mathcal{F})$ $(= \bigoplus H^i(L,\mathcal{F}))$, and each $H^i(L,\mathcal{F})$ is a graded $\mathbb{Z}$-module. Khovanov's theory is powerful for classical links; for example, Bar-Natan [1] and Wehrli [15] showed that Khovanov's cohomology is stronger than the Jones polynomial, and Rasmussen [12] gave a combinatorial proof of the Milnor conjecture by using a variant of Khovanov's theory defined by Lee [11].

Jacobsson [7] and Khovanov [9] proved that Khovanov's theory is functorial for link cobordisms in the following sense: a link cobordism $S \subset \mathbb{R}^3 \times [0,1]$ between classical links L_0 and L_1 induces a map $\phi_S : H(L_0,\mathcal{F}) \to H(L_1,\mathcal{F})$, well-defined up to overall minus sign, under ambient isotopy of S rel ∂S. Here the map ϕ_S is a graded map of degree $\chi(S)$, where $\chi(S)$ is the Euler characteristic of S.

A *surface-knot* F is a closed connected oriented surface embedded piecewise linearly and locally flatly in $\mathbb{R}^4$, and can be considered as a link cobordism between empty links. Then the induced map $\phi_F : H(\emptyset,\mathcal{F}) \to H(\emptyset,\mathcal{F})$, up to overall minus sign, gives an invariant of the surface-knot F. Since the cohomology group $H(\emptyset,\mathcal{F})$ of the empty link $\emptyset$ is $\mathbb{Z}$, the map ϕ_F is an endomorphism of $\mathbb{Z}$. Hence we obtain an invariant of the surface-knot F defined as $\mid \phi_F(1) \mid \in \mathbb{Z}$, and denote it by $KJ(F)$. This invariant is called the *Khovanov-Jacobsson number* in [4].

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As far as the author knows, there are a few results on the computation of the Khovanov-Jacobsson numbers of surface-knots; see [4], for example. It follows from a simple observation that $KJ(F) = 0$ for any surface-knot F with $\chi(F) \neq 0$ and that $KJ(F) = 2$ for a trivial T^2-knot F (a surface-knot with $\chi(F) = 0$). It seems to be hard to compute the Khovanov-Jacobsson number in general, but Carter, Saito and Satoh [4] proved that $KJ(F) = 2$ for any T^2-knot F obtained from a spun/twist-spun S^2-knot by attaching a 1-handle. They also proved that $KJ(F) = 2$ for any pseudo-ribbon T^2-knot F.

In this paper, we define an invariant $BN(F) \in \mathbb{Z}[t]$ of a surface-knot F by using a variant of Khovanov's theory defined by Bar-Natan [2]. This invariant is a generalization of the Khovanov-Jacobsson number such that

$$BN(F)|_{t=0} = KJ(F).$$

The main result of this paper is that the invariant $BN(F)$ of any surface-knot F is determined by its genus, and hence it turns out that the Khovanov-Jacobsson number is trivial for any T^2-knot.[1]

Theorem 1.1. *For any surface-knot F of genus g $(g \geq 0)$, we have the following:*

(i) *If g is an even integer, then we have $BN(F) = 0$.*
(ii) *If g is an odd integer, then we have $BN(F) = 2^g t^{(g-1)/2}$.*

Corollary 1.2. *For any T^2-knot F, we have $KJ(F) = 2$.*

This paper is organized as follows. In Section 2, we define the surface-knot invariant $BN(F)$. Section 3 is devoted to the proof of Theorem 1.1. Throughout this paper we rely on the reader's familiarity with [8], and refer the reader to [8], [1], [2], [11, Section 2], [12, Section 2 and 4], [14, Appendix], for example.

2. The surface-knot invariant derived from Bar-Natan's theory

Bar-Natan [2] defined several variants of Khovanov's theory. Let V' be a free graded $\mathbb{Z}[t]$-module of rank two generated by $\mathbf{v}_+$ and $\mathbf{v}_-$ with

$$\deg(t) = -4, \ \deg(\mathbf{v}_+) = 1 \ \text{and} \ \deg(\mathbf{v}_-) = -1.$$

We give V' a Frobenius algebra structure with a multiplication m', a comultiplication Δ', a unit ι', and a counit ϵ' defined by

$$\begin{aligned}
&m'(\mathbf{v}_+ \otimes \mathbf{v}_+) = \mathbf{v}_+, && \Delta'(\mathbf{v}_+) = \mathbf{v}_+ \otimes \mathbf{v}_- + \mathbf{v}_- \otimes \mathbf{v}_+,\\
&m'(\mathbf{v}_+ \otimes \mathbf{v}_-) = m'(\mathbf{v}_- \otimes \mathbf{v}_+) = \mathbf{v}_-, && \Delta'(\mathbf{v}_-) = \mathbf{v}_- \otimes \mathbf{v}_- + t\mathbf{v}_+ \otimes \mathbf{v}_+,\\
&m'(\mathbf{v}_- \otimes \mathbf{v}_-) = t\mathbf{v}_+, && \\
&\iota'(1) = \mathbf{v}_+, && \epsilon'(\mathbf{v}_+) = 0 \quad \epsilon'(\mathbf{v}_-) = 1.
\end{aligned}$$

The structure maps m', Δ', ι' and ϵ' are graded maps of degree -1, -1, 1 and 1 respectively.

One of his cohomology theories, implicitly defined in [2, Section 9.2], is based on a $(1+1)$-dimensional TQFT $\mathcal{F}'$, a monoidal functor from oriented $(1+1)$-cobordisms to graded $\mathbb{Z}[t]$-modules, associated to V'. The Frobenius algebra V' defines $\mathcal{F}'$ by assigning $\mathbb{Z}[t]$ to an empty 1-manifold, V' to a single circle, $V' \otimes V'$ to a disjoint

[1]The author has subsequently learned that Jacob Rasmussen [13] has a different proof of Corollary 1.2 using Lee's theory.

union of two circles, and so on. The structure maps are assigned to elementary cobordisms such that

$\mathcal{F}'$ () $= m' : V' \otimes V' \to V'$, $\mathcal{F}'$ () $= \iota' : \mathbb{Z}[t] \to V'$,

$\mathcal{F}'$ () $= \Delta' : V' \to V' \otimes V'$, $\mathcal{F}'$ () $= \epsilon' : V' \to \mathbb{Z}[t]$.

We denote the cohomology group of an oriented link L by $H(L, \mathcal{F}')$ $(= \bigoplus H^i(L, \mathcal{F}'))$, and each $H^i(L, \mathcal{F}')$ is a graded $\mathbb{Z}[t]$-module. We note that the cohomology theory associated to $\mathcal{F}'$ is essentially the same as that associated to the Frobenius system $\mathcal{F}_3$ in [10], but the notational conventions are slightly different.

Bar-Natan proved that the cohomology theory associated to $\mathcal{F}'$ is also functorial for link cobordisms up to sign indeterminacy (cf. [2], [10, Proposition 6]). Given a surface-knot F, the induced graded map $\psi_F : H(\emptyset, \mathcal{F}') \to H(\emptyset, \mathcal{F}')$ of degree $\chi(F)$ becomes an endomorphism of $\mathbb{Z}[t]$. Hence we obtain an invariant of the surface-knot F defined as

$$| \psi_F(1) | \in \mathbb{Z}[t],$$

and denote it by $BN(F)$. Since Bar-Natan's Frobenius algebra V' recovers Khovanov's Frobenius algebra V by adding the relation $t = 0$ (cf. [2], [10]), we have

$$BN(F)|_{t=0} = KJ(F).$$

We remark here that Bar-Natan's theory also recovers Lee's theory [11] by adding the relation $t = 1$.

3. Proof

For a surface-knot F, taking an arbitrary point p of F and cutting off a small neighborhood of p which is homeomorphic to the standard disk pair (D^4, D^2), we obtain a link cobordism between an empty link and a trivial knot. Then we can define the following two maps:

$$\psi_F^{(\bigcirc \to \emptyset)} : H(\bigcirc, \mathcal{F}') \to H(\emptyset, \mathcal{F}') \quad \text{and} \quad \psi_F^{(\emptyset \to \bigcirc)} : H(\emptyset, \mathcal{F}') \to H(\bigcirc, \mathcal{F}'),$$

where $\bigcirc$ stands for a trivial knot diagram and the cohomology group $H(\bigcirc, \mathcal{F}')$ of a trivial knot is V'. Here these two maps satisfy

$$\psi_F = \psi_F^{(\bigcirc \to \emptyset)} \circ \iota' = \epsilon' \circ \psi_F^{(\emptyset \to \bigcirc)}.$$

For the connected sum $F_1 \# F_2$ of two surface-knots F_1 and F_2, the map $\psi_{F_1 \# F_2}$ can be decomposed into the composite of two maps such that

$$\psi_{F_1 \# F_2} = \psi_{F_2}^{(\bigcirc \to \emptyset)} \circ \psi_{F_1}^{(\emptyset \to \bigcirc)}.$$

The following two lemmas are direct consequences of the fact that

$$(m' \circ \Delta')(\mathbf{v}_+) = 2\mathbf{v}_- \quad \text{and} \quad (m' \circ \Delta')(\mathbf{v}_-) = 2t\mathbf{v}_+.$$

We note that the map $m' \circ \Delta'$ corresponds to a link cobordism between trivial knots induced by a trivial T^2-knot with two holes.

Lemma 3.1. *If the surface-knot F of genus $2m+1$ $(m \geq 0)$ is trivial, then we have $BN(F) = 2(4t)^m$.*

Lemma 3.2. *If the surface-knot F of genus $2m$ $(m \geq 0)$ is trivial, then we have*

$$\psi_F^{(\bigcirc \to \emptyset)}(\mathbf{v}_-) = \pm(4t)^m.$$

Proof of Theorem 1.1. Since the map ψ_F induced by a surface-knot F is a graded map of degree $\chi(F)$ and the degree of t is -4, it is easy to see the following:

- If the genus of a surface-knot F is $2m$ $(m \geq 0)$, then we have $BN(F) = 0$.
- If the genus of a surface-knot F is $2m+1$ $(m \geq 0)$, then there exists some nonnegative integer a such that $BN(F) = at^m$.

It is sufficient to prove that the above integer a is equal to 2^{2m+1} for any surface-knot F of genus $2m+1$.

It follows from $BN(F) = at^m$ that

$$\psi_F^{(\emptyset \to \bigcirc)}(1) = \pm a t^m \mathbf{v}_-.$$

Let Σ_g denote a trivial surface-knot of genus g. We consider the connected sum $F\#\Sigma_{2m'}$ of F and $\Sigma_{2m'}$ for a nonnegative integer m'. By Lemma 3.2, we have

$$\psi_{F\#\Sigma_{2m'}}(1) = \left(\psi_{\Sigma_{2m'}}^{(\bigcirc \to \emptyset)} \circ \psi_F^{(\emptyset \to \bigcirc)}\right)(1) = \pm a t^m (4t)^{m'},$$

and hence we have $BN(F\#\Sigma_{2m'}) = at^m(4t)^{m'}$.

We recall here two facts in the theory of surface-knots:

- It is known in [5] that any surface-knot becomes trivial by attaching a finite number of 1-handles, and the minimal number of such 1-handles is called the *unknotting number* in [6].
- Any 1-handle on a surface-knot is ribbon-move equivalent to a trivial 1-handle. (This fact is implicitly used in the proof of [4, Theorem 1].)

By the above facts, if we take the integer m' such that $2m'$ is greater than the unknotting number of F, then the surface-knot $F\#\Sigma_{2m'}$ is ribbon-move equivalent to $\Sigma_{2(m+m')+1}$. When two surface-knots are related by ribbon-moves, it is not difficult to see that the induced maps on the cohomology groups are the same (cf. [4], [2]). Hence we have $BN(F\#\Sigma_{2m'}) = 2(4t)^{m+m'}$ by Lemma 3.1. This implies $a = 2^{2m+1}$. □

Remark 3.3. We can obtain a result similar to Theorem 1.1 for the cohomology theory associated to the universal Frobenius system $\mathcal{F}_5$ defined in [10].

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