THE MOLCHANOV–VAINBERG LAPLACIAN

PHILIPPE POULIN

(Communicated by Mikhail Shubin)

ABSTRACT. It is well known that the Green function of the standard discrete Laplacian on $l^2(\mathbb{Z}^d)$,

$$\Delta_{st}\psi(n) = (2d)^{-1} \sum_{|n-m|=1} \psi(m),$$

exhibits a pathological behavior in dimension $d \geq 3$. In particular, the estimate

$$\langle \delta_0 | (\Delta_{st} - E - i0)^{-1} \delta_n \rangle = O(|n|^{-\frac{d-1}{2}})$$

fails for $0 < |E| < 1 - 2/d$. This fact complicates the study of the scattering theory of discrete Schrödinger operators. Molchanov and Vainberg suggested the following alternative to the standard discrete Laplacian,

$$\Delta\psi(n) = 2^{-d} \sum_{|n-m|=\sqrt{d}} \psi(m),$$

and conjectured that the estimate

$$\langle \delta_0 | (\Delta - E - i0)^{-1} \delta_n \rangle = O(|n|^{-\frac{d-1}{2}})$$

holds for all $0 < |E| < 1$. In this paper we prove this conjecture.

1. INTRODUCTION

In recent years there has been considerable interest in the scattering theory of discrete random Schrödinger operators on the lattice $\mathbb{Z}^d$ [B], [BBP], [CK], [CS], [HK], [JL1], [JL2], [JM], [JMP], [K], [MV1], [MV2], [RS], [SV]. Typical models considered in the literature have the general form

$$(1.1) \quad H_\omega = -\Delta_{st} + \sum_{n \in \Gamma} V_\omega(n) \langle \delta_n | \cdot \rangle \delta_n,$$

where δ is the Kronecker delta on $\mathbb{Z}^d$ and $V_\omega(n)$ are independent random variables indexed by $n \in \Gamma \subseteq \mathbb{Z}^d$. Δ_{st} is the *standard* discrete Laplacian,

$$\Delta_{st}\psi(n) = \frac{1}{2d} \sum_{|n-m|=1} \psi(m),$$

where $\psi \in l^2(\mathbb{Z}^d)$ and $|n| = \sqrt{\sum_j n_j^2}$. The symbol of Δ_{st} is $\frac{1}{d} \sum_{j=1}^d \cos x_j$, and its spectrum is absolutely continuous and equal to $[-1, 1]$. Concerning the random

Received by the editors April 20, 2005 and, in revised form, July 12, 2005.

2000 *Mathematics Subject Classification*. Primary 47B39, 34L40.

Key words and phrases. Discrete Laplacian, Green function.

This research was supported in part by NSERC.

©2006 American Mathematical Society
 Reverts to public domain 28 years from publication

potential V_ω in (1.1), the following two cases are distinguished in the literature: sparse random potentials, for which

$$\#\{n \in \Gamma ; |n| \leq R\} / R^d \rightarrow 0$$

as $R \rightarrow \infty$, and decaying random potentials, for which $\mathbb{E}(|V_\omega(n)|^2) \rightarrow 0$ as $|n| \rightarrow \infty$.

Practically all recent works on the scattering theory of Hamiltonians (1.1) make use of the stationary scattering theory. An important ingredient of the stationary methods is an *a priori* estimate on the Green function

$$G_E(n) := \langle \delta_0 | (\Delta_{st} - E - i0)^{-1} \delta_n \rangle$$

for energies E in the spectrum of Δ_{st} . In contrast to the standard continuous Laplacian on $L^2(\mathbb{R}^d)$, the Green function $G_E(n)$ is not necessarily $O(|n|^{-\frac{d-1}{2}})$ for $d \geq 3$. The reason for this is that the constant energy surfaces

$$\{x \in \mathbb{T}^d ; \frac{1}{d} \sum_{j=1}^d \cos x_j = E\}$$

are not convex for $|E| < 1 - \frac{2}{d}$. Hence, while the expected decay holds for $1 - \frac{2}{d} < |E| < 1$, it fails otherwise [SV]. This property of the standard Laplacian is an unfortunate obstacle in the development of the scattering theory of (1.1).

On physical grounds the choice of the standard discrete Laplacian is primarily dictated by custom and convenience, due to its relation to the standard random walk on $\mathbb{Z}^d$. Indeed, equivalent choices are possible.¹ Since Δ_{st} is ill suited for the study of the scattering theory, Molchanov and Vainberg [MV1] suggested the following alternative, which we will call the *Molchanov-Vainberg Laplacian*:

$$\Delta\psi(n) = 2^{-d} \sum_{|m-n|=\sqrt{d}} \psi(m).$$

In other words, $\Delta\psi(n)$ is equal to a constant times the sum of the $\psi(m)$'s for m varying over the vertices of the hypercube of side 2 centered at n . These vertices are the neighbours of n along the full diagonals. The symbol of Δ is

$$\Phi(x) := \prod_{j=1}^d \cos x_j,$$

as the reader may easily verify.² Consequently, the spectrum of Δ is absolutely continuous and equal to $[-1, 1]$. Since the Fourier transform is unitary, the Green function of Δ for $z \notin \mathbb{R}$ is

$$G_z(n) := \langle \delta_0 | (\Delta - z)^{-1} \delta_n \rangle = (2\pi)^{-d} \int_{\mathbb{T}^d} \frac{e^{in \cdot x}}{\Phi(x) - z} dx.$$

Molchanov and Vainberg [MV1] conjectured that the constant energy surfaces of Δ ,

$$\Gamma(E) := \{x \in \mathbb{T}^d ; \prod_{j=1}^d \cos x_j = E\},$$

¹By *equivalent*, we mean that by taking the scaling limit, $l^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{R}^d)$, one recovers the continuous Laplacian.

²In other words $\mathcal{F}\Delta\mathcal{F}^{-1} = \Phi$, where $(\mathcal{F}\psi)(x) = (2\pi)^{-\frac{d}{2}} \sum_{n \in \mathbb{Z}^d} e^{in \cdot x} \psi(n)$ is the usual Fourier transform acting on $\psi \in l^2(\mathbb{Z}^d)$ and Φ denotes the operator of multiplication by the function $\Phi(x)$ on $L^2(\mathbb{T}^d)$. This result follows from Euler's formula for the cosine.

are strictly convex and that the bound $G_{E+i0}(n) = O(|n|^{-\frac{d-1}{2}})$ holds. In this paper we prove this conjecture.

Let $D_\varepsilon = \mathbb{C}_+ \cup [-1 + \varepsilon, -\varepsilon] \cup [\varepsilon, 1 - \varepsilon]$, where $0 < \varepsilon < \frac{1}{2}$. Our main result is:

Theorem 1.1. (1) *For $0 < |E| < 1$ the surface $\Gamma(E)$ is strictly convex.*
 (2) *The function $\mathbb{C}_+ \ni z \mapsto G_z(n)$ extends continuously to D_ε and satisfies*

$$\sup_{E \in D_\varepsilon} |G_E(n)| \leq C_\varepsilon |n|^{-\frac{d-1}{2}}.$$

Applications of Theorem 1.1 to the scattering theory of random Schrödinger operators with sparse random potentials will be discussed in a forthcoming article [JP].

I would like to thank E. Kritchovski, P. Koosis, M. Merkli, S. Starr and especially B. Vainberg for useful discussion and comments, and V. Jakšić for his help and wonderful teaching.

2. LEVEL SURFACES OF Φ

2.1. Basic properties. For $0 < |t| < 1$, $\Gamma(t)$ is a regular smooth surface, since for all $x \in \Gamma(t)$

$$\|\nabla \Phi(x)\|^2 = t^2 \sum_{j=1}^d \tan^2 x_j \neq 0.$$

Consider the covering $\hat{\Gamma}(t) := \{x \in \mathbb{R}^d; \Phi(x) = t\}$ of $\Gamma(t)$. If $t = 0$, $\hat{\Gamma}(t)$ consists of hyperplanes dividing the space into open hypercubes, which we call the *cells*. For $0 < |t| < 1$, half of them contain a connected component of $\hat{\Gamma}(t)$. These enclosing cells are separated by an even number of the above hyperplanes; their choice depends on the sign of t . Since $\Gamma(t)$ is obtained from $\hat{\Gamma}(t)$ by restricting it to the torus, one concludes that $\Gamma(t)$ consists of 2^{d-1} congruent connected components.

Let $\mathcal{D}(t)$ be a component of $\Gamma(t)$ and $N(x) = \frac{\nabla \Phi(x)}{\|\nabla \Phi(x)\|}$.

Lemma 2.1. *For any $\omega \in S^{d-1}$ there exists exactly one point $x = x(\omega, t) \in \mathcal{D}(t)$ such that $N(x) = \omega$. Moreover, $x(\omega, t)$ extends to a holomorphic function in a neighbourhood of $S^{d-1} \times \{t\}$.*

Proof. $N(x) = \omega$ iff x satisfies under the constraint $\Phi(x) = t$ the relations

$$-t \tan x_j = C \omega_j, \quad j = 1, \dots, d,$$

for a scalar $C > 0$. Then, $x_j = \arctan \frac{-C \omega_j}{t}$, where the branch of $\arctan$ is fixed by the choice of $\mathcal{D}(t)$. Moreover, $\sec^2 x_j = 1 + (\frac{C \omega_j}{t})^2$, so the constraint implies

$$(2.1) \quad t^2 \left(1 + \frac{C^2 \omega_1^2}{t^2}\right) \dots \left(1 + \frac{C^2 \omega_d^2}{t^2}\right) = 1.$$

This equation has a unique positive root $C = C(\omega, t)$. The desired x is then given by $x_j = \arctan \frac{-C(\omega, t) \omega_j}{t}$. Finally, $x(\omega, t)$ is holomorphic since the left side of (2.1) has a nonvanishing derivative in C . $\square$

Note that the inverse, (N, Φ) , of x also extends to a holomorphic function in a neighbourhood of $\mathcal{D}(t)$.

2.2. Strict convexity. In this section we prove part (1) of Theorem 1.1.

A regular smooth surface $\mathcal{S}$ may be expressed locally as the graph of a smooth function $h: U \subseteq \mathbb{R}^{d-1} \rightarrow \mathbb{R}$. The surface is *strictly convex* in a neighbourhood of $(u, h(u))$ if and only if the Hessian $D^2h(u)$ is positive definite. It is *strictly convex* iff this property holds everywhere.

Theorem 2.2. *For $0 < |t| < 1$, any component of $\Gamma(t)$ is strictly convex.*

Proof. We treat the case $0 < t < 1$, the other case being similar. W.l.o.g. the component under consideration is given by $\{x \in]-\frac{\pi}{2}, \frac{\pi}{2}[^d; \prod_{j=1}^d \cos x_j = t\}$. Let $m = d - 1$, $u = (x_1, \dots, x_m)$, and $h = x_d$. The equation defining the previous component becomes $\cos x_1 \dots \cos x_m \cos h = t$ or, each factor being positive,

$$(2.2) \quad \ln \cos x_1 + \dots + \ln \cos x_m + \ln \cos h - \ln t = 0.$$

Since the component is symmetric with respect to the hyperplanes $x_j = 0$ and $x_j = x_l$, where $j, l \in \{1, \dots, d\}$ are distinct, it suffices to show the result for the fundamental domain $h \leq x_1 \leq \dots \leq x_m \leq 0$. There, $h \neq 0$ since $t \neq 1$. The derivative with respect to h of the left side of (2.2) is $-\tan h$, which does not vanish. Consequently, an implicit function $h = h(u, t)$ satisfying (2.2) in a neighbourhood of an arbitrary point of the fundamental domain exists, is analytic, and induces a local parametrization of the previous component.

We thus have to show that the Hessian $D_u^2 h(u, t)$ is positive definite. Differentiating (2.2) with respect to x_j gives

$$-\tan x_j - (\tan h) \partial_{x_j} h = 0,$$

where $h = h(u, t)$. Differentiating this last equation with respect to x_l gives

$$\begin{aligned} -\sec^2 x_j - (\sec^2 h)(\partial_{x_j} h)^2 - (\tan h) \partial_{x_j}^2 h &= 0 & \text{if } l = j, \\ -(\sec^2 h) \partial_{x_j} h \partial_{x_l} h - (\tan h) \partial_{x_j} \partial_{x_l} h &= 0 & \text{otherwise.} \end{aligned}$$

Let $a_{jl} = -(\tan h) \partial_{x_j} \partial_{x_l} h$. Since $-\tan h > 0$, it suffices to show that $[a_{jl}]$ is positive definite. By the above,

$$a_{jl} = \begin{cases} \sec^2 x_j + (\sec^2 h)(\partial_{x_j} h)^2 & \text{if } l = j, \\ (\sec^2 h) \partial_{x_j} h \partial_{x_l} h & \text{otherwise.} \end{cases}$$

Clearly, $(\sec^2 h)[\partial_{x_j} h \partial_{x_l} h]$ is positive, while the diagonal matrix of entries $\sec^2 x_j$ is strictly positive. Thus $[a_{jl}] > 0$, and the result follows. $\square$

2.3. Analytic parametrization. In this section we limit our considerations to a specific cell in $\mathbb{R}^d$, say $]-\frac{\pi}{2}, \frac{\pi}{2}[^d$. Exactly one component, $\mathcal{D}(t)$, of $\Gamma(t)$ is enclosed in this cell for $0 < t < 1$. We show that $\mathcal{D}(t)$ inherits the C^ω -structure of the sphere. This last is specified by the following system of local parametrizations: each hyperplane of coordinates divides S^{d-1} into two hemispheres whose projections on D^{d-1} are denoted γ_j^{-1} ; one then constitutes $\{\gamma_j: D^{d-1} \rightarrow S^{d-1}\}_{j=1}^{2d}$.

By Lemma 2.1 the following system then yields a C^ω -structure on $\mathcal{D}(t)$:

$$\{\sigma_j(\cdot, t): D^{d-1} \rightarrow \mathcal{D}(t), \sigma_j(u, t) = x(\gamma_j(u), t)\}_{j=1}^{2d}.$$

Thus,

Corollary 2.3. *Each component $\mathcal{D}(t)$ of $\Gamma(t)$ is C^ω -diffeomorphic to S^{d-1} .*

Let us compute the element of surface. To this end we consider the determinant $M_j^{(l)}(u, t)$ of the submatrix of format $d - 1 \times d - 1$ obtained from $[\partial_{u_k} \sigma_j^{(l)}(u, t)]_{k,l}$ by removing its l^{th} column, where $\sigma_j = (\sigma_j^{(1)}, \dots, \sigma_j^{(d)})$. Let

$$D_j(u, t) = (M_j^{(1)}(u, t), -M_j^{(2)}(u, t), \dots, (-1)^{d-1} M_j^{(d)}(u, t)).$$

By definition the element of surface is $\|D_j(u, t)\| du$.

Lemma 2.4. $\|D_j(u, t)\| = \|\nabla \Phi(\sigma_j(u, t))\| |\det D\sigma_j(u, t)|$, where the derivative of σ_j is taken with respect to (u, t) .

Proof. The chain rule applied to $\Phi(\sigma_j(u, t)) = t$ gives

$$\nabla \Phi(\sigma_j(u, t)) D\sigma_j(u, t) = [0 \ \cdots \ 0 \ 1],$$

which we abbreviate as $\nabla \Phi D\sigma_j = e_d^t$. Since $(u, t) \mapsto \sigma_j(u, t)$ is bijective, analytic, and has an analytic inverse, $D\sigma_j$ is invertible. Thus,

$$\nabla \Phi = e_d^t (D\sigma_j)^{-1} = \frac{1}{\det D\sigma_j} e_d^t (\text{adj } D\sigma_j),$$

where adj stands for the classical adjoint. The result follows from $e_d^t (\text{adj } D\sigma_j) = (-1)^{d-1} D_j(u, t)$. $\square$

The whole $\mathcal{D}(t)$ except a set of measure zero may be covered using two parametrizations, σ_j and σ_k , associated with opposite hemispheres, $\gamma_j(D^{d-1}), \gamma_k(D^{d-1}) \subset S^{d-1}$, and admitting holomorphic extensions. Using such parametrizations for each component of $\Gamma(t)$, one obtains:

Corollary 2.5. *For a fixed $0 < |t| < 1$, let $f(x)$ be holomorphic in a neighbourhood of $\Gamma(t)$. Then, the Fourier transform*

$$(\mathbf{F}f)(n, t) := \int_{\Gamma(t)} e^{in \cdot x} f(x) dx$$

is holomorphic at t , where $n \in \mathbb{Z}^d$ and dx denotes the element of surface.

We close this section by presenting a local parametrization of $\mathcal{D}(t)$ used in the sequel.

Let $\mathcal{R}(t)$ be the interior of $\mathcal{D}(t)$. For an arbitrarily fixed $\omega \in S^{d-1}$, let Ω be a point in $\mathcal{R}(t)$ incident to the line through $x(\omega, t)$ and parallel to ω , and let $\omega^\perp$ be the hyperplane perpendicular to ω through Ω . Let R be the rigid motion sending e_d to ω and O to Ω . Finally, let $V = \omega^\perp \cap \mathcal{R}(t)$ and $\Upsilon = R^{-1}V$.

By convexity of $\mathcal{D}(t)$ the ray parallel to ω emanating from a given $v \in V$ intersects $\mathcal{D}(t)$ at exactly one point, $Q(v)$, whose normal is not parallel to $\omega^\perp$. Using the natural embedding $\mathbb{R}^{d-1} \subset \mathbb{R}^d$, the composition $Q \circ R$ then gives a parametrization $\tau: \Upsilon \subset \mathbb{R}^{d-1} \rightarrow \mathcal{D}(t)$ compatible with the C^ω -structure on $\mathcal{D}(t)$.

Note that $\Upsilon \rightarrow R^{-1}\mathcal{D}(t)$, $v \mapsto (v, \omega \cdot \tau(v))$ is a local parametrization of $R^{-1}\mathcal{D}(t)$. Since the curvature is invariant under rigid motion, it follows from Section 2.2 that

$$(2.3) \quad \det D_v^2(\omega \cdot \tau(v)) > 0.$$

3. DECAY OF THE RESOLVENT

In this section we prove part (2) of Theorem 1.1. Our proof, based on the stationary phase method, uses of the following classical result (cf. [P], [S]).

Proposition 3.1. *Let $I(r, t) = \int_{\mathbb{R}^d} e^{ir\varphi(x, t)} f(x, t) dx$, where $r > 0$, $t \in T$, T is a compact set in $\mathbb{R}^m$, m is arbitrary, φ and f are smooth on $\mathbb{R}^d \times T$, φ is real valued, and $f(x, t) = 0$ when x is outside a compact subset $K \subseteq \mathbb{R}^d$.*

- (1) *If $\nabla_x \varphi(x, t) \neq 0$ for all $t \in T$ and $x \in K$, then $I(r, t) = O(r^{-\infty})$.*
- (2) *If, for all $t \in T$, $\varphi(x, t)$ admits exactly one critical point $x_0 = x_0(t)$ in K (i.e. $\nabla_x \varphi(x_0, t) = 0$), which is not degenerate (i.e. $\det D_x^2 \varphi(x_0, t) \neq 0$), then $I(r, t) = O(r^{-\frac{d}{2}})$.*

These estimates hold when $r \rightarrow \infty$ and are uniform in $t \in T$. Moreover, (1) holds for $I(r, t)$ over $\mathbb{T}^d$ under the same assumptions except that the compactness of $\text{supp } f$ is replaced with the periodicity of f .

Suppose $T = S \times [-\delta, \delta]$ for a compact S and $\delta > 0$. For $s \in S$ and $\tau \in [-\delta, \delta]$ one may define the Cauchy principal value

$$J(r, s) = \text{p.v.} \int_{|\tau| \leq \delta} \frac{I(r, (s, \tau))}{\tau} d\tau,$$

for which a similar conclusion holds (cf. [P], [V]):

Proposition 3.2. *Let $t = (s, \tau) \in T$. Under the assumption (1) of the above proposition, $J(r, s) = O(r^{-\infty})$. Under the assumption (2), if in addition $\partial_\tau \varphi(x, s, \tau) \neq 0$ when $x \in K$, then $J(r, s) = O(r^{-\frac{d}{2}})$. Both estimates hold when $r \rightarrow \infty$ and are uniform in s .*

In the sequel $r\omega$ stands for the polar form of $n \in \mathbb{Z}^d$. For a fixed nonzero $-1 < E < 1$, let χ be a smooth compactly supported function on $\mathbb{T}^d$ satisfying

$$\chi(x) = \begin{cases} 1 & \text{if } |\Phi(x) - E| < \frac{\delta}{2}, \\ 0 & \text{if } |\Phi(x) - E| > \delta, \end{cases}$$

where $0 < \delta < \text{dist}(E, \{-1, 0, 1\})$. Finally, let $\phi = \frac{1}{\|\nabla \Phi\|}$. Then,

Theorem 3.3. *$G_{E+i0}(n)$ exists and is equal to $(2\pi)^{-d}$ times*

$$\pi i(\mathbf{F}\phi)(n, E) + \text{p.v.} \int_{|t-E| \leq \delta} \frac{(\mathbf{F}\chi\phi)(n, t)}{t - E} dt + O(r^{-\infty})$$

when $r \rightarrow \infty$, uniformly in ω and uniformly in E on each compact.

Proof. Consider the second term in the following decomposition, where $z \in \mathbb{C}_+$:

$$(2\pi)^d G_z(n) = \int_{\mathbb{T}^d} \frac{\chi(x) e^{in \cdot x}}{\Phi(x) - z} dx + \int_{\mathbb{T}^d} \frac{(1 - \chi(x)) e^{in \cdot x}}{\Phi(x) - z} dx.$$

By the dominated convergence theorem its limit exists when $z \rightarrow E$. Moreover, integration by parts shows that this limit is $O(|n|^{-\infty})$ (cf. Proposition 3.1).

It then suffices to analyze the first term. One may express it as a sum of integrals over the open cells of $\mathbb{T}^d$ and then break each cell into two halves covered by surfaces of the form $\mathcal{D}(t)^+ = \sigma_j(D^{d-1}, t)$ and $\mathcal{D}(t)^- = \sigma_k(D^{d-1}, t)$ respectively, where γ_j

and γ_k parametrize opposite hemispheres (see Section 2.3). For one of these halves, say $\mathcal{H}^+ = \bigcup_{0 < t < 1} \mathcal{D}(t)^+$, the change of variables $x = \sigma_j(u, t)$ gives

$$\begin{aligned} \int_{\mathcal{H}^+} \frac{\chi(x) e^{in \cdot x}}{\Phi(x) - z} dx &= \int_0^1 \int_{\mathbb{D}^{d-1}} \frac{\chi(\sigma_j(u, t)) e^{in \cdot \sigma_j(u, t)}}{\Phi(\sigma_j(u, t)) - z} |\det D\sigma_j(u, t)| du dt \\ &= \int_{E-\delta}^{E+\delta} \frac{1}{t-z} \int_{\mathcal{D}(t)^+} \phi(x) \chi(x) e^{in \cdot x} dx dt. \end{aligned}$$

Reuniting all pieces together,

$$\begin{aligned} \int_{\mathbb{T}^d} \frac{\chi(x) e^{in \cdot x}}{\Phi(x) - z} dx &= \int_{E-\delta}^{E+\delta} \frac{1}{t-z} \int_{\Gamma(t)} \phi(x) \chi(x) e^{in \cdot x} dx dt \\ &= \int_{E-\delta}^{E+\delta} \frac{1}{t-z} (\mathbf{F}\phi\chi)(n, t) dt. \end{aligned}$$

Since $(\mathbf{F}\phi\chi)(n, t)$ is holomorphic at $t = E$, the result follows. $\square$

It remains to establish the decay of both nonnegligible terms in the last theorem. In what follows t varies in a neighbourhood of a compact interval $T \subset]-1, 1[$ not containing 0. Again, our considerations are limited to the component $\mathcal{D}(t)$ of $\Gamma(t)$ enclosed in a fixed cell. We define $\mathcal{S} := \bigcup_{t \in T} \mathcal{D}(t)$.

Theorem 3.4. *If $f(x)$ is smooth in a neighbourhood of $\mathcal{S}$, then $(\mathbf{F}f)(n, t) = O(r^{-\frac{d-1}{2}})$ uniformly in $\omega \in \mathbb{S}^{d-1}$ and $t \in T$ when $r \rightarrow \infty$.*

Proof. By compactness of $\mathbb{S}^{d-1}$ it suffices to prove the result for ω in a closed neighbourhood of an arbitrarily fixed ω_0 , say, for $|\omega - \omega_0| \leq \varepsilon$, where ε is arbitrarily small.

Let $\{\sigma_j(\cdot, t)\}_{j=1}^{2d}$ be the system of local parametrizations described in Section 2.3, where $\sigma_j(u, t) = x(\gamma_j(u), t)$. One assumes w.l.o.g. that γ_1 parametrizes a hemisphere containing ω_0 , and γ_2 , the opposite hemisphere. For an open interval $0 \notin I \subset]-1, 1[$ containing T , let $U_j = \sigma_j(\mathbb{D}^{d-1}, I)$, so $\{U_j\}_{j=1}^{2d}$ is an open covering of $\mathcal{S}$. Note that $x(\omega_0, t) \in U_1$ and $x(-\omega_0, t) \in U_2$. By continuity of x , $x(\omega, t)$ and $x(-\omega, t)$ varies in compact sets $K_1 \subset U_1$ and $K_2 \subset U_2$ respectively when t varies in T and ω varies in $|\omega - \omega_0| \leq \varepsilon$ for a sufficiently small ε . Consequently, there exists a partition of unity $\{\chi_j\}_{j=1}^{2d}$ subordinate to $\{U_j\}_{j=1}^{2d}$ satisfying $\chi_j|_{K_j} = 1$ for $j = 1, 2$. Restricting the Fourier transform to $\mathcal{D}(t)$, one obtains

$$(3.1) \quad (\mathbf{F}f)(n, t) = \sum_j \int_{\mathbb{D}^{d-1}} e^{ir\omega \cdot \sigma_j(u, t)} \chi_j(\sigma_j(u, t)) f(\sigma_j(u, t)) \|D_j(u, t)\| du.$$

Let $\varphi_j(u) = \omega \cdot \sigma_j(u, t)$ be the phase. φ_j is stationary if and only if its gradient is 0, in which case $\omega \cdot \partial_{u_k} \sigma_j(u, t) = 0$ for each $k = 1, \dots, d-1$, i.e. $\omega \perp \mathcal{D}(t)$. Thus, $x(\pm\omega, t)$ are the only stationary phase points. Since they lie in K_1 and K_2 respectively, they affect only the first two integrals, separately.

Let us prove that $x(\omega, t)$ is not degenerate, the proof for $x(-\omega, t)$ being similar. The parametrization (Y, τ) defined at the end of Section 2.3 covers a neighbourhood of $x(\omega, t)$. Thus, there exists a v_0 such that $x(\omega, t) = \tau(v_0)$. On the other hand, there exists a u_0 such that $x(\omega, t) = \sigma_1(u_0, t)$. Let $\mathcal{T} = \sigma_1(\cdot, t)^{-1} \circ \tau$, so the relation $\omega \cdot \tau(v) = (\varphi_1 \circ \mathcal{T})(v)$ holds locally. There, letting $u = \mathcal{T}(v)$,

$$D_v^2(\omega \cdot \tau(v)) = D_u^2 \varphi_1(u) (D_v \mathcal{T}(v))^2 + D_u \varphi_1(u) D_v^2 \mathcal{T}(v).$$

Since φ_1 is stationary at u_0 , (2.3) and the previous relation give $\det D_u^2 \varphi_1(u_0) > 0$, as desired.

Proposition 3.1 then completes the proof. $\square$

The previous theorem takes care of the first term in the decomposition provided by Theorem 3.3, taking $f = \phi$ and $t = E$. In order to study the second term, let E vary in an arbitrary interval $[a, b] \subset]-1, 1[$ not containing zero. One may assume that δ is less than the distance between $[a, b]$ and $\{-1, 0, 1\}$, and apply the previous proof to $T = [a - \delta, b + \delta]$ and $f = \chi\phi$.

Theorem 3.5. *Suppose $f(x)$ is smooth in a neighbourhood of $\mathcal{S}$. Then,*

$$\text{p. v.} \int_{|t-E| \leq \delta} \frac{(\mathbf{F}f)(n, t)}{t - E} dt = O(r^{-\frac{d-1}{2}})$$

uniformly in $\omega \in S^{d-1}$ and $E \in [a, b]$ when $r \rightarrow \infty$.

Proof. Again, it is sufficient to show the result for $|\omega - \omega_0| \leq \varepsilon$, where $\omega_0 \in S^{d-1}$ is arbitrarily fixed. The relation (3.1) then holds, which we abbreviate

$$(\mathbf{F}f)(n, t) = \sum_j \int_{D^{d-1}} e^{ir\varphi_j(u, \omega, t)} f_j(u, t) du,$$

so $\varphi_j(u, \omega, t) = \omega \cdot \sigma_j(u, t)$. Since $(\Phi \circ \sigma_j)(u, t) = t$, the chain rule gives $\nabla \Phi(\sigma_j(u, t)) \cdot \partial_t \sigma_j(u, t) = 1$. At $x(\pm\omega, t)$, $\nabla \Phi = \pm\omega \|\nabla \Phi\|$, so the last equation gives $\partial_t \varphi_j = \pm\omega \cdot \partial_t \sigma_j = \|\nabla \Phi\|^{-1}$.

Let us introduce a family of smooth cutoff functions

$$\theta_j(u, \omega, t) = \begin{cases} 1 & \text{if } \partial_t \varphi_j(u, \omega, t) \geq (2/3) \|\nabla \Phi(\sigma_j(u, t))\|^{-1}, \\ 0 & \text{if } \partial_t \varphi_j(u, \omega, t) \leq (1/3) \|\nabla \Phi(\sigma_j(u, t))\|^{-1}. \end{cases}$$

By what precedes, $\theta_j = 1$ at the stationary phase points. Thus, only the first two terms of the first summation, for which we have assured $\partial_t \varphi_j > 0$, are affected by a stationary phase point in the following relation:

$$(\mathbf{F}f)(n, t) = \sum_j \int_{D^{d-1}} e^{ir\varphi_j} \theta_j f_j du + \sum_j \int_{D^{d-1}} e^{ir\varphi_j} (1 - \theta_j) f_j du.$$

Proposition 3.2 then completes the proof. $\square$

Joining Theorems 2.2, 3.3, 3.4, and 3.5, we have proven Theorem 1.1.

REFERENCES

- [B] Bourgain, J.: On random Schrödinger operators on $\mathbb{Z}^2$. *Discrete and Continuous Dynamical Systems* **8**, 1 (2002). MR1877824 (2003f:47063)
- [BBP] Bentosela, F., Briet, Ph., Pastur, L.: On the spectral and wave propagation properties of the surface Maryland model. *J. Math. Phys.* **44** (2003), no. 1, 1–35. MR1946689 (2003k:81049)
- [CK] Christ, M., Kiselev, A.: Scattering and wave operators for one-dimensional Schrödinger operators with slowly decaying nonsmooth potentials. *Geom. Funct. Anal.* **12** (2002), no. 6, 1174–1234. MR1952927 (2003m:47019)
- [CS] Chahrour, A., Sahbani, J.: On the spectral and scattering theory of the Schrödinger operator with surface potential. *Rev. Math. Phys.* **12**, 561 (2000). MR1763841 (2001b:81026)
- [HK] Hundertmark, D., Kirsch, W.: Spectral theory of sparse potentials. In *Stochastic processes, physics and geometry: new interplays, I (Leipzig, 1999)*, CMS Conf. Proc. **28**, 213 (2000). MR1803388 (2002b:81029)

- [JL1] Jakšić, V., Last, Y.: Corrugated surfaces and a.c. spectrum. *Rev. Math. Phys.* **12**, 1465 (2000). MR1809458 (2001m:47143)
- [JL2] Jakšić, V., Last, Y.: Surface states and spectra. *Commun. Math. Phys.* **218**, 459 (2001). MR1828849 (2002g:81030)
- [JM] Jakšić, V., Molchanov, S.: Wave operators for the surface Maryland model. *J. Math. Phys.* **41**, 4452 (2000). MR1765613 (2002a:81052)
- [JMP] Jakšić, V., Molchanov, S., Pastur, L.: On the propagation properties of surface waves. In *Wave Propagation in Complex Media*, IMA Vol. Math. Appl. **96**, 143 (1998). MR1489748
- [JP] Jakšić, V., Poulin, P.: In preparation.
- [K] Krishna, M.: Anderson model with decaying randomness-extended states. *Proc. Indian Acad. Sci. (MathSci.)* **100**, 285 (1990). MR1081712 (91m:82069)
- [MV1] Molchanov, S., Vainberg, B.: Scattering on the system of the sparse bumps: multidimensional case. *Appl. Anal.* **71**, 167 (1999). MR1690097 (2000a:47071)
- [MV2] Molchanov, S., Vainberg, B.: Spectrum of multidimensional Schrödinger operators with sparse potentials. Analytic and computational methods in scattering and applied mathematics Chapman Hall/CRC Res. Notes Math. **417**, 231 (2000). MR1756700 (2001b:47054)
- [P] Poulin, P.: Ph.D. Thesis, McGill University. In preparation.
- [RS] Rodnianski, I., Schlag, W.: Classical and quantum scattering for a class of long range random potentials. *Int. Math. Res. Not.* 2003, no. 5, 243–300. MR1941087 (2003k:81248)
- [S] Stein, E.: *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*. Princeton University Press, Princeton 1993. MR1232192 (95c:42002)
- [SV] Shaban, W., Vainberg, B.: Radiation conditions for the difference Schrödinger operators. *Appl. Anal.* 80 (2001), no. 3-4, 525–556. MR1914696 (2003d:47049)
- [V] Vainberg, B.: *Asymptotic Methods in Equations of Mathematical Physics*. Gordon and Breach Science Publishers, New York 1982. MR1054376 (91h:35081)

DEPARTMENT OF MATHEMATICS AND STATISTICS, MCGILL UNIVERSITY, MONTRÉAL, QUÉBEC, CANADA H3A-2K6

E-mail address: ppoulin@math.mcgill.ca