POINTWISE CONVERGENCE OF BOUNDED CASCADE SEQUENCES

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ABSTRACT. The cascade algorithm plays an important role in computer graphics and wavelet analysis. For an initial function ϕ_0 , a cascade sequence $(\phi_n)_{n=0}^\infty$ is constructed by the iteration $\phi_n = C_a \phi_{n-1}$, $n = 1, 2, \dots$, where C_a is defined by $C_a g = \sum_{\alpha \in \mathbb{Z}} a(\alpha) g(2 \cdot -\alpha)$, $g \in L_p(\mathbb{R})$. In this paper, under a condition that the sequence $(\phi_n)_{n=0}^\infty$ is bounded in $L_\infty(\mathbb{R})$, we prove that the following three statements are equivalent: (i) $(\phi_n)_{n=0}^\infty$ converges a.e. $x \in \mathbb{R}$. (ii) For a.e. $x \in \mathbb{R}$, there exist a positive constant c and a constant $\gamma \in (0, 1)$ such that $|\phi_{n+1}(x) - \phi_n(x)| \leq c\gamma^n$ $\forall n = 1, 2, \dots$ (iii) For some $p \in [1, \infty)$, $(\phi_n)_{n=0}^\infty$ converges in $L_p(\mathbb{R})$. An example is presented to illustrate our result.

1. INTRODUCTION

We are interested in the functional equation of form

$$\phi = \sum_{\alpha \in \mathbb{Z}} a(\alpha) \phi(2 \cdot -\alpha),$$

where $a = (a(\alpha))_{\alpha \in \mathbb{Z}}$, called a refinement mask, is a finitely supported sequence on $\mathbb{Z}$. The equation is known as a refinement equation. A nonzero function ϕ is a refinable function corresponding to a if it satisfies the above equation.

Associated with the above refinement equation is the cascade operator C_a defined by

$$(1) \quad C_a g = \sum_{\alpha \in \mathbb{Z}} a(\alpha) g(2 \cdot -\alpha), \quad g \in L_p(\mathbb{R}).$$

For any initial function ϕ_0 , we construct a sequence by the iteration

$$(2) \quad \phi_n = C_a \phi_{n-1} = C_a^n \phi_0, \quad n = 1, 2, \dots$$

This sequence is called a cascade sequence generated by ϕ_0 . Such a construction method of sequences is known as the cascade algorithm. It is the most powerful tool to solve the refinement equations.

The cascade algorithm plays an important role in computer graphics and wavelet analysis. The convergence of the subdivision scheme and cascade algorithm in

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$W_p(\mathbb{R})$ and the Sobolev space $W_p^s(\mathbb{R})$ has been studied by many authors. We mention here some work, but without any completeness at all. Cavaretta *et al.* [2] already found necessary and sufficient conditions ensuring that the cascade algorithm converges uniformly to a continuous limit function. In L_2 the convergence had been shown by Strang [21], Lawton *et al.* [14], [20] and [16]. Jia [9] considered the convergence of cascade algorithms in the univariate setting for general L_p -spaces. For the univariate vector case we refer to Jia *et al.* [12] and to Micchelli *et al.* [18]. Chen *et al.* [6] and Li [15] studied this problem in the general setting. The convergence of cascade algorithms for perturbed refinement masks was discussed in [7]. Recently, with the help of the cascade algorithm, an easily implemented algorithm for the construction of smooth refinable functions in $L_p(\mathbb{R})$, $1 \leq p < \infty$, was presented in [3].

However, as far as we know, there has been very little discussion on L_∞ refinable functions or the pointwise convergence of cascade sequences in $L_\infty(\mathbb{R})$ in literature. The properties of L_∞ refinable functions are somewhat different from those of refinable functions in $W_p(\mathbb{R})$. For example, if a compactly supported refinable function $\phi \in W_p(\mathbb{R})$, then

$$\|\phi - \phi(\cdot - h)\|_{W_p(\mathbb{R})} = O(h^s), \quad \text{as } h \rightarrow 0,$$

where $s > 0$ is a constant. It is an interesting question of whether the analogous conclusion holds for the L_∞ case. The author established a result about pointwise regularity of an arbitrary L_∞ refinable function in [4]. This result may be interpreted as a partially positive answer to the above question in the pointwise case.

In this paper we study the pointwise convergence of a bounded cascade sequence in $L_\infty(\mathbb{R})$. Our main result is the following.

Theorem 1. *Let ϕ_0 be a compactly supported function in $L_\infty(\mathbb{R})$. Suppose that the cascade sequence $(\phi_n)_{n=0}^\infty$ is bounded in $L_\infty(\mathbb{R})$. Then the following statements are equivalent:*

- (i) $(\phi_n)_{n=0}^\infty$ converges a.e. $x \in \mathbb{R}$.
- (ii) For a.e. $x \in \mathbb{R}$, there exist a positive constant c and a constant $\gamma \in (0, 1)$ such that $|\phi_{n+1}(x) - \phi_n(x)| \leq c\gamma^n \forall n = 1, 2, \dots$
- (iii) For some $p \in [1, \infty)$, $(\phi_n)_{n=0}^\infty$ converges in $L_p(\mathbb{R})$.

The paper is organized as follows. In Section 2 we consider the pointwise convergence in a setting more general than the cascade algorithm. As a corollary of the results of Section 2, the proof of Theorem 1 is given in Section 3. We also generalize Theorem 1 to the sequence of derivatives. Finally, an example is presented to illustrate our results.

2. CONVERGENCE IN THE SETTING OF REFINABLE CURVES

We define a shift operator τ on $[0, 1]$ by

$$\tau x = \begin{cases} 2x, & 0 \leq x \leq 1/2, \\ 2x - 1, & 1/2 < x \leq 1. \end{cases}$$

Let $x \in [0, 1]$. There are $d_j \in \{0, 1\}$, $j = 1, 2, \dots$, such that $x = \sum_{j=1}^\infty 2^{-j} d_j$. This is a dyadic expansion of x , and d_j is its j -th digit. The shift operator may be

represented in terms of dyadic expansions as follows:

$$(3) \quad \tau x = \sum_{j=1}^{\infty} 2^{-j} d_{j+1} \quad \text{for } x = \sum_{j=1}^{\infty} 2^{-j} d_j.$$

If $x \notin \{t \in (0, 1) : t = k/2^m, 0 < k < 2^m, m = 1, 2, \dots\}$, the dyadic expansion of x is unique. Consequently, for almost every $x \in [0, 1]$, we may use $d_j(x)$ to denote its j -th digit.

To study refinable functions, it is sometimes convenient to work in a more general setting, the setting of refinable curves. Suppose that A_0 and A_1 are $N \times N$ matrices. A vector Φ of N functions on $[0, 1]$ is referred to as a refinable curve (see e.g. [17]) if it satisfies

$$(4) \quad \Phi(x) = A_{d_1(x)} \Phi(\tau x), \quad \text{a.e. on } x \in [0, 1].$$

For an initial vector Φ_0 of N functions defined on $[0, 1]$, the cascade sequence $(\Phi_n)_{n=0}^{\infty}$ generated by Φ_0 is given by the iteration

$$(5) \quad \Phi_n(x) = A_{d_1(x)} \Phi_{n-1}(\tau x), \quad \text{a.e. on } x \in [0, 1].$$

Consequently,

$$(6) \quad \Phi_n(x) = A_{d_1(x)} \cdots A_{d_n(x)} \Phi_0(\tau^n x), \quad \text{a.e. on } x \in [0, 1].$$

Let $\|\cdot\|$ be a norm of $\mathbb{C}^N$. It derives a norm of the space of vectors F of functions in $L_p([0, 1])$ as follows:

$$\|F\|_p = \left(\int_0^1 \|F(x)\|^p dx \right)^{1/p}.$$

We say that the sequence $(\Phi_n)_{n=0}^{\infty}$ converges in L_p if it converges in the norm $\|\cdot\|_p$. Clearly the definition is independent of the choice of the norm on $\mathbb{C}^N$.

Suppose that A_0 and A_1 are $N \times N$ matrices. A linear subspace $V \subseteq \mathbb{C}^N$ is called an $\{A_0, A_1\}$ -invariant subspace if it is invariant under each $A_d, d = 0, 1$. For a subset set $S \subseteq \mathbb{C}^N$, denote by $V(S)$ the smallest $\{A_0, A_1\}$ -invariant subspace containing S .

In our study, a special kind of set $S = S_{E, \Psi}$ given by

$$(7) \quad S_{E, \Psi} = \{\Psi(x) : x \in E\}$$

will be used, where $E \subseteq [0, 1]$ and Ψ is a vector of N functions on $[0, 1]$.

Afterwards we use the same notation $\|\cdot\|$ to denote a norm of $\mathbb{C}^N$ and a matrix norm. The exact meaning should be clear from the context.

Lemma 2. Choose $\Phi_0 \in (L_{\infty}([0, 1]))^N$. Then the cascade sequence $(\Phi_n)_{n=0}^{\infty}$ is bounded in $(L_{\infty}([0, 1]))^N$ if and only if there are a subset $E \subseteq [0, 1]$ with $\mu(E) = 1$ and a matrix norm $\|\cdot\|$ satisfying $\|A_d|_{V(S_{E, \Phi_0})}\| \leq 1, d = 0, 1$, where S_{E, Φ_0} is given as in (7).

Proof. We first verify the necessity. By assumption we have a subset $D \subseteq [0, 1]$ with $\mu(D) = 1$ and a constant c such that $\|\Phi_n(x)\| \leq c \quad \forall x \in D$ and $n = 1, 2, \dots$. Let $E = \bigcap_{n=1}^{\infty} \tau^n(D)$. Then $\mu(E) = 1$. Moreover, we have by (6)

$$\|A_{d_1} \cdots A_{d_n} \Phi_0(t)\| \leq c \quad \forall t \in E, d_1, \dots, d_n \in \{0, 1\} \text{ and } n = 1, 2, \dots$$

Let $S = S_{E, \Phi_0}$ be defined by (7) and $\Sigma = \{A_d|_{V(S)} : d = 0, 1\}$. By the last inequality, Σ is product bounded. In other words, all products generated by Σ are bounded uniformly. The necessity follows from [1].

Conversely, suppose that there are a subset $E \subseteq [0, 1]$ with $\mu(E) = 1$ and a matrix norm $\|\cdot\|$ satisfying $\|A_d|_{V(S_{E, \Phi_0})}\| \leq 1, d = 0, 1$. Let $D = \bigcap_{n=1}^{\infty} \tau^{-n}(E)$. Then $\mu(D) = 1$ and for any $x \in D, \tau^n x \in E$.

Let $n_0 = \dim V(S_{E, \Phi_0})$. It is clear that $A_{d_1} \cdots A_{d_{n_0}} \Phi_0(t) \in V(S_{E, \Phi_0})$ for any $t \in E$. Consequently, for $n \geq n_0$ and $x \in D$, we have, by (6) and the assumption on $A_d, d = 0, 1$, that $\|\Phi_n(x)\| \leq \|\Phi_{n_0}(\tau^{n-n_0}x)\| \leq \|\Phi_{n_0}\|_{(L_{\infty}[0,1])^N}$. Therefore, the sequence (Φ_n) is bounded in $(L_{\infty}[0,1])^N$. The proof is complete. $\square$

Before proceeding further, let us introduce the notion of joint spectral radius for the convenience of readers. Suppose that V is a *finitely dimensional* vector space equipped with a vector norm $\|\cdot\|$. For a linear operator A on V , define

$$\|A\| := \max\{\|Av\| : \|v\| = 1\}.$$

Let $\mathcal{A}$ be a finite set of linear operators on V . For a positive integer n we denote by $\mathcal{A}^n$ the n -th Cartesian power of $\mathcal{A}$:

$$\mathcal{A}^n = \{(A_1, \dots, A_n) : A_1, \dots, A_n \in \mathcal{A}\}.$$

For $1 \leq p < \infty$, let

$$\|\mathcal{A}^n\|_p := \left(\sum_{(A_1, \dots, A_n) \in \mathcal{A}^n} \|A_1 \cdots A_n\|^p \right)^{1/p},$$

and for $p = \infty$ define

$$\|\mathcal{A}^n\|_{\infty} := \max\{\|A_1 \cdots A_n\| : (A_1, \dots, A_n) \in \mathcal{A}^n\}.$$

For $1 \leq p \leq \infty$, the p -norm joint spectral radius of $\mathcal{A}$ is defined to be

$$\rho_p(\mathcal{A}) := \lim_{n \rightarrow \infty} \|\mathcal{A}^n\|_p^{1/n}.$$

It is easily seen that this limit indeed exists and

$$(8) \quad \rho_p(\mathcal{A}) = \inf_{n \geq 1} \|\mathcal{A}^n\|_p^{1/n}.$$

Clearly, $\rho_p(\mathcal{A})$ is independent of the choice of the vector norm on V .

The uniform joint spectral radius was introduced in [19]. The mean joint spectral radius was introduced in [22]. For $1 < p < \infty$, the p -joint spectral radius was introduced in [9] and applied to the study of L_p convergence of cascade algorithms.

Theorem 3. *Let $\Phi_0 \in (L_1([0, 1]))^N$. Suppose that the sequence $(\Phi_n)_{n=0}^{\infty}$ converges in $(L_1([0, 1]))^N$. Then for any matrix norm $\|\cdot\|$, there are a set $F \subseteq [0, 1]$ with $\mu(F) = 1$ and a positive integer k such that*

$$(9) \quad \sum_{d_1, \dots, d_k} \|B_{d_1} \cdots B_{d_k}\| < 2^k,$$

where B_d is the restriction of A_d on $V(S)$, i.e., $B_d = A_d|_{V(S)}, d = 0, 1$, and $S = S_{F, \Phi_1 - \Phi_0}$ is given as in (7).

Proof. For any n and $d_j \in \{0, 1\}, 1 \leq j \leq n$, define a subset $T_{d_1 \dots d_n}$ by

$$T_{d_1 \dots d_n} = \left\{ x : \sum_{j=1}^n 2^{-j-n} d_j \leq x \leq 2^{-n} + \sum_{j=1}^n 2^{-j-n} d_j \right\}.$$

It follows from $\tau^n(T_{d_1, \dots, d_n}) = [0, 1]$ and (6) that

$$(10) \quad \Phi_n(x) = A_{d_1} \cdots A_{d_n} \Phi_0(\tau^n x), \quad x \in T_{d_1, \dots, d_n}.$$

Consequently, for $x \in T_{d_1 \dots d_n d_{n+1}} \subset T_{d_1 \dots d_n}$,

$$\begin{aligned}\Phi_{n+1}(x) - \Phi_n(x) &= A_{d_1} \cdots A_{d_n} (A_{n+1} \Phi_0(\tau^{n+1}x) - \Phi_0(\tau^n x)) \\ &= A_{d_1} \cdots A_{d_n} (\Phi_1(\tau^n x) - \Phi_0(\tau^n x)),\end{aligned}$$

where the second equality is true due to $d_1(\tau^n x) = d_{n+1}(x) \forall x \in T_{d_1 \dots d_n d_{n+1}}$. Integrating on this subset gives

$$\int_{T_{d_1 \dots d_n d_{n+1}}} \|\Phi_{n+1}(x) - \Phi_n(x)\| dx = 2^{-n} \int_{\frac{d_{n+1}}{2}}^{\frac{d_{n+1}+1}{2}} \|A_{d_1} \cdots A_{d_n} (\Phi_1(x) - \Phi_0(x))\| dx.$$

Summing all the equalities over $d_1, \dots, d_n, d_{n+1} \in \{0, 1\}$ we have

$$\|\Phi_{n+1} - \Phi_n\|_1 = \int_0^1 2^{-n} \sum_{d_1, \dots, d_n} \|A_{d_1} \cdots A_{d_n} (\Phi_1(x) - \Phi_0(x))\| dx.$$

The convergence of $(\Phi_n)_{n=0}^\infty$ in $(L_1([0, 1]))^N$ and Fatou's Lemma imply

$$\liminf_{n \rightarrow \infty} 2^{-n} \sum_{d_1, \dots, d_n} \|A_{d_1} \cdots A_{d_n} (\Phi_1(x) - \Phi_0(x))\| = 0 \quad \text{a.e.}$$

Let F be the set of all the points $x \in [0, 1]$ such that the above equality holds for x . Clearly, $\mu(F) = 1$. For any $x \in F$, by the last equality and the method of [8], Lemma 2.4, we can conclude that $\rho_1(\{A_0|_{V(S_{\{x\}, \Phi_1 - \Phi_0})}, A_1|_{V(S_{\{x\}, \Phi_1 - \Phi_0})}\}) < 2$. Moreover, it is known from [5], Theorem 1, that

$$\begin{aligned}\rho_1(\{A_0|_{V(S_{F, \Phi_1 - \Phi_0})}, A_1|_{V(S_{F, \Phi_1 - \Phi_0})}\}) \\ = \max_{x \in F} \rho_1(\{A_0|_{V(S_{\{x\}, \Phi_1 - \Phi_0})}, A_1|_{V(S_{\{x\}, \Phi_1 - \Phi_0})}\}).\end{aligned}$$

Therefore, $\rho_1(\{B_0, B_1\}) < 2$. By the definition of joint spectral radius, we obtain the existence of some k such that (9) holds. The proof is complete. $\square$

For a fixed k -tuple $(d_1^0, \dots, d_k^0) \in \{0, 1\}^k$, a positive integer n and $x \in [0, 1]$, how many i 's are there with $0 \leq i \leq [n/k] - 1$ such that

$$(11) \quad (d_{ik+1}(x), d_{ik+2}(x), \dots, d_{i(k+k)}(x)) = (d_1^0, \dots, d_k^0),$$

where $[t]$ for $t \in \mathbb{R}$ denotes the largest integer smaller than or equal to t . The number of such i 's is just the cardinality of the set

$$\{0 \leq i \leq [n/k] - 1 : \tau^{ik}x \in T_{d_1^0 \dots d_k^0}\},$$

since it is easily seen that (11) is equivalent to $\tau^{ik}x \in T_{d_1^0 \dots d_k^0}$. For simplicity we denote by $Q(x, n)$ the cardinality of above set.

By use of the ergodic theorem, the following result is established in [4].

Lemma 4. *Let a k -tuple $(d_1^0, \dots, d_k^0) \in \{0, 1\}^k$ be fixed and let $Q(x, n)$ be defined as above. Then there exists a subset $F_{d_1^0 \dots d_k^0} \subseteq [0, 1]$ satisfying the following conditions:*

- (i) $\mu(F_{d_1^0 \dots d_k^0}) = 1$.
- (ii) *For any $x \in F_{d_1^0 \dots d_k^0}$, there are a positive integer n_0 and a constant c , both dependent of x , such that $Q(x, n) \geq cn$, $n \geq n_0$.* $\square$

We are in a position to establish an equivalence for the pointwise convergence of a cascade sequence.

Theorem 5. *Suppose that the sequence $(\Phi_n)_{n=0}^\infty$ is bounded in $(L_\infty([0, 1]))^N$. Then the following statements are equivalent:*

- (i) $(\Phi_n)_{n=0}^\infty$ converges a.e. on $[0, 1]$.
- (ii) For a.e. $x \in [0, 1]$, there exist a positive constant c and a constant $\gamma \in (0, 1)$ such that $\|\Phi_{n+1}(x) - \Phi_n(x)\| \leq \gamma^n$ for sufficiently large n .

Proof. We only need to prove (i) $\Rightarrow$ (ii). Suppose (i) is true. By bounded dominated convergence theorem, the sequence $(\Phi_n)_{n=0}^\infty$ converges in $L_1([0, 1])$. By a result of [1] there are a subset $E \subseteq [0, 1]$ with $\mu(E) = 1$ and a matrix norm satisfying $\|A_d|_{V(S_{E, \Phi_0})}\| \leq 1$ for $d = 0, 1$. For this matrix norm, Theorem 3 ensures the existence of some k such that (9) holds for $B_d = A_d|_{V(S_{F, \Phi_1 - \Phi_0})}$, where $F \subseteq [0, 1]$ with $\mu(F) = 1$. It implies that there exists a k tuple $(d_1^0, \dots, d_k^0)$ satisfying

$$(12) \quad \eta := \|B_{d_1^0} \cdots B_{d_k^0}\| < 1.$$

By (6) we have for a.e. x

$$\Phi_{n+1}(x) - \Phi_n(x) = A_{d_1(x)} \cdots A_{d_n(x)} (\Phi_1(\tau^n x) - \Phi_0(\tau^n x)).$$

Let $G = \bigcap_{n=0}^\infty \tau^n(F)$. Then $\mu(G) = 1$ and $\tau^n(G) \subseteq G \subseteq F$ for any $n = 1, 2, \dots$. Therefore, for any $\varepsilon_1, \dots, \varepsilon_m \in \{0, 1\}$ and $x \in G$, we have

$$A_{\varepsilon_1} \cdots A_{\varepsilon_m} (\Phi_1(\tau^n x) - \Phi_0(\tau^n x)) \in V(S_{F, \Phi_1 - \Phi_0}).$$

Suppose that i satisfies $\tau^{ik}x \in T_{d_1^0 \dots d_k^0}$. Note that $(A_{d_1^0} \cdots A_{d_k^0})|_{V(S_{F, \Phi_1 - \Phi_0})} = B_{d_1^0} \cdots B_{d_k^0}$. Then the block of product $A_{d_{ik+1}(x)} \cdots A_{d_{ik+k}(x)}$ has norm η on $V(S_{F, \Phi_1 - \Phi_0})$.

We now consider the remaining $n - kQ(x, n)$ matrices in the product $A_{d_1(x)} \cdots A_{d_n(x)}$. Lemma 2 tells us that each $A_{d_i(x)}$ has a norm not larger than 1 on $V(S_{E, \Phi_0})$. On the other hand, for any $x \in E$, by construction of E , $\tau^n x \in E \forall n = 0, 1, \dots$. Therefore, for any $\varepsilon_1, \dots, \varepsilon_m \in \{0, 1\}$, we have

$$\begin{aligned} & A_{\varepsilon_1} \cdots A_{\varepsilon_m} (\Phi_1(\tau^n x) - \Phi_0(\tau^n x)) \\ &= A_{\varepsilon_1} \cdots A_{\varepsilon_m} A_{d_{n+1}(x)} \Phi_0(\tau^{n+1} x) - A_{\varepsilon_1} \cdots A_{\varepsilon_m} \Phi_0(\tau^n x) \in V(S_{E, \Phi_0}). \end{aligned}$$

In summary, for any $x \in E \cap G$ we have

$$\|\Phi_{n+1}(x) - \Phi_n(x)\| \leq \eta^{Q(x, n)} \|\Phi_1(\tau^n x) - \Phi_0(\tau^n x)\|.$$

The statement (ii) follows from conclusion (ii) of Lemma 4. The proof is complete. $\square$

In the deduction of (i) $\Rightarrow$ (ii) from the proof of Theorem 5, only the condition that $(\Phi_n)_{n=0}^\infty$ converges in $(L_1([0, 1]))^N$ is needed. Therefore, we actually have proved the following result.

Corollary 6. *Suppose that the sequence $(\Phi_n)_{n=0}^\infty$ is bounded in $(L_\infty([0, 1]))^N$. Then $(\Phi_n)_{n=0}^\infty$ converges a.e. on $[0, 1]$ if and only if for some $p \in [1, \infty)$, $(\Phi_n)_{n=0}^\infty$ converges in $(L_p([0, 1]))^N$.*

3. PROOF AND GENERALIZATION OF THEOREM 1

Proof of Theorem 1. For a finitely supported refinement mask a , we assume without loss of generality that $a(\alpha) = 0 \forall \alpha < 1$ or $\alpha > N - 1$, where N is a positive integer. We construct two $N \times N$ matrices $A_d, d = 0, 1$, as follows:

$$A_d = \left(a(2\alpha - \beta + d) \right)_{0 \leq \alpha, \beta \leq N-1}.$$

For a compactly supported function ϕ_0 , we construct a cascade sequence $(\phi_n)_{n=0}^\infty$ by (2). It is easily seen that there exists an integer n_0 such that $\text{supp} \phi_n \subseteq [0, N] \forall n \geq n_0$.

Let $\Phi_n(x) = (\phi_n(x + \alpha))_{\alpha=0}^{N-1}, x \in [0, 1]$. Then they satisfy (5) with A_d given above and for all $n \geq n_0 + 1$. Moreover, $(\Phi_n)_{n=0}^\infty$ is bounded in $L_\infty([0, 1])$. Clearly, Theorem 1 follows from Theorem 5 and Corollary 6. The proof is complete. $\square$

We now discuss the pointwise convergence of k -th derivatives. Let $(\phi_n)_{n=0}^\infty$ be a cascade sequence constructed by (2). Then the sequence $(\phi_n^{(k)})_{n=0}^\infty$ itself is also a cascade sequence with the mask $a = (a(\alpha))$ replaced by the mask $2^k a$. Theorem 1 applies in this case. We have established the following result.

Theorem 7. *Let ϕ_0 be a compactly supported function in $L_\infty(\mathbb{R})$ and let k be a positive integer. Suppose that the sequence $(\phi_n^{(k)})_{n=0}^\infty$ of k -th derivatives is bounded in $L_\infty(\mathbb{R})$. Then the following statements are equivalent:*

- (i) $(\phi_n^{(k)})_{n=0}^\infty$ converges a.e. $x \in \mathbb{R}$.
- (ii) For a.e. $x \in \mathbb{R}$, there exist a positive constant c and a constant $\gamma \in (0, 1)$ such that $|\phi_{n+1}^{(k)}(x) - \phi_n^{(k)}(x)| \leq c\gamma^n \forall n = 1, 2, \dots$
- (iii) For some $p \in [1, \infty)$, $(\phi_n^{(k)})_{n=0}^\infty$ converges in $L_p(\mathbb{R})$.

We end the paper with an example.

Example 8. Let a mask a be given by $\tilde{a}(z) = \frac{(1+z^3)^k}{2^{k-2}(1+z)}, k \geq 2$, where $\tilde{a}$, for a finitely supported sequence a , is defined by $\tilde{a}(z) = \sum_{\alpha \in \mathbb{Z}} a(\alpha)z^\alpha$. If we choose an initial function $\phi_0 = \underbrace{\chi_{[0,3]} * \chi_{[0,3]} * \dots * \chi_{[0,3]}}_{k \text{ functions}}$, then the sequence $(\phi_n^{(k-2)})_{n=1}^\infty$ is

bounded in $L_\infty(\mathbb{R})$ and converges a.e. on $\mathbb{R}$. Consequently, there exists a refinable function ϕ corresponding to mask a with $\phi^{(k-2)} \in L_\infty(\mathbb{R})$.

Proof. Let b be a finitely supported sequence defined by $\tilde{b}(z) = (1+z^3)^k/2^{k-1}$. Then ϕ_0 is a refinable function corresponding to b .

For any function g , it follows from $a(\alpha) + a(\alpha - 1) = 2b(\alpha) \forall \alpha \in \mathbb{Z}$ that

$$C_a g - C_a g(\cdot - 1) = 2(C_b g - C_b g(\cdot - 1/2)).$$

By induction we have

$$C_a^n g - C_a^n g(\cdot - 1) = 2^n (C_b^n g - C_b^n g(\cdot - 2^{-n})), \quad n = 1, 2, \dots$$

In particular we choose $g = \phi_0 = \underbrace{\chi_{[0,3]} * \chi_{[0,3]} * \dots * \chi_{[0,3]}}_{k \text{ functions}}$. Then $\phi_0 = C_b^n \phi_0$

and therefore $\phi_n = C_a^n \phi_0$ satisfies

$$\phi_n - \phi_n(\cdot - 1) = 2^n (\phi_0 - \phi_0(\cdot - 2^{-n})), \quad n = 1, 2, \dots$$

Since $\phi_0^{(k-1)} \in L_\infty(\mathbb{R})$, it follows from the last equality that

$$(13) \quad \|\phi_n^{(k-2)} - \phi_n^{(k-2)}(\cdot - 1)\|_{L_\infty(\mathbb{R})} \leq \|\phi_0^{(k-1)}\|_{L_\infty(\mathbb{R})} \quad \forall n = 1, 2, \dots$$

Moreover, for any $1 \leq p < \infty$, we have the following convergence in $L_p(\mathbb{R})$:

$$\phi_n^{(k-2)} - \phi_n^{(k-2)}(\cdot - 1) = 2^n(\phi_0^{(k-2)} - \phi_0^{(k-2)}(\cdot - 2^{-n})) \rightarrow \phi_0^{(k-1)}.$$

Let $E \subset \mathbb{R}$ be a compact set. For any f defined on $\mathbb{R}$ with $\text{supp } f \subseteq E$, we define a function $g = f - f(\cdot - 1)$. Then the functions $f(x + \alpha)$, $\alpha \in \mathbb{Z}$, may be represented in terms of $g(\cdot - \beta)$, $\beta \in \mathbb{Z}$, as $f(x + \alpha) = \sum_{\beta \geq 0} g(x + \alpha - \beta)$. The series converges due to the fact that g is a compactly supported function. By the equality

$$\|f\|_{L_p(\mathbb{R})}^p = \sum_{\alpha \in \mathbb{Z}} \int_0^1 |f(x + \alpha)|^p dx = \sum_{\alpha \in \mathbb{Z}} \int_0^1 \left| \sum_{\beta \geq 0} g(x + \alpha - \beta) \right|^p,$$

we can find a constant c , dependent of E and p , such that for any $f \in L_p(\mathbb{R})$, $1 \leq p < \infty$, with $\text{supp } f \subseteq E$

$$(14) \quad \|f\|_{L_p(\mathbb{R})} \leq c \|g\|_{L_p(\mathbb{R})}.$$

As is known, $(\phi_n^{(k-2)} - \phi_n^{(k-2)}(\cdot - 1))_{n=0}^\infty$ is a Cauchy sequence in $L_p(\mathbb{R})$, $1 \leq p < \infty$, so is $(\phi_n^{(k-2)})_{n=0}^\infty$ by (14).

Clearly, when $p = \infty$, there exists a constant c such that the inequality (14) holds for any f with $\text{supp } f \subseteq E$. Therefore, for $n = 1, 2, \dots$, $\|\phi_n^{(k-2)}\|_{L_\infty(\mathbb{R})} \leq c \|\phi_0^{(k-1)}\|_{L_\infty(\mathbb{R})}$ by (13) and (14). The conclusion of the example follows from Theorem 7. $\square$

It was proved in [11] by a different method that there exists a refinable function in $L_p(\mathbb{R})$, $1 \leq p < \infty$, corresponding to mask a when $k = 2$.

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