

DISINTEGRATION OF PROJECTIVE MEASURES

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ABSTRACT. In this paper, we study a class of quasi-invariant measures on paths generated by discrete dynamical systems. Our main result characterizes the subfamily of these measures which admit a certain disintegration. This is a disintegration with respect to a random walk Markov process which is indexed by the starting point of the paths. Our applications include wavelet constructions on Julia sets of rational maps on the Riemann sphere.

1. INTRODUCTION

We prove a general theorem about endomorphisms of measure spaces, but it is motivated by our desire to use wavelet-type time/scale-self similarity in operator theory, and more generally outside the traditional context of wavelet theory; see, e.g., [BrJo02]. Hence our present setup is chosen with these general applications in mind. What is typical is that the starting point is geometric, and that a Hilbert space is not given at the outset. The best choice of Hilbert space in turn is dictated by the applications at hand.

Since wavelet decompositions use a special case of a general disintegration process for measures via transition operators [Rue89, Wal01], we feel that it is of interest to offer a direct and self-contained statement and proof of a necessary and sufficient condition for disintegration.

In its general form, the subject of disintegration of measures has its roots in the pioneering work by David Ruelle [Rue89] on transfer operators. This viewpoint was continued by Peter Walters [Wal01] in ergodic theory, and in operator algebras by a series of papers by Jean Renault and co-workers, on the use of groupoids in representation theory and in dynamics; see [Ren03, KuRe03, AnRe01, Ren98, Ren80]. While our present setup and proofs are self contained, in contrast the groupoid approach is realized via representations of a certain non-abelian algebra of operators (a C^* -algebra); see also [BJO04]. Hence the other approaches use representations of non-abelian algebras, which in turn entail more restrictive assumptions on the endomorphism and on the transfer operator which is the starting point of the analysis. Jean Renault's approach is via a groupoid C^* -algebra [Ren80]. So in this approach, the first step is from endomorphism to groupoid.

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Our present approach is rather direct, starting with a given endomorphism of a measure space. Yet our result Theorem 3.4 below belongs to the same circle of ideas as that of a related disintegration theorem of Renault et al.; see, e.g., [KuRe03, Theorems 2.8 and 3.5]. However, turning to our present setup, there are new elements, e.g., path space measures: We begin with a construction of families of path space measures, and there is no natural groupoid which is represented on our path space X_∞ (see section 3 below). So Renault’s approach does not apply directly.

Our starting point is an endomorphism in a measure space X . One might add that it is then possible to get a corresponding representation of Renault’s groupoid built on X , or rather Renault’s groupoid C^* -algebra. With this, we note that our construction is then in fact a dilation of Renault’s representation. Having representations opens up ways of doing decomposition and disintegration; see also [BJO04]. This viewpoint further opens up some intriguing possibilities, and applications, which however we leave to a future paper.

2. ENDOMORPHISMS AND MEASURES

Let X be a compact Hausdorff space, and let $r : X \rightarrow X$ be a given finite-to-one continuous endomorphism, mapping X onto itself.

As an example, $r = r(z)$ could be a rational mapping of the Riemann sphere. In that case, we take X to be the corresponding Julia set, or X could be the state-space of a subshift associated to a $0 - 1$ matrix.

Due to work of Brolin [Bro] and Ruelle [Rue89], it is known that, for these examples, the dynamical system (X, r) carries a unique maximal entropy or minimal free energy measure μ . Specifically, for each point x in X , μ distributes the “mass” equally on the finite number of solutions y to $r(y) = x$. The measure μ is obtained by iterating this over the paths of inverse images under r .

In earlier papers, [BrJo05], [DuJo04a], [DuJo04b], we showed that this structure in fact admits a rich family of wavelet bases. This wavelet construction is on a Hilbert space which is different from, but analogous to, $L^2(\mathbb{R})$, and the more familiar case of multiwavelets on the real line.

These are the wavelets corresponding to scaling x to Nx by a fixed integer N , $N \geq 2$. In that case, $X = \mathbb{T}$, the circle in \mathbb{C} , and $r(z) = z^N$. So even in the “classical” case, there is a “unitary dilation” from $L^2(X)$ to $L^2(\mathbb{R})$. In this construction, the Haar measure on \mathbb{T} “turns” into the Lebesgue measure on \mathbb{R} . In our general construction, we pass from (X, r, μ) to a Hilbert space built on a system $(X_\infty, \hat{r}, \hat{\mu})$. Just as in the familiar case of $L^2(\mathbb{R})$, we shall be concerned with the interplay between the initial space and the dilated one. Our consideration of quasi-invariant measures is motivated by wavelet filters in the more familiar case of $L^2(\mathbb{R})$ wavelet analysis. See for example [Dau92] and [BrJo02]. We stress that it is the dilated Hilbert space which carries wavelet bases. These bases are in turn parametrized by a family of functions V on X , or on \mathbb{T} in the classical case.

Our work so far on examples shows that this viewpoint holds promise for a deeper understanding of the harmonic analysis of Julia sets, and of other iteration systems.

In these cases, the analogue of $L^2(\mathbb{R})$ involves quasi-invariant measures $\hat{\mu}$ on a space X_∞ , built from X in a way that closely follows the analogue of passing

from \mathbb{T} to \mathbb{R} in wavelet theory; see [DuJo04a] for details. But the induced measure $\hat{\mu}$ on X_∞ is now a path-space measure. The analogue of the unitary action of translations by the integers \mathbb{Z} on $L^2(\mathbb{R})$, or equivalently multiplication in the Fourier dual, will be described in general below. For the Julia examples, it is the operator of multiplication by z on $L^2(X, \mu)$, i.e., it is a normal operator. We get a corresponding covariant system on X_∞ , where multiplication by $f(z)$ is unitarily equivalent to multiplication by the composition $f(r(z))$. But this is now dilated onto the Hilbert space $L^2(X_\infty)$, and our dilated Hilbert space is defined from our path-space measures $\hat{\mu}$.

Hence all the issues from wavelet analysis of the mid-1980s, especially the pyramid algorithms, have analogues in this wider context. In other work, we use this to construct multiresolutions on the Julia sets and on other iteration systems, and our r -scale multiresolutions appear to reveal interesting spectral theory for Julia sets. Also, it seems further to offer a promising interplay between geometry and spectrum. None of this is yet well understood.

The purpose of this paper is to give an interconnection between the measures μ_0 on X and a family of V -quasi-invariant measures $\hat{\mu}$ on X_∞ . We characterize those V -quasi-invariant measures $\hat{\mu}$ on X_∞ which admit a path-space decomposition in terms of the initial measure μ_0 at the start of the paths. This is useful because it gives the V -quasi-invariant measure a more concrete form; see [DuJo04a].

We recall the definitions: If \mathfrak{B} is a sigma-algebra on a set X and $r : X \rightarrow X$ is a finite-to-one, onto and measurable map, we denote it by

$$X_\infty(r) := \{(x_0, x_1, \dots) \in \prod_{n \in \mathbb{N}_0} X \mid r(x_{n+1}) = x_n, \text{ for all } n \in \mathbb{N}_0\}.$$

We denote the projections by $\theta_n : X_\infty(r) \ni (x_0, x_1, \dots) \mapsto x_n \in X$. The union of the pull-backs $\theta_n^{-1}(\mathfrak{B})$ generates a sigma-algebra \mathfrak{B}_∞ . The map r extends to a measurable bijection $\hat{r} : X_\infty(r) \rightarrow X_\infty(r)$ defined by

$$\hat{r}(x_0, x_1, \dots) = (r(x_0), x_0, x_1, \dots).$$

Let $V : X \rightarrow [0, \infty)$ be a measurable, bounded function. We say that a measure μ_0 on (X, \mathfrak{B}) has the *fixed-point property* if

$$(2.1) \quad \int_X V f \circ r d\mu_0 = \int_X f d\mu_0, \quad (f \in L^\infty(X, \mu_0)).$$

We say that a measure $\hat{\mu}$ on $(X_\infty(r), \mathfrak{B}_\infty)$ is *V -quasi-invariant* if

$$d(\hat{\mu} \circ \hat{r}) = V \circ \theta_0 d\hat{\mu}.$$

We recall the result from [DuJo04b].

Theorem 2.1. *There exists a one-to-one correspondence between measures μ_0 on X with the fixed-point property and V -quasi-invariant measures $\hat{\mu}$ on $X_\infty(r)$, given by*

$$\mu_0 = \hat{\mu} \circ \theta_0^{-1}.$$

Our main result, Theorem 3.4 below, applies to a class of measures on some extended spaces X_∞ , generalized solenoids. Our setup is as follows: Start with (X, r, V, μ_0) where X is a measure space with an endomorphism r , V is a weight function as specified, and μ_0 is a V -strongly invariant measure. In wavelet applications (see [DuJo04b]), V may have the form $V = |m_0|^2$ where m_0 is a wavelet low-pass filter. In the dyadic case, $X = \mathbb{T}$ (the circle or the one-torus) and

$r : z \mapsto z^2$. Under suitable conditions on the so-called filter function m_0 , we show in [DuJo04b] that (X_∞, μ_∞) is measure theoretically equivalent to the real line $(\mathbb{R}, \text{Lebesgue measure})$, and so, in this case, the dilated Hilbert space is merely the familiar $L^2(\mathbb{R})$ of classical wavelet theory [Dau92]. In [DuJo04a], we show that the same idea applies to the case when the system (X, r, V, μ_0) is as follows: $X := X(r)$ is the Julia set of some fixed rational mapping $r(z)$ of one complex variable. (We also denote by r the restriction to X of the rational mapping $r(z)$.) Using a theorem of Brolin [Bro] about measures on Julia sets, we further describe in [DuJo04a] a class of weight functions V and measures μ_0 on X which satisfy the V -fixed point property considered in Theorem 3.1. We also describe in [DuJo04a] how our dilation construction then provides wavelet bases in the dilation Hilbert space $L^2(X_\infty, \mu_\infty)$. However, in our work in [DuJo04a, DuJo04b] some cases are left open. This limits our approach to dilation/wavelet constructions for dynamical systems. The reason is that at the time we lacked a general disintegration formula for the relevant measure spaces. It is the purpose of the present paper to remedy this, and to prove such a theorem, specifically Theorem 3.4 below. We now turn to the path-space measures which are used in our theorem.

The significance of our path-space measures $\{P_x \mid x \in X\}$ is that they support our disintegration from Theorem 3.4 below. This is further elaborated in [DuJo04a, DuJo04b]. In the wavelet case, starting with $V = |m_0|^2$, let P_x be the path-space measures from Proposition 3.3. We gave conditions in [DuJo04a] which imply that for a.e. x , P_x is supported on the union U of certain finite orbits for r . We outline how this union U is naturally in a bijective correspondence with a copy of the integers \mathbb{Z} . This turns out to be consistent with the natural way the group \mathbb{Z} acts by translation in wavelet theory of the Hilbert space $L^2(\mathbb{R})$.

Starting with V and the corresponding P_x , we then get an important function $h(x) := P_x(\mathbb{Z})$. It turns out to be a fundamental eigenfunction of wavelet theory. In fact the function h is known [BrJo02] to be a fundamental minimal eigenfunction for the wavelet transfer operator R_V (also called a Ruelle-Perron-Frobenius operator), studied first by W. Lawton in [Law90]. The function h determines a number of important properties of a wavelet basis; see, e.g., [Dau92, Chapter 6] and [BrJo02] for additional details on this point. We further show in [DuJo04b] that $R_V h = h$ holds, and further that h is an infinite-dimensional generalization of the familiar Perron-Frobenius right-eigenvector. This entails a suitable renormalization so that the Perron-Frobenius eigenvalue is 1. The measures μ_0 from Theorem 3.1 satisfying our V -fixed point property may then be viewed as generalizations of Perron-Frobenius left-eigenvectors. See also [BrJo02] and [Rue89] for details on infinite-dimensional Perron-Frobenius theory.

3. DISINTEGRATION

Let X be a non-empty set, \mathfrak{B} a sigma-algebra of subsets of X , and $r : X \rightarrow X$ an onto, finite-to-one, and \mathfrak{B} -measurable map.

We will assume in addition that we can label measurably the branches of the inverse of r . By this, we mean that the following conditions are satisfied:

$$(3.1) \quad \text{The map } \mathbf{c} : X \ni x \mapsto \#r^{-1}(x) < \infty \text{ is measurable.}$$

We denote

$$A_i := \{x \in X \mid \mathbf{c}(x) = \#r^{-1}(x) \geq i\} \quad (i \in \mathbb{N}).$$

Equation (3.1) implies that the sets A_i are measurable. Also they form a decreasing sequence and, since the map is finite to one,

$$X = \bigcup_{i=1}^{\infty} (A_{i+1} \setminus A_i).$$

Then, we assume that there exist measurable maps $\tau_i : A_i \rightarrow X$, $i \in \{1, 2, \dots\}$, such that

$$(3.2) \quad r^{-1}(x) = \{\tau_1(x), \dots, \tau_{c(x)}(x)\} \quad (x \in X),$$

$$(3.3) \quad \tau_i(A_i) \in \mathfrak{B}, \text{ for all } i \in \{1, 2, \dots\}.$$

Thus $\tau_1(x), \dots, \tau_{c(x)}(x)$ is a list without repetition of the “roots” of x , $r^{-1}(x)$.

From (3.2) we also obtain that

$$(3.4) \quad \tau_i(A_i) \cap \tau_j(A_j) = \emptyset, \text{ if } i \neq j,$$

and

$$(3.5) \quad \bigcup_{i=1}^{\infty} \tau_i(A_i) = X.$$

In the sequel, we will use the following notation: For a function $f : X \rightarrow \mathbb{C}$, we denote

$$f \circ \tau_i(x) := \begin{cases} f(\tau_i(x)), & \text{if } x \in A_i, \\ 0, & \text{if } x \in X \setminus A_i. \end{cases}$$

Theorem 3.1. *Let (X, \mathfrak{B}) and $r : X \rightarrow X$ be as above. Let $V : X \rightarrow [0, \infty)$ be a bounded \mathfrak{B} -measurable map. For a measure μ_0 on (X, \mathfrak{B}) , the following affirmations are equivalent:*

- (i) *The measure μ_0 has the fixed-point property (3.8).*
- (ii) *There exists a non-negative, \mathfrak{B} -measurable map Δ (depending on V and μ_0) such that*

$$(3.6) \quad \sum_{r(y)=x} \Delta(y) = 1, \text{ for } \mu_0\text{-a.e. } x \in X,$$

and

$$(3.7) \quad \int_X V f d\mu_0 = \int_X \sum_{r(y)=x} \Delta(y) f(y) d\mu_0(x), \text{ for all } f \in L^\infty(X, \mu_0).$$

Moreover, when the affirmations are true, Δ is unique up to $V d\mu_0$ -measure zero.

Proof. We first check the implication (ii) \Rightarrow (i). We have, for $f \in L^\infty(X, \mu_0)$,

$$\begin{aligned} \int_X V f \circ r d\mu_0 &= \int_X \sum_{r(y)=x} \Delta(y) f(r(y)) d\mu_0(x) \\ &= \int_X f(x) \sum_{r(y)=x} \Delta(y) d\mu_0(x) = \int_X f d\mu_0, \end{aligned}$$

and (2.1) follows.

For the implication (i) \Rightarrow (ii) we will need a lemma.

Lemma 3.2. *Let μ_0 be a measure on (X, \mathfrak{B}) that satisfies (2.1). Then, for each $f \in L^1(X, \mu_0)$ there exists a function which we denote by $R_{\mu_0}(Vf) \in L^1(X, \mu_0)$ such that*

$$(3.8) \quad \int_X Vf g \circ r d\mu_0 = \int_X R_{\mu_0}(Vf) g d\mu_0 \quad (g \in L^\infty(X, \mu_0)).$$

The operator $f \mapsto R_{\mu_0}(Vf)$ has the following properties:

$$(3.9) \quad R_{\mu_0}(V)(= R_{\mu_0}(V \cdot \mathbf{1})) = \mathbf{1}, \text{ pointwise } \mu_0\text{-a.e.},$$

$$(3.10) \quad R_{\mu_0}(Vfh \circ r) = R_{\mu_0}(Vf)h \quad (f, h \in L^\infty(X, \mu_0)),$$

$$(3.11) \quad R_{\mu_0}(Vf) \geq 0 \text{ if } f \geq 0.$$

Proof of Lemma 3.2. The positive functional

$$g \mapsto \int_X Vf g \circ r d\mu_0 \quad (g \in L^\infty(X, \mu_0))$$

defines a measure on (X, \mathfrak{B}) which is absolutely continuous with respect to μ_0 . Indeed, if $E \in \mathfrak{B}$ has $\mu_0(E) = 0$, then by (2.1)

$$\int_X V \chi_E \circ r d\mu_0 = \int_X \chi_E d\mu_0 = 0,$$

therefore $r^{-1}(E)$ has $V d\mu_0$ -measure zero so

$$\int_X Vf \chi_E \circ r d\mu_0 = 0.$$

Denoting by $R_{\mu_0}(Vf)$ the Radon-Nikodym derivative, the relation (3.8) is obtained.

Taking $f = \mathbf{1}$, we obtain, for all $g \in L^\infty(X, \mu_0)$,

$$\int_X R_{\mu_0}(V) g d\mu_0 = \int_X V g \circ r d\mu_0 = \int_X g d\mu_0,$$

where we used (2.1) in the last step. Thus (3.9) follows.

Next, for all $g \in L^\infty(X, \mu_0)$ we have

$$\int_X R_{\mu_0}(Vfh \circ r) g d\mu_0 = \int_X Vf h \circ r g \circ r d\mu_0 = \int_X R_{\mu_0}(Vf) h g d\mu_0,$$

and this implies (3.10).

Now take $f \geq 0$, $f \in L^1(X, \mu_0)$. For all $g \in L^\infty(X, \mu_0)$ with $g \geq 0$, we have

$$\int_X R_{\mu_0}(Vf) g d\mu_0 = \int_X Vf g \circ r d\mu_0 \geq 0,$$

and (3.11) follows. \square

We return to the proof of the theorem. We prove that there exist some non-negative, measurable function Δ on X , such that

$$(3.12) \quad (R_{\mu_0}(Vf))(x) = \sum_{i=1}^{\infty} \Delta(\tau_i(x)) f(\tau_i(x)), \text{ for } \mu_0\text{-a.e. } x \in X \quad (f \in L^1(X, \mu_0)).$$

Note that every $f \in L^1(X, \mu_0)$ can be written as

$$(3.13) \quad f = \sum_{i=1}^N (f \circ \tau_i) \circ r \chi_{\tau_i(A_i)}.$$

Indeed, if $x \in X$, then, by (3.5), $x \in \tau_i(A_i)$ for some $i \in \{1, 2, \dots\}$, and

$$x \in r^{-1}(r(x)) = \{\tau_1(r(x)), \dots, \tau_{\mathfrak{c}(r(x))}(r(x))\}.$$

Therefore $x = \tau_i(r(x))$ and (3.13) follows. Note also that, for all $x \in X$, the sum in (3.13) is actually finite. It has at most $\mathfrak{c}(x)$ terms.

The function $f \circ \tau_i$ is measurable because τ_i is.

Applying R_{μ_0} and using (3.10) we obtain that

$$(3.14) \quad R_{\mu_0}(Vf) = \sum_{i=1}^{\infty} R_{\mu_0}(V \chi_{\tau_i(A_i)}) f \circ \tau_i.$$

A little argument is needed for this, to deal with the infinite sum. If S_n are the partial sums $S_n := \sum_{i=1}^n \chi_{\tau_i(A_i)} f \circ \tau_i \circ r$, then they converge pointwise and dominate to the limit f . The definition of R_{μ_0} shows that $R_{\mu_0}(V S_n)$ converges weakly to $R_{\mu_0}(V f)$. However, the sum in $R_{\mu_0}(V f)$ is pointwise finite, because any $x \in X$ belongs to only finitely many A_i 's, so $f \circ \tau_i(x) = 0$, except for finitely many i 's. Hence (3.14) holds.

Then define

$$(3.15) \quad \Delta(\tau_i(x)) = (R_{\mu_0}(V \chi_{\tau_i(A_i)}))(x) \quad (x \in A_i).$$

Since $(\tau_i(X))_{i=1, \infty}$ form a partition of X , (3.15) defines the function Δ on X .

We claim that Δ is measurable. Take E to be a Borel subset of \mathbb{R} . Then

$$\begin{aligned} \Delta^{-1}(E) &= \bigcup_{i=1}^{\infty} (\Delta^{-1}(E) \cap \tau_i(A_i)) \\ &= \bigcup_{i=1}^{\infty} \left(\tau_i \left((\Delta \circ \tau_i)|_{A_i}^{-1}(E) \right) \right) \\ &= \bigcup_{i=1}^{\infty} \left(\tau_i \left((R_{\mu_0}(V \chi_{\tau_i(A_i)}))^{-1}(E) \right) \right) \\ &= \bigcup_{i=1}^{\infty} \left(r^{-1} \left((R_{\mu_0}(V \chi_{\tau_i(A_i)}))^{-1}(E) \right) \cap \tau_i(A_i) \right), \end{aligned}$$

and since the functions $R_{\mu_0}(V \chi_{\tau_i(A_i)})$ are measurable, it follows that Δ is too.

Now the equations (3.14) and (3.15) imply (3.12). Plugging (3.12) into (3.8) we obtain (3.7).

$$\sum_{i=1}^{\infty} \Delta \circ \tau_i = R_{\mu_0}(V \mathbf{1}) = 1,$$

where we used (3.9) in the last step. This proves (3.6).

To prove the uniqueness, let Δ' be another such function. Considering in (3.7) functions f supported on $\chi_{\tau_i(X)}$, we obtain that

$$\Delta(\tau_i(x)) = \Delta'(\tau_i(x)), \quad \text{for } \mu_0\text{-a.e. } x \in A_i.$$

But

$$\begin{aligned} \{x \in X \mid \Delta(x) \neq \Delta'(x)\} &= \bigcup_{i=1}^{\infty} \{x \in \tau_i(A_i) \mid \Delta(\tau_i(r(x))) \neq \Delta'(\tau_i(r(x)))\} \\ &= \bigcup_{i=1}^{\infty} (\tau_i(A_i) \cap r^{-1}(\{y \in X, \mid \Delta(\tau_i(y)) \neq \Delta'(\tau_i(y))\})). \end{aligned}$$

However, the argument at the beginning of the proof shows that if a set E has $\mu_0(E) = 0$, then $r^{-1}(E)$ has $V d\mu_0$ -measure 0. Thus each term of the union has $V d\mu_0$ -measure 0, and the theorem is proved. \square

3.1. Path measures. Each point in $(x_0, x_1, \dots) \in X_\infty(r)$ is determined by a point $x_0 \in X$ and a choice of a root $x_1 \in r^{-1}(x_0)$, a root $x_2 \in r^{-1}(x_1) \subset r^{-2}(x_0)$, and so on.

If the point x_0 is fixed, then to specify a point in $X_\infty(r)$ is to specify the choice of roots, or the choice of a path

$$x_0 \xleftarrow{r} x_1 \xleftarrow{r} \dots \xleftarrow{r} x_n \xleftarrow{r} x_{n+1} \xleftarrow{r} \dots$$

We will denote the set of these paths by

$$\Omega_{x_0} := \{(x_1, x_2, \dots) \mid (x_0, x_1, x_2, \dots) \in X_\infty(r)\}.$$

The set can be regarded as a subset of $X_\infty(r)$, namely the subset of points in $X_\infty(r)$ that have the first coordinate equal to x_0 . It can also be regarded as the projective limit of the following diagram:

$$r^{-1}(x_0) \xleftarrow{r} r^{-2}(x_0) \xleftarrow{r} \dots \xleftarrow{r} r^{-n}(x_0) \xleftarrow{r} r^{-(n+1)}(x_0) \xleftarrow{r} \dots$$

We can construct a Radon measure P_{x_0} on Ω_{x_0} using Δ to assign probabilities to the choices of roots.

Proposition 3.3. *Let (X, \mathfrak{B}) and $r : X \rightarrow X$ and be as above, and let $D : X \rightarrow [0, \infty)$ be a measurable function with the property that*

$$(3.16) \quad \sum_{r(y)=x} D(y) = 1.$$

Denote

$$(3.17) \quad D^{(n)} := D \cdot D \circ r \cdot \dots \cdot D \circ r^{n-1} \quad (n \in \mathbb{N}), \quad D^{(0)} := 1.$$

Then for each $x_0 \in X$, there exists a Radon probability measure P_{x_0} on Ω_{x_0} such that, if f is a bounded measurable function on Ω_{x_0} which depends only on the first n coordinates x_1, \dots, x_n , then

$$(3.18) \quad \int_{\Omega_{x_0}} f(\omega) dP_{x_0}(\omega) = \sum_{r^n(x_n)=x_0} D^{(n)}(x_n) f(x_1, \dots, x_n).$$

Proof. To check the consistency, take a measurable function f which depends only on the first n coordinates x_1, \dots, x_n . We regard it as a function which depends on the first $n+1$ coordinates and check that the two formulas in (3.18) (one for n , one

for $n + 1$) give the same result:

$$\begin{aligned} \int_{\Omega_{x_0}} f(\omega) dP_{x_0}(\omega) &= \sum_{r^{n+1}(x_{n+1})=x_0} D^{(n+1)}(x_{n+1})f(x_1, \dots, x_{n+1}) \\ &= \sum_{r^n(x_n)=x_0} f(x_1, \dots, x_n)D^{(n)}(x_n) \sum_{r(x_{n+1})=x_n} D(x_{n+1}) \\ &= \sum_{r^n(x_n)=x_0} D^{(n)}(x_n)f(x_1, \dots, x_n), \end{aligned}$$

so P_{x_0} is well defined. Since the functions which depend only on finitely many coordinates form a dense subalgebra of bounded functions (according to the Stone-Weierstrass theorem), we can use Riesz' theorem to obtain the probability measure P_{x_0} on Ω_{x_0} . The application of Stone-Weierstrass is justified because when the point x is fixed, we may use Tychonoff-compactness of the usual infinite product $\Omega := \mathbb{Z}_N \times \mathbb{Z}_N \times \dots$. But since we may have a variable number of preimages, we must first construct a dense subalgebra of bounded functions, and then the measures (P_x) by using the structure of the measure space $X_\infty(r)$ defined above. \square

3.2. Main theorem. The next theorem shows that the V -quasi-invariant measure $\hat{\mu}$ on $X_\infty(r)$ associated to μ_0 can be disintegrated through the measures P_{x_0} associated to the function Δ .

Theorem 3.4. *Let (X, \mathfrak{B}) . $r : X \rightarrow X$ and $V : X \rightarrow [0, \infty)$ be as above. Let μ_0 be a measure on (X, \mathfrak{B}) with the fixed point property (2.1). Let $\Delta : X \rightarrow [0, 1]$ be the function associated to V and μ_0 as in Theorem 3.1, and let $\hat{\mu}$ be the unique V -quasi-invariant measure on $X_\infty(r)$ with*

$$\hat{\mu} \circ \theta_0^{-1} = \mu_0,$$

as in Theorem 2.1. For Δ , we define the measures P_{x_0} as in Proposition 3.3. Then, for all bounded measurable functions f on $X_\infty(r)$,

$$(3.19) \quad \int_{X_\infty(r)} f d\hat{\mu} = \int_X \int_{\Omega_{x_0}} f(x_0, \omega) dP_{x_0}(\omega) d\mu_0(x_0).$$

Proof. By density, it is enough to prove the theorem for functions f on $X_\infty(r)$ which depend only on finitely many coordinates. So, assume $f = g \circ \theta_n$, with g bounded and \mathfrak{B} measurable. Then $g \circ \theta_n$ will depend only on x_0 and x_1, \dots, x_n ,

$$g \circ \theta_n(x_0, x_1, \dots, x_n) = g(x_n).$$

From [DuJo04b, Theorem 3.3], we recall that if $\mu_n := \hat{\mu} \circ \theta_n^{-1}$, then $d\mu_n = V^{(n)} d\mu_0$. Thus

$$\int_{X_\infty(r)} g \circ \theta_n d\hat{\mu} = \int_X g d\mu_n = \int_X V^{(n)} g d\mu_0.$$

Applying Lemma 3.2, we obtain by induction that

$$\begin{aligned} \int_X V^{(n)} g d\mu_0 &= \int_X V g V^{(n-1)} \circ r d\mu_0 = \int_X R_{\mu_0}(V g) V^{(n-1)} d\mu_0 \\ &= \dots = \int_X R_{\mu_0}^n(V g) d\mu_0. \end{aligned}$$

Also, using induction on (3.12), we obtain that

$$\begin{aligned} (R_{\mu_0}^n(Vg))(x_0) &= \sum_{r^n(x_n)=x_0} \Delta^{(n)}(x_n)g(x_n) \\ &= \int_{\Omega_{x_0}} (g \circ \theta_n)(x_0, \omega) dP_{x_0}(\omega). \end{aligned}$$

Combining these equalities we obtain (3.19). \square

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