

IRREDUCIBLE CHARACTERS WHICH ARE ZERO ON ONLY ONE CONJUGACY CLASS

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(Communicated by Jonathan I. Hall)

ABSTRACT. Suppose that G is a finite solvable group which has an irreducible character χ which vanishes on exactly one conjugacy class. Then we show that G has a homomorphic image which is a nontrivial 2-transitive permutation group. The latter groups have been classified by Huppert. We can also say more about the structure of G depending on whether χ is primitive or not.

1. INTRODUCTION

Let χ be an irreducible character of a finite group G . A well-known theorem of Burnside [9, page 40] shows that when χ is nonlinear it takes the value 0 on at least one conjugacy class of G . Groups having an irreducible character that vanishes on exactly one class were studied by Zhmud' in [11] (see also [1]). Chillag [2, Lemma 2.4] has proved that if the restriction of χ to the derived group G' is reducible and χ vanishes on exactly one class of G , then G is a Frobenius group with a complement of order 2 and an abelian odd-order kernel.

Our purpose in this paper is to show that, if an irreducible character χ of a finite solvable group G vanishes on exactly one conjugacy class, then G has a homomorphic image which is a nontrivial 2-transitive permutation group. The latter groups have been classified by Huppert: they have degree p^d where p is prime, and are subgroups of the extended affine group $A\Gamma L(1, p^d)$ except for six exceptional degrees (see Remark 8 below). We can also say more about the structure of G depending on whether χ is primitive or not.

2. MAIN RESULTS

We shall initially assume that our character is faithful, and make the following assumptions:

- (*) G is a finite group with a faithful irreducible character χ which is 0 on only one class which we denote by \mathcal{C} . Furthermore, G has a chief factor K/L which is an elementary abelian p -group of order p^d such that the restriction χ_K is irreducible, but χ_L is not.

Since χ must be nonlinear, the latter condition clearly holds whenever G is solvable, but for the present we shall not assume solvability.

Received by the editors August 4, 2005.

2000 *Mathematics Subject Classification*. Primary 20C15, 20D10, 20B20.

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Proposition 1. *Suppose $(*)$ holds. Then $\mathcal{C} = K \setminus L$, $K = \langle \mathcal{C} \rangle$ and L is equal to $L_0 := \{u \in G \mid u\mathcal{C} = \mathcal{C}\}$. In particular, \mathcal{C} consists of p -elements (since L does not contain a Sylow p -subgroup of K). Moreover, either:*

- (i) d is even, $\chi_L = p^{d/2}\phi$ for some nontrivial linear character ϕ of L , and $L = Z(G)$; or
- (ii) $\chi_L = \phi_1 + \dots + \phi_{p^d}$ is the sum of p^d distinct G -conjugate irreducible characters of L , and so χ is imprimitive and $p^d \mid \chi(1)$.

Proof. Since χ_K is irreducible, the theorem of Burnside quoted above shows that $\mathcal{C} \subseteq K$. On the other hand, since χ_L is reducible, [1, Theorem 21.1] shows that $\mathcal{C} \cap L = \emptyset$. Hence $\mathcal{C} \subseteq K \setminus L$.

Now since K/L is an abelian chief factor, and χ_K is irreducible, it follows from [9, (6.18)] that either (i) d is even, $\chi_L = p^{d/2}\phi$ for some $\phi \in \text{Irr}(L)$; or (ii) $\chi_L = \phi_1 + \dots + \phi_{p^d}$ is the sum of p^d distinct G -conjugate irreducible characters ϕ_i . We shall consider these two cases separately.

In case (i) we note that, since $\mathcal{C} \cap L = \emptyset$, the irreducible character ϕ does not take the value 0. Thus Burnside's theorem implies that $\phi(1) = 1$. This implies that if ρ is a representation affording χ , then ρ is scalar on L . Since χ is assumed to be faithful, L is contained in the centre $Z(G)$ of G . On the other hand, for each $z \in Z(G)$, $\rho(z)$ is a scalar of the form $\zeta 1$. Thus for each $x \in \mathcal{C}$ we have $\chi(zx) = \text{trace } \rho(zx) = \text{trace } \zeta \rho(x) = \zeta \chi(x) = 0$ and so $zx \in \mathcal{C}$. Therefore $z = (zx)x^{-1} \in \langle \mathcal{C} \rangle \leq K$ for all $z \in Z(G)$. This shows that $Z(G)$ is a normal subgroup of G satisfying $L \leq Z(G) \leq K$. Since K/L is a chief factor and χ_K is a nonlinear irreducible character of K , we conclude that $Z(G) = L$. Finally, since $K/Z(G)$ is abelian, [9, (2.30)] shows that $K \setminus L = \mathcal{C}$.

In case (ii) χ_K is an irreducible constituent of $(\phi_1)^K$ and so comparison of degrees shows that $\chi_K = (\phi_1)^K$. Thus χ_K is 0 everywhere outside of the normal subgroup L , and so $K \setminus L = \mathcal{C}$ in this case as well.

Finally since $|\mathcal{C} \cup \{1\}| > \frac{1}{2}|K|$, then $K = \langle \mathcal{C} \rangle$. Finally, it is easily seen that L_0 is a normal subgroup of G , and that $L_0 \subseteq \mathcal{C}\mathcal{C}^{-1}$ and so $L_0 \leq K$. Since $\mathcal{C} = K \setminus L$ is a union of cosets of L , we see that $L \leq L_0$. On the other hand, $\mathcal{C} \not\subseteq L_0$ since \mathcal{C} is not a subgroup. Therefore $L_0 \triangleleft G$ and $L \leq L_0 < K$; hence $L_0 = L$ as claimed. \square

Corollary 2. *Under the hypothesis $(*)$ every normal subgroup N of G either contains K (when χ_N is irreducible) or is contained in L (when χ_N is reducible). In particular, K/L is the unique chief factor such that χ_K is irreducible and χ_L is reducible and K/L is the socle of G/L . Since K has a nonlinear irreducible character, K is not abelian and so $L \neq 1$.*

Remark 3. Both cases (i) and (ii) in Proposition 1 can actually occur. The group $\text{SL}(2, 3)$ has three primitive characters of degree 2 which satisfy $(*)$ (case (i) with $|K| = 8$ and $|L| = 2$ for each character), and S_4 has an imprimitive character of degree 3 which satisfies $(*)$ (case (ii) with $|K| = 12$ and $|L| = 4$).

Proposition 4. *Suppose that the hypothesis $(*)$ and case (i) of Proposition 1 hold. Then $L = Z(G)$ has order p , K is an extra special p -group and χ is primitive.*

Proof. Let $z \in L$. Then for any $x \in \mathcal{C}$ we have $zx \in \mathcal{C}$ and so $zx = y^{-1}xy$ for some $y \in G$. Since K/L is an elementary abelian p -group, $z^p x^p = (zx)^p = y^{-1}x^p y = x^p$, and so $z^p = 1$. Thus L is of exponent p . Since $L = Z(G)$ is represented faithfully as a group of scalar matrices by a representation affording χ , it follows that L is

cyclic and hence $|L| = p$. Because K is nonabelian, $K' = \Phi(K) = L = Z(K)$ and so K is an extraspecial p -group.

We finally show that χ is primitive. Indeed, otherwise there is a maximal subgroup H in G and $\psi \in \text{Irr}(H)$ such that $\chi = \psi^G$. The formula for an induced character shows that ψ^G is 0 on each conjugacy class disjoint from H . As is well known every proper subgroup of a finite group is disjoint from some conjugacy class, and so we conclude that \mathcal{C} is the unique class such that $\mathcal{C} \cap H = \emptyset$. By Proposition 1 this implies that $H \cap K \leq L$. Thus $K \not\leq H$, and so $G = HK$ by the maximality of H . Hence

$$\chi(1) = \psi^G(1) \geq |G : H| = |K : H \cap K| \geq |K : L| = p^d.$$

Since $\chi(1) = p^{d/2}$, we obtain a contradiction. Thus χ is primitive. \square

Proposition 5. *Suppose that the hypothesis $(*)$ and case (ii) of Proposition 1 hold (so χ is imprimitive). Then there exists a subgroup M of index p^d in G such that $\chi = \psi^G$ for some $\psi \in \text{Irr}(M)$, $G = MK$ and $M \cap K = L = \text{core}_G(M)$.*

Proof. As noted in the proof of Proposition 1 χ_L is a sum of p^d distinct irreducible constituents ϕ_i . Because χ_K is irreducible, these constituents are K -conjugates (as well as G -conjugates). Let $M := I_G(\phi_1)$ be the inertial subgroup fixing the constituent ϕ_1 . Then $|G : M| = p^d$ and $G = MK$ because K acts transitively on the set of ϕ_i . Clearly $L \leq M$. Since $|K : L| = p^d = |G : M| = |K : M \cap K|$, we conclude that $M \cap K = L$. On the other hand, since ψ^G is 0 on any class which does not intersect M , the hypothesis on χ shows that $\mathcal{C} = G \setminus \bigcup_{y \in G} y^{-1}My$. Now $u \in \text{core}_G(M) = \bigcap_{y \in G} y^{-1}My$ and $x \in \mathcal{C}$ implies that ux does not lie in any $y^{-1}My$, and hence $ux \in \mathcal{C}$. Thus with the notation of Proposition 1, $\text{core}_G(M) \leq L_0 = L$. Since L is a normal subgroup contained in M , the reverse inequality is also true, and so $\text{core}_G(M) = L$. \square

The proof of the next result requires a theorem of Isaacs [10, Theorem 2] which states:

Let H be a finite group with centre Z and let K be a normal subgroup of H with $Z = Z(K)$. Suppose that H centralizes K/Z and $|\text{Hom}(K/Z, Z)| \leq |K/Z|$. Then $H/Z = K/Z \times C_H(K)/Z$.

Proposition 6. *Under the hypothesis $(*)$ the centralizer $C_G(K/L)$ equals K .*

Proof. If χ is primitive, then Proposition 4 shows that the hypotheses of Isaacs' theorem are satisfied for $H := C_G(K/L)$ (the condition $|\text{Hom}(K/Z, Z)| \leq |K/Z|$ is trivial since the irreducibility of χ_H implies that Z is cyclic). Also, since χ_K is irreducible, $C_G(K) = Z(G) = L$, and so Isaacs' theorem shows that $H/L = K/L \times C_H(K)/L = K/L$, as required.

If χ is imprimitive, then using the notation of Proposition 5 we can show that $M \cap H = L$, where $H := C_G(K/L)$. Indeed, it is clear from Proposition 5 that $L \leq M \cap H$. To prove the reverse inequality suppose that $u \in M \cap H$. Then for each $x \in K$ we have $xu = yux$ for some $y \in L$. Choose i such that $\phi_1^x = \phi_i$. Then $\phi_i^u = \phi_1^{xu} = \phi_1^{yux} = \phi_1^x = \phi_i$. Hence u fixes ϕ_1^x and therefore lies in $x^{-1}Mx$. Since this is true for all $x \in K$, it follows from Proposition 5 that $u \in \text{core}_G(M) = L$. Thus $M \cap H = L$ as claimed. Finally $H = H \cap MK = (H \cap M)K = LK = K$, as required. \square

Corollary 7. *Under the hypothesis (*) G acts transitively by conjugation on the nontrivial elements of the vector space K/L , and the kernel of this action is K . Thus G/K is isomorphic to a subgroup of $GL(d, p)$ which is transitive on the nonzero elements of the underlying vector space.*

Remark 8. Huppert [8, Chapter XII Theorem 7.3] has classified all solvable subgroups S of $GL(d, p)$ which are transitive on the nonzero vectors of the underlying vector space. Apart from six exceptional cases (where $p^d = 3^2, 5^2, 7^2, 11^2, 23^2$ or 3^4), the underlying vector space can be identified with the Galois field $GF(p^d)$ in such a way that S is a subgroup of the group $\Gamma L(1, p^d)$ consisting of all transformations of the form $\xi \mapsto \alpha \xi^t$, where α is a nonzero element of the field and t is an automorphism of the field. The group $\Gamma L(1, p^d)$ is metacyclic of order $(p^d - 1)d$. A classification for nonsolvable groups has been carried out by Hering [5], [6]. It is considerably more complicated to state and prove, but among other things it shows that such groups have only a single nonsolvable composition factor (a summary is given in [8, page 386]).

Since the latter half of hypothesis (*) is certainly satisfied in a solvable group, we can specialize to solvable groups and drop the condition that χ is faithful in order to obtain the following theorem.

Theorem 9. *Let G be a finite solvable group which has an irreducible character χ which takes the value 0 on only one conjugacy class \mathcal{C} . Let $K := \langle \mathcal{C} \rangle$. Then:*

- (a) *$K = G^{(k)}$ for some $k \geq 0$.*
- (b) *There is a unique normal subgroup L of G such that K/L is a chief factor of G and $K \setminus L = \mathcal{C}$ (we set $|K : L| = p^d$).*
- (c) *G/K acts transitively on the set $(K/L)^\#$ of nontrivial elements of the vector space K/L and so is one of the groups classified by Huppert.*
- (d) *If χ is primitive, then $K/\ker \chi$ is an extra special group of order p^{d+1} with centre $L/\ker \chi$.*
- (e) *If χ is imprimitive, then G/L is a 2-transitive Frobenius group of degree p^d .*

Remark 10. We also note that (c) and Huppert's classification show that the integer k in (a) is bounded. Indeed, since $\Gamma L(1, p^d)$ is metacyclic, $k = 1$ or 2 except in the six exceptional cases. Computations using GAP [4] show that in the remaining cases $k \leq 4$.

Proof. (a) Let k be the largest integer such that $K \leq G^{(k)}$. By Corollary 2 we know that the restriction $\chi_{G^{(k+1)}}$ is reducible, and so $G^{(k+1)} \leq L$. Therefore $G^{(k+1)} \leq L < K \leq G^{(k)}$, and so $G^{(k)} \leq C_G(K/L)$. Hence $K = G^{(k)}$ by Proposition 6.

(b), (c) and (d) follow from Proposition 1, Corollary 7 and Proposition 4.

(e) Let M be the subgroup defined in Proposition 5. Since χ is induced from a character on M , its restriction χ_M must be reducible, and so [1, page 145] shows that

$$2 \leq [\chi_M, \chi_M] \leq 1 + \frac{|\mathcal{C} \setminus M|}{|M|} = 1 + \frac{|\mathcal{C}|}{|M|}.$$

Hence $|\mathcal{C}| \geq |M|$. Since $G = MK$ and G/K acts transitively on $(K/L)^\#$ we conclude using Proposition 5 that

$$\frac{|K|}{|L|} - 1 = p^d - 1 \leq |G : K| = |M : M \cap K| = |M : L| \leq \frac{|\mathcal{C}|}{|L|}.$$

However, $|\mathcal{C}| / |L| = p^d - 1$ by Proposition 1, so equality must hold throughout. Thus $|M : L| = p^d - 1$. Hence M/L acts regularly on $(K/L)^\#$ and so $G/L = (M/L)(K/L)$ is a 2-transitive Frobenius group. \square

Remark 11. Not all groups having an irreducible character which takes 0 on a single conjugacy class satisfy the second half of hypothesis (*). For example, the Atlas [3] shows that A_5 has three characters with this property and its central cover $2 \cdot A_5$ also has three. The group $L_2(7)$ has two characters with the required property, and each of the groups $L_2(2^k)$ ($k = 3, 4, \dots$) appears to have one such character (of degree 2^k). It would be interesting to know if these were the only simple groups with this property, or whether a group with such a character can have more than one nonabelian composition factor (see Remark 8). Another question which can be asked is what can be said about the kernel of such a character; evidently this kernel is contained in the normal subgroup $L_0 := \{u \in G \mid u\mathcal{C} = \mathcal{C}\}$.

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