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EQUIVARIANT DEFORMATIONS OF LEBRUN'S SELF-DUAL METRICS WITH TORUS ACTION

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ABSTRACT. We investigate $U(1)$-equivariant deformations of C. LeBrun's self-dual metric with torus action. We explicitly determine all $U(1)$-subgroups of the torus for which one can obtain $U(1)$-equivariant deformations that do not preserve the whole of the torus action. This gives many new self-dual metrics with $U(1)$-action which are not conformally isometric to LeBrun metrics. We also count the dimension of the moduli space of self-dual metrics with $U(1)$-action obtained in this way.

INTRODUCTION

In [4] C. LeBrun explicitly constructed a family of self-dual metrics on $n\mathbf{CP}^2$, the connected sum of n copies of complex projective planes, where n is an arbitrary positive integer. His construction starts by giving distinct n points on the upper half-space $\mathscr{H}^3$ with the usual hyperbolic metric. Once these n points are given, everything proceeds in a canonical way. Namely a principal $U(1)$-bundle over the punctured $\mathscr{H}^3$ together with a connection is canonically constructed, and then on the total space of this $U(1)$-bundle a self-dual metric is naturally and explicitly introduced, for which the $U(1)$-action becomes isometric. Then finally by choosing an appropriate conformal gauge (which is also concretely given), the self-dual metric is shown to extend to a compactification, yielding the desired self-dual metric on $n\mathbf{CP}^2$. Thus LeBrun metrics on $n\mathbf{CP}^2$ are naturally parametrized by the set of different n points on $\mathscr{H}^3$.

If the n points are located in a general position, the corresponding LeBrun metric admits only a $U(1)$- isometry (coming from the principal bundle structure). However, when the n points are put in a collinear position, meaning that the n points lie on the same geodesic on the hyperbolic $\mathscr{H}^3$, then the rotations around the geodesic can be lifted to the total space and it gives another $U(1)$-isometry of the LeBrun metric. We call this kind of self-dual metric on $n\mathbf{CP}^2$ the LeBrun metric with torus action. By a characterization theorem of LeBrun [5], the LeBrun metric with torus action is preserved under deformation keeping the torus action.

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In this note, following a suggestion of LeBrun [5, p. 123, Remark], we investigate $U(1)$-isometric deformation of LeBrun metrics with torus action, where $U(1)$ is a subgroup of the torus. In particular, *we determine all $U(1)$-subgroups of the torus for which one can obtain a $U(1)$-equivariant deformation, such that full torus symmetry does not survive.* Note that on $2\mathbf{CP}^2$ every self-dual metric of a positive scalar curvature is LeBrun metric with torus action [8] and such a subgroup in this problem cannot exist for $n = 2$ (and also for $n = 1$). Of course, LeBrun's original $U(1)$-subgroup (coming from the principal bundle structure), which acts semi-freely on $n\mathbf{CP}^2$, has the desired property for $n \geq 3$. We show that involving this subgroup, *there are precisely n numbers of $U(1)$-subgroups for which there exists the required equivariant deformation.* We give these subgroups concretely and observe that the remaining $(n-1)$ subgroups give non-LeBrun self-dual metric. Also, we count the dimension of the moduli space of the resulting family of self-dual metrics with a non-semi-free $U(1)$-isometry. Finally, we discuss some examples.

1. Computation of the torus action on a cohomology group

1.1. Our proof of the main result is via twistor space. So let Z be the twistor space of a LeBrun metric with torus action on $n\mathbf{CP}^2$. In order to investigate $U(1)$-equivariant deformations of this metric, we calculate the torus action on the cohomology group $H^1(\Theta_Z)$ which is relevant to deformation of complex structure of Z. In this subsection, to this end, we recall the explicit construction of Z due to LeBrun [4]. We need to be careful in resolving singularities of a projective model of the twistor space, since in [4] it is assumed that the semi-free $U(1)$-action does not extend to torus action, and since, under the existence of torus action, there are $n!$ possible ways of (small) resolutions, and most of them do not yield a twistor space

First let $Q = \mathbf{CP}^1 \times \mathbf{CP}^1$ be a quadratic surface and $\mathscr{E} \to Q$ a rank-3 vector bundle

$$\mathscr{E} = \mathscr{O}(n-1, 1) \oplus \mathscr{O}(1, n-1) \oplus \mathscr{O} \to Q,$$

where $\mathscr{O}(k, l)$ denotes the line bundle over Q whose bidegree is (k, l). Let (ξ_0, ξ_1) (resp. (η_0, η_1)) be a homogeneous coordinate on the first (resp. the second) factor of Q, and set $U_0 = \{(\xi_0, \xi_1) \,|\, \xi_0 \neq 0\}, V_0 = \{(\eta_0, \eta_1) \,|\, \eta_0 \neq 0\}$. On U_0 (resp. V_0) we use a non-homogeneous coordinate $u = \xi_1/\xi_0$ (resp. $v = \eta_1/\eta_0$). We choose a trivialization of $\mathscr{E}$ over $U_0 \times V_0 \subset Q$, and let (x, y, z) be the resulting fiber coordinate on $\mathscr{E}|_{U_0 \times V_0}$. Thus on the total space of $\mathscr{E}|_{U_0 \times V_0}$ we can use (u, v, x, y, z) as a global coordinate. Then let X be a compact (or complete) algebraic variety in $\mathbf{P}(\mathscr{E})$ defined by

$$xy = z^2 \prod_{i=1}^{n} (v - a_i u), \tag{1.1}$$

where $a_1, a_2, \cdots, a_n$ are positive real numbers satisfying $a_1 < a_2 < \cdots < a_n$. ((1.1) is an equation on $\mathbf{P}(\mathscr{E}|_{U_0 \times V_0})$, but it can be naturally compactified in $\mathbf{P}(\mathscr{E})$.) X has an obvious conic bundle structure over Q whose discriminant locus is $C_1 \cup C_2 \cup \cdots \cup C_n$, where C_i is a $(1, 1)$-curve in Q defined by $v = a_i u$. Further, the point $(x, y, z) = (0, 0, 1) \in \mathbf{P}(\mathscr{E})$ lying over the fiber over the point $(u, v) = (0, 0)$ is called a compound A_{n-1}-singularity of X. Similarly, by the choice of the degree of the direct summand in $\mathscr{E}$, the point $(0, 0, 1) \in \mathbf{P}(\mathscr{E})$ over $(u, v) = (\infty, \infty)$ is also a compound A_{n-1}-singularity of X. We denote these two singularities of X by p_0 and $\overline{p}_0$. These are all the singularities of X.

We have to define a real structure. In terms of the above coordinate (u, v, x, y, z) on $\mathbf{P}(\mathscr{E}|_{U_0 \times V_0})$ it is defined by

$$\sigma : (u, v\,; x, y, z) \mapsto \left(\frac{1}{\overline{v}}, \frac{1}{\overline{u}}\,; \frac{\overline{y}}{\overline{u}^{n-1}\overline{v}}, \frac{\overline{x}}{\overline{u}\,\overline{v}^{n-1}}, \overline{z}\right), \tag{1.2}$$

which preserves X, and interchanges the two singular points p_0 and $\overline{p}_0$ of X.

Next we give a small resolution of p_0. To give it explicitly we write $\tilde{x} = x/z$ and $\tilde{y} = y/z$. Then in an affine neighborhood of p_0 in $\mathbf{P}(\mathscr{E})$, X is defined by $\tilde{x}\tilde{y} = \prod_{i=1}^{n}(v - a_i u)$. The small resolution of p_0 is a composition of $(n-1)$ blowing-ups, where the center is 2-dimensional in each step. As the first step we take a blow-up of X along $\tilde{x} = v - a_1 u = 0$, yielding a new space X_1 and a morphism $X_1 \to X$. Since this center is contained in X, the exceptional locus E_1 arises only over p_0 and it is isomorphic to $\mathbf{CP}^1$. Introducing a new coordinate $\tilde{x}_1$ by $\tilde{x} = \tilde{x}_1(v - a_1 u)$ on E_1, the new space X_1 is locally defined by $\tilde{x}_1\tilde{y} = \prod_{i \geq 2}(v - a_i u)$, thereby having a compound A_{n-2}-singularity at the origin. The second step is to blow-up X_1 along $\tilde{x}_1 = v - a_2 u = 0$, giving a new space X_2 with a compound A_{n-3}-singularity at the new origin. After repeating this process $(n-1)$ times, the singularity p_0 is resolved, and the exceptional locus is a string of $(n-1)$ smooth rational curves. This is how to obtain a small resolution of p_0. Once a resolution of p_0 is given, another singularity $\overline{p}_0$ is naturally resolved by reality. Let $Y \to X$ be the small resolution of p_0 and $\overline{p}_0$ obtained in this way. (Y is non-singular.)

Obviously, other small resolutions of p_0 can be obtained for each permutation of n letters $\{1, 2, \cdots, n\}$. But keeping in mind that we have assumed $a_1 < a_2 < \cdots < a_n$ and that the curve $x = y = v - a_i u$ $(1 \leq i \leq n)$, which is over a discriminant locus $C_i \subset Q$, has to be a twistor line over the isolated fixed point of the torus action on $n\mathbf{CP}^2$, it is easily seen that if we take the resolution associated to a permutation other than $\{1, 2, \cdots, n-1, n\}$ (giving the small resolution above) and $\{n, n-1, \cdots, 2, 1\}$, then the resulting space does not become a twistor space even after the blowing-down process, explained below.

Next we explain the final step for obtaining the twistor space. The conic bundle $X \to Q$ has two distinct sections $E = \{x = z = 0\}$ and $\overline{E} = \{y = z = 0\}$, which are conjugate of each other. These sections are disjoint from p_0 and $\overline{p}_0$, and their normal bundles in X are $\mathscr{O}(-1, 1-n)$ and $\mathscr{O}(1-n, -1)$, respectively. Clearly the small resolution $Y \to X$ does not have any effect around E and $\overline{E}$, so that it does not change the normal bundles. Hence (if $n > 2$) both E and $\overline{E}$ (considered as divisors on Y) can be naturally contracted to $\mathbf{CP}^1$ along mutually different directions. Let $\mu : Y \to Z$ be this contraction and put $C_0 = \mu(E)$, $\overline{C}_0 = \mu(\overline{E})$. Then the normal bundle of C_0 and $\overline{C}_0$ in Z is $\mathscr{O}(1-n)^{\oplus 2}$. This Z is the twistor space of a LeBrun metric with torus action.

Finally a $\mathbf{C}^* \times \mathbf{C}^*$-action on the twistor space Z has to be introduced. On $\mathbf{P}(\mathscr{E})$ it is explicitly given by

$$(u, v, x, y, z) \mapsto (su, sv, tx, s^n t^{-1} y, z), \quad (s, t) \in \mathbf{C}^* \times \mathbf{C}^*, \tag{1.3}$$

which preserves X and fixes p_0 and $\overline{p}_0$. When restricted to $U(1) \times U(1)$ this action commutes with the real structure (1.2).

1.2. In the sequel we write $G = \mathbf{C}^* \times \mathbf{C}^* = \{(s, t)\}$ for simplicity. To calculate G-action on $H^1(\Theta_Z)$, we introduce various G-equivariant exact sequences related to this cohomology group. Our calculation in this subsection is similar to that of LeBrun in [6] with some simplifications. We note that the dimensions of the cohomology groups $H^i(\Theta_Z)$ are different from LeBrun's case in [6] for $i = 0, 1$.

Let $\pi : Y \to Q$ be the projection which is the composition of the small resolution $Y \to X$ and the projection $X \to Q$. We have the following exact sequence of sheaves of $\mathscr{O}_Y$-modules:

$$0 \longrightarrow \Theta_{Y/Q} \longrightarrow \Theta_Y \longrightarrow \pi^*\Theta_Q \longrightarrow \mathscr{G} \longrightarrow 0, \tag{1.4}$$

where $\Theta_{Y/Q}$ and $\mathscr{G}$ denote the kernel and the cokernel of the natural homomorphism $\Theta_Y \longrightarrow \pi^*\Theta_Q$, respectively. We decompose (1.4) into the following two short exact sequences:

$$0 \longrightarrow \Theta_{Y/Q} \longrightarrow \Theta_Y \longrightarrow \mathscr{F} \longrightarrow 0, \tag{1.5}$$

$$0 \longrightarrow \mathscr{F} \longrightarrow \pi^*\Theta_Q \longrightarrow \mathscr{G} \longrightarrow 0, \tag{1.6}$$

where $\mathscr{F}$ denotes the image sheaf of $\Theta_Y \longrightarrow \pi^*\Theta_Q$. On the other hand we have a natural isomorphism $\Theta_{Y/Q} \simeq \mathscr{O}_Y(E + \overline{E})$ and an exact sequence

$$0 \longrightarrow \mathscr{O}_Y \longrightarrow \mathscr{O}_Y(E + \overline{E}) \longrightarrow \mathscr{O}_E(E) \oplus \mathscr{O}_{\overline{E}}(\overline{E}) \longrightarrow 0. \tag{1.7}$$

As already explained we have $\mathscr{O}_E(E) \simeq \mathscr{O}_E(-1, 1-n)$ and $\mathscr{O}_{\overline{E}}(\overline{E}) \simeq \mathscr{O}_{\overline{E}}(1-n, -1)$. By taking the direct image of (1.7), we obtain an exact sequence

$$0 \longrightarrow \mathscr{O}_Q \longrightarrow \pi_*\mathscr{O}_Y(E + \overline{E}) \longrightarrow \mathscr{O}_Q(-1, 1-n) \oplus \mathscr{O}_Q(1-n, -1) \longrightarrow 0, \tag{1.8}$$

since $R^1\pi_*\mathscr{O}_Y = 0$. Because the relevant extension group

$$H^1(\mathscr{O}_Q(1, n-1) \oplus \mathscr{O}_Q(n-1, 1))$$

vanishes, (1.8) splits and we get $\pi_*\mathscr{O}_Y(E+\overline{E}) \simeq \mathscr{O}_Q \oplus \mathscr{O}_Q(-1, 1-n) \oplus \mathscr{O}_Q(1-n, -1)$. From this we obtain

$$H^i(\Theta_{Y/Q}) \simeq H^i(\pi_*\mathscr{O}_Y(E + \overline{E})) \simeq H^i(\mathscr{O}_Q \oplus \mathscr{O}_Q(-1, 1-n) \oplus \mathscr{O}_Q(1-n, -1)),$$

which vanishes if $i \geq 1$. Therefore by (1.5) we obtain

$$H^i(\Theta_Y) \simeq H^i(\mathscr{F}) \quad \text{for } i \geq 1. \tag{1.9}$$

On the other hand we have $H^i(Y, \mathscr{G}) \simeq \bigoplus_{i=1}^n H^i(C_i, N_{C_i/Q}) \simeq \bigoplus_{i=1}^n H^i(\mathscr{O}_{C_i}(2))$ for any $i \geq 0$. Thus we obtain from (1.6) an exact sequence

$$0 \longrightarrow H^0(\mathscr{F}) \longrightarrow H^0(\Theta_Q) \longrightarrow \bigoplus_{i=1}^{n} H^0(N_{C_i/Q}) \longrightarrow H^1(\mathscr{F}) \longrightarrow 0 \tag{1.10}$$

and $H^i(\mathscr{F}) \simeq H^i(\pi^*\Theta_Q) \simeq H^i(\Theta_Q) = 0$ for $i \geq 2$. In particular, by (1.9), we obtain

$$H^i(\Theta_Y) = 0 \quad \text{for } i \geq 2. \tag{1.11}$$

Since any C_i is a member of the pencil of G-invariant $(1,1)$-curves on Q, the image of the map $H^0(\Theta_Q) \longrightarrow \bigoplus_{i=1}^n H^0(N_{C_i/Q})$ in (1.10) is $6 - 1 = 5$-dimensional. (This is more concretely shown in the proof of Proposition 1.1 below.) It follows that $H^1(\Theta_Y)$ is $(3n-5)$-dimensional.

Associated to the blowing-down map $\mu : Y \to Z$ we have a natural isomorphism

$$\Theta_{Y, E+\overline{E}} \simeq \mu^*\Theta_{Z, C_0 + \overline{C}_0},$$

where for a complex manifold A and its complex submanifold B, $\Theta_{A,B}$ denotes the sheaf of holomorphic vector fields on A which are tangent to B in general. On the other hand we readily have $H^i(\Theta_{Y, E+\overline{E}}) \simeq H^i(\Theta_Y)$ for any $i \geq 0$ and

$H^i(\mu^*\Theta_{Z,C_0+\overline{C}_0}) \simeq H^i(\Theta_{Z,C_0+\overline{C}_0})$ for any $i \geq 0$. Consequently we obtain a natural isomorphism

$$H^i(\Theta_Y) \simeq H^i(\Theta_{Z,C_0+\overline{C}_0}) \quad \text{for any } i \geq 0. \tag{1.12}$$

Hence by (1.11) we obtain $H^i(\Theta_{Z,C_0+\overline{C}_0}) = 0$ for $i \geq 2$. Therefore by an obvious exact sequence

$$0 \longrightarrow \Theta_{Z,C_0+\overline{C}_0} \longrightarrow \Theta_Z \longrightarrow N_{C_0/Z} \oplus N_{\overline{C}_0/Z} \longrightarrow 0, \tag{1.13}$$

(1.12) and $N_{C_0/Z} \simeq \mathscr{O}(1-n)^{\oplus 2} \simeq N_{\overline{C}_0/Z}$, we get an exact sequence

$$0 \longrightarrow H^1(\Theta_Y) \longrightarrow H^1(\Theta_Z) \longrightarrow H^1(N_{C_0/Z}) \oplus H^1(N_{\overline{C}_0/Z}) \longrightarrow 0. \tag{1.14}$$

It follows that the dimension of $H^1(\Theta_Z)$ is $(3n-5) + 2 \cdot 2(n-2) = 7n - 13$. Also we obtain from the long exact sequence and (1.11) that $H^2(\Theta_Z) = 0$.

1.3. Now we have finished preliminaries for calculating the torus action on the cohomology group. By the exact sequence (1.14) which is obviously G-equivariant, it suffices to calculate G-actions on $H^1(\Theta_Y)$ and $H^1(N_{C_0/Z}) \oplus H^1(N_{\overline{C}_0/Z})$, respectively. To put the result in simple form, we use the following notation for expressing torus actions: if a complex vector space V of finite dimension k is acted by the torus $G = \mathbf{C}^* \times \mathbf{C}^* = \{(s,t)\}$, V can be decomposed essentially in a unique way into the direct sum of 1-dimensional G-invariant subspaces V_i, $1 \leq i \leq k$. For each V_i, G-action on V_i takes the form $v_i \mapsto s^{m_i}t^{n_i}v_i$ for some integers m_i and n_i. Under this situation we write the G-action on V by $\{(m_1,n_1),(m_2,n_2),\cdots,(m_k,n_k)\}$. Then our result is as follows.

Proposition 1.1. *Let Z be the twistor space of a LeBrun metric with torus action on $n\mathbf{CP}^2$, $n \geq 3$. Then the natural action of the torus on the cohomology group $H^1(Z,\Theta_Z) \simeq \mathbf{C}^{7n-13}$ is the direct sum of the following three representations of the torus:*

$$\{\underbrace{(0,0),\cdots,(0,0)}_{n-1},\underbrace{(1,0),\cdots,(1,0)}_{n-2},\underbrace{(-1,0),\cdots,(-1,0)}_{n-2}\} \tag{1.15}$$

on $H^1(\Theta_Y) \simeq \mathbf{C}^{3n-5}$, and

$$\{\underbrace{(1-n,1),(2-n,1),\cdots,(-2,1)}_{n-2},\underbrace{(2-n,1),(3-n,1),\cdots,(-1,1)}_{n-2}\} \tag{1.16}$$

on $H^1(N_{C_0/Z}) \simeq \mathbf{C}^{2n-4}$, and

(1.17)
$$\{\underbrace{(n-1,-1),(n-2,-1),\cdots,(2,-1)}_{n-2},\underbrace{(n-2,-1),(n-3,-1),\cdots,(1,-1)}_{n-2}\}$$

on $H^1(N_{\overline{C}_0/Z}) \simeq \mathbf{C}^{2n-4}$.

Proof. First we prove that the torus action on $H^1(\Theta_Y)$ is as in (1.15). We use the exact sequence (1.10) which is also a torus-equivariant sequence. We first determine the image of the homomorphism $\alpha : H^0(\Theta_Q) \to \bigoplus_{i=1}^n H^0(N_i)$ in (1.10), where we write $N_i = N_{C_i/Q}$ for simplicity. Viewing $H^0(\Theta_Q)$ as the Lie algebra of $\mathrm{Aut}_0(Q) \simeq \mathrm{PSL}(2,\mathbf{C}) \times \mathrm{PSL}(2,\mathbf{C})$, α can be concretely given as follows: for any $X \in sl(2,\mathbf{C}) \oplus sl(2,\mathbf{C})$, let $\{A(t) \,|\, t \in \mathbf{C}\}$ be the 1-parameter subgroup in $\mathrm{PSL}(2,\mathbf{C}) \times \mathrm{PSL}(2,\mathbf{C})$ generated by X. For any point $q \in C_i$, we associate the tangent vector at q of the $A(t)$-orbit through q. Consequently we obtain a tangent vector along C_i, which is a holomorphic section of $\Theta_Q|_{C_i}$. Then projecting this onto N_i, we obtain an element of $H^0(N_i)$. This is $\alpha(X)$. In the sequel we choose a basis of $sl(2,\mathbf{C})\oplus sl(2,\mathbf{C})$, and for each member of the basis we calculate their images under α.

Before concretely calculating the image of α, we give, for each C_i $(1\le i\le n)$, a direct sum decomposition $\Theta_Q|_{C_i}\simeq\Theta_{C_i}\oplus N_i$ (namely, a splitting of $0\longrightarrow\Theta_{C_i}\longrightarrow\Theta_Q|_{C_i}\longrightarrow N_i\longrightarrow 0$). For this, let (u,v) be a non-homogeneous coordinate on Q as in (2.1), and let $\tau_i\in H^0(\Theta_{C_i})$ and $\nu_i\in H^0(\Theta_Q|_{C_i})$ be holomorphic vector fields defined by

$$\tau_i=\frac{\partial}{\partial u}+a_i\frac{\partial}{\partial v},\quad \nu_i=a_i\frac{\partial}{\partial u}-\frac{\partial}{\partial v}.$$

Because a_i is real, τ_i and ν_i cannot be parallel and ν_i can be regarded as a (holomorphic) non-zero section of ν_i. Then we obtain a direct sum decomposition $\Theta_Q|_{C_i}\simeq\Theta_{C_i}\oplus N_i$. Explicitly, if $\gamma=g(\partial/\partial u)+h(\partial/\partial v)$ is a holomorphic section of $\Theta_Q|_{C_i}$, we have

$$\gamma=\alpha\tau_i+\beta\nu_i,\quad \alpha=\frac{g+a_ih}{1+a_i^2},\quad \beta=\frac{a_ig-h}{1+a_i^2}. \tag{1.18}$$

Moreover, we can take $\{\nu_i,u\nu_i,u^2\nu_i\}$ as a basis of $H^0(N_i)$.

As a basis of $sl(2,\mathbf{C})$ we choose

$$A=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad B=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad C=\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

Corresponding 1-parameter subgroups are

$$\begin{pmatrix}e^t&0\\0&e^{-t}\end{pmatrix},\quad \begin{pmatrix}1&t\\0&1\end{pmatrix},\quad \begin{pmatrix}1&0\\t&1\end{pmatrix},$$

respectively, where $t\in\mathbf{C}$. Then if we choose as a basis of $sl(2,\mathbf{C})\oplus sl(2,\mathbf{C})$

$$(A,O),\ (B,O),\ (C,O),\ (O,A),\ (O,B),\ (O,C), \tag{1.19}$$

where O is the zero matrix, and if $\gamma_{i1},\gamma_{i2},\cdots,\gamma_{i6}\in H^0(N_i)$ denotes the image of the above 6 generators of $sl(2,\mathbf{C})\oplus sl(2,\mathbf{C})\simeq H^0(\Theta_Q)$ by the homomorphism $H^0(\Theta_Q)\to H^0(N_i)$, respectively, then we obtain by using (1.18)

$$\begin{aligned}\gamma_{i1}&=\frac{a_i}{1+a_i^2}u\nu_i,\quad \gamma_{i2}=-\frac{a_i}{1+a_i^2}u^2\nu_i,\quad \gamma_{i3}=\frac{a_i}{1+a_i^2}\nu_i,\\ \gamma_{i4}&=-\frac{a_i}{1+a_i^2}u\nu_i,\quad \gamma_{i5}=\frac{a_i^2}{1+a_i^2}u^2\nu_i,\quad \gamma_{i6}=-\frac{1}{1+a_i^2}\nu_i.\end{aligned} \tag{1.20}$$

Thus the image of each member of (1.19) by α is $\gamma_k:=\sum_{i=1}^n\gamma_{ik}\in\bigoplus_{i=1}^n H^0(N_i)$, $1\le k\le 6$, respectively. Obviously $\gamma_1=-\gamma_4$, and it is easily verified that $\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6$ are linearly independent (in $\bigoplus H^0(N_i)$). Thus we have obtained

$$\mathrm{Image}(\alpha)=\langle\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6\rangle\subset\bigoplus_{i=1}^n H^0(N_i). \tag{1.21}$$

Now we are able to calculate G-action on $H^0(N_i)$. Recall that by (1.3) we have $(u,v)\mapsto(su,sv)$ for $(u,v)\in Q$. (In particular a subgroup $\{(s,t)\in G\,|\,s=1\}$ acts trivially on Q.) It follows that

$$\nu_i\mapsto s\nu_i,\ u\nu_i\mapsto u\nu_i,\ u^2\nu_i\mapsto s^{-1}u^2\nu_i\quad\text{for}\quad s\in\mathbf{C}^* \tag{1.22}$$

for each basis of $H^0(N_i)$. The G-action on $\bigoplus H^0(N_i)$ is the direct sum of these n representations. Needless to say, $\{\nu_i, u\nu_i, u^2\nu_i \,|\, 1 \le i \le n\}$ is a basis of $\bigoplus_{i=1}^n H^0(N_i)$. Instead of this basis, it is easily seen by carefully looking at (1.20) that we can take, as a basis of $\bigoplus_{i=1}^n H^0(N_i)$,

$$\left\{\gamma_i,\, \nu_j,\, u\nu_k,\, u^2\nu_l \,|\, 2 \le i \le 6,\, 3 \le j \le n,\, 2 \le k \le n,\, 3 \le l \le n\right\}.$$

Combining this with (1.21), we obtain

$$\begin{aligned}(1.23)\quad \left(\bigoplus_{i=1}^n H^0(N_i)\right) / \operatorname{Image}\{\alpha : H^0(\Theta_Q) \to \bigoplus_{i=1}^n H^0(N_i)\} \\ \simeq \left\{\nu_j,\, u\nu_k,\, u^2\nu_l \,|\, 3 \le j \le n,\, 2 \le k \le n,\, 3 \le l \le n\right\}.\end{aligned}$$

Hence by (1.14) we have obtained that the G-action on $H^1(\mathscr{F}) \simeq H^1(\Theta_Y)$ is given by (1.15).

Our next task is to calculate G-action on $H^0(N_{C_0/Z})$. For this, we first consider the following two divisors:

$$D_0 := \{u = 0\} \cap X \subset \mathbf{P}(\mathscr{E}), \quad D_\infty = \{u = \infty\} \cap X \subset \mathbf{P}(\mathscr{E})$$

in X, which are clearly G-invariant. Obviously D_0 and D_∞ are disjoint. If we use the same symbols to denote the corresponding G-invariant divisors in Y and Z, $D_0 \subset Z$ and $D_\infty \subset Z$ intersect transversally along C_0. (Note that by the blowing-down $\mu : Y \to Z$ the divisor E is blown-down along fibers of the projection to the second factor of $E \simeq Q \simeq \mathbf{CP}^1 \times \mathbf{CP}^1$. On the other hand we do not need to be careful of the small resolution $Y \to X$ since E and $\overline{E}$ are disjoint from the singular points of X.) Therefore by setting $\Gamma_0 = D_0 \cap E \subset X$ and $\Gamma_\infty = D_\infty \cap E \subset X$, we have

$$(1.24)\qquad N_{C_0/Z} \simeq N_{C_0/D_0} \oplus N_{C_0/D_\infty} \simeq N_{\Gamma_0/D_0} \oplus N_{\Gamma_\infty/D_\infty}.$$

Moreover we have $N_{\Gamma_0/D_0} \simeq \mathscr{O}(1-n) \simeq N_{\Gamma_\infty/D_\infty}$ since $N_{E/X} \simeq \mathscr{O}(-1, 1-n)$. Thus it suffices to determine G-actions on $H^1(N_{\Gamma_0/D_0})$ and $H^1(N_{\Gamma_\infty/D_\infty})$, respectively. For these, we use Čech representation of elements of $H^1(\mathscr{O}(1-n))$. First we calculate G-action on $H^1(N_{\Gamma_0/D_0})$. The point $(u, v, x, y, z) = (0, 0, 0, 1, 0) \in \mathbf{P}(\mathscr{E})$ lies on Γ_0 and is a G-fixed point. We use v as a non-homogeneous coordinate on Γ_0. Then over $\mathbf{C} \subset \Gamma_0$ on which v is valid, one can use z/y as a fiber coordinate of N_{Γ_0/D_0}. Then by (1.3) G-action on the total space of N_{Γ_0/D_0} is given by

$$(v, (z/y)) \mapsto (sv, s^{-n}t\,(z/y)), \quad (s, t) \in G.$$

On the other hand, any element of $H^1(\mathscr{O}(1-n))$ is represented by a linear combination of the following $n-2$ sections of $\mathscr{O}(1-n)$ over $\mathbf{C}^*$:

$$\zeta_k : v \mapsto v^{-k},\ v \in \mathbf{C}^*,\ 1 \le k \le n-2.$$

Then since $s^{-n}t \cdot v^{-k} = s^{k-n}t \cdot (sv)^{-k}$, ζ_k is mapped to $s^{k-n}t \cdot \zeta_k$ by $(s, t) \in G$. Thus in the notation we have introduced before Proposition 1.1, we obtain that the G-action on $H^1(N_{C_0/D_0}) \simeq H^1(N_{\Gamma_0/D_0})$ is given by

$$(1.25)\qquad \{(k-n, 1) \,|\, k = 1, 2, \cdots, n-2\}.$$

Next we calculate G-action on $H^1(N_{\Gamma_\infty/D_\infty})$ in a similar way. As a G-fixed point on Γ_∞ we choose a point $(u, v, x, y, z) = (\infty, 0, 0, 1, 0)$ and again use v as a non-homogeneous coordinate on Γ_∞. Then as a fiber coordinate of $N_{\Gamma_\infty/D_\infty}$ we can use

$z/(u^{-1}y)$. (The multiplication of u^{-1} comes from $y \in \mathscr{O}(1, n-1)$.) Again by (1.3), in this coordinate the G-action on the total space of $N_{\Gamma_\infty/D_\infty}$ is given by

$$\left(v, z/(u^{-1}y)\right) \mapsto \left(sv,\ s^{1-n}t \cdot (z/(u^{-1}y))\right).$$

Since $s^{1-n}t \cdot u^{-k} = s^{k-n+1}t \cdot (su)^{-k}$ this time, we have that ζ_k is multiplied by $s^{k-n+1}t$ by $(s,t) \in G$. It follows that G-action on $H^1(N_{C_\infty/D_\infty}) \simeq H^1(N_{\Gamma_\infty/D_\infty})$ is given by

$$\{(k-n+1, 1) \mid k = 1, 2, \cdots, n-2\}. \tag{1.26}$$

By (1.25) and (1.26), we obtain that the G-action on $H^1(N_{C_0/Z})$ is as in (1.16).

Finally, the G-action on $H^1(N_{C_\infty/Z})$ is known to be given by (1.17) by taking $\overline{D}_0 = \{v = 0\} \cap X$ and $\overline{D}_\infty = \{v = \infty\} \cap X$ instead of D_0 and D_∞ in the above argument. □

The statement of Proposition 1.1 and its proof also work perfectly for the case $n = 1$ and $n = 2$, but in these cases it brings not much information.

2. Equivariant deformations of the metric and examples

2.1. Proposition 1.1 is not so useful in itself. In this subsection, by using Proposition 1.1, we give a geometric characterization of $U(1)$-subgroups for which there exists a $U(1)$-equivariant deformation which does not preserve full torus symmetry. Let $E_1 + E_2 + \cdots + E_{n-1} \subset Y$ be the exceptional curve of the small resolution of $p_0 \in X$ given in (2.1), where $E_i \simeq \mathbf{CP}^1$ is the exceptional curve obtained in the i-th blow-up (along the 2-dimensional center we have explicitly given), so that E_i and E_j $(i \neq j)$ intersect and iff $|i-j| = 1$. Because any E_i is not affected by the blowing-down $\mu : Y \to Z$, we use the same notation to represent the corresponding rational curves in Z. Clearly C_0 and $\overline{C}_0$ are disjoint from $E_1 + \cdots + E_{n-1} \subset Z$. The curve $\{y = u = v = 0\}$ in X connects p_0 and $\overline{E}$. Let $B_0 \subset Z$ be the strict transform of this curve. B_0 connects $\overline{C}_0$ and E_1. Similarly the rational curve $\{x = u = v = 0\} \subset X$ connects p_0 and E, and its strict transform in Z is denoted by B_n which connects E_{n-1} and C_0. In this way we obtain a string of $(n+3)$ smooth rational curves

$$\overline{C}_0 + B_0 + E_1 + E_2 + \cdots + E_{n-1} + B_n + C_0, \tag{2.1}$$

where only two adjacent curves intersect. Adding the conjugate curves $\overline{B}_0 + \overline{E}_1 + \cdots + \overline{E}_{n-1} + \overline{B}_n$ to (2.1), we obtain a cycle of $(2n+4)$ rational curves in Z. Obviously this cycle of rational curves are G-invariant, and the intersection points of the irreducible components are (isolated) G-fixed points of Z. Moreover, this cycle is the basel locus of the pencil of G-invariant divisors in $|(-1/2)K_Z|$. Note that the image of this cycle onto $n\mathbf{CP}^2$ by the twistor fibration is a cycle of torus invariant $(n+2)$ spheres, on which some of the $U(1)$-subgroup of the torus acts trivially.

Elements of the torus $U(1) \times U(1) \subset G$ fixing any point of C_0 form a $U(1)$-subgroup, which we denote by K_0. By reality, K_0 automatically fixes any point of $\overline{C}_0$. Similarly let $K_i \subset U(1) \times U(1)$, $1 \le i \le n-1$, be the $U(1)$-subgroup fixing any point of E_i (and hence $\overline{E}_i$). In this way we have obtained n numbers of $U(1)$-subgroups in the torus (so that in particular we do not consider the $U(1)$-subgroup fixing B_0 and B_n among the cycles above).

Proposition 2.1. *Let K be any $U(1)$-subgroup in the torus. Then LeBrun's metric with torus action on $n\mathbf{CP}^2$, $n \geq 3$, can be K-equivariantly deformed into a self-dual metric with only K-isometry if and only if $K = K_i$ for some i, $0 \leq i \leq n-1$. Moreover, the dimension of the moduli spaces of resulting self-dual metrics with just $U(1)$-isometry obtained in this way are as follows:*

- $(3n-6)$*-dimensional for K_0-equivariant deformations,*
- n*-dimensional for K_i-equivariant deformations for $i = 1$ or $n-1$,*
- $(n+2)$*-dimensional for K_i-equivariant deformations for $2 \leq i \leq n-2$.*

Furthermore, in the second and the third cases, the self-dual metric is not conformally isometric to the LeBrun metric. (Note that if $n = 3$ the third item does not occur.)

Proof. The G-action on C_0 and the exceptional curves E_i $(1 \leq i \leq n-1)$ can be readily computed by using (1.3) and explicit small resolution given in (2.1). Consequently we obtain that the subgroups K_i are explicitly given by

$$K_0 = \{(s,t) \in U(1) \times U(1) \mid s = 1\},$$

$$K_i = \{(s,t) \in U(1) \times U(1) \mid t = s^i\},\ 1 \leq i \leq n-1.$$

Then comparing these with the result in Proposition 1.1, we obtain that for a $U(1)$-subgroup $K \subset U(1) \times U(1)$, the K-fixed subspace $H^1(\Theta_Z)^K$ contains $H^1(\Theta_Z)^{U(1)\times U(1)}$ as a proper subspace if and only if $K = K_i$ for some i, $0 \leq i \leq n-1$. Noting that $H^1(\Theta_Z)^K$ is the tangent space of the Kuranishi family of K-equivariant deformations of Z (since $H^2(\Theta_Z) = 0$), it follows that Z admits a K-equivariant deformation which does not preserve the full torus symmetry if and only if $K = K_i$ for some i, $0 \leq i \leq n-1$. Since the $U(1) \times U(1)$-action on $H^1(\Theta_Z)$ commutes with the natural real structure induced by that on Z, the situation remains unchanged even after restricting to the real part of $H^1(\Theta_Z)$; namely Z admits a K-equivariant deformation which preserves the real structure but does not preserve the full torus symmetry if and only if $K = K_i$ for some i, $0 \leq i \leq n-1$. This implies that LeBrun's twistor space admits a non-torus equivariant, K-equivariant deformation as a twistor space if and only if $K = K_i$ for some $0 \leq i \leq n-1$. Going down on $n\mathbf{CP}^2$, we obtain the first claim of the proposition.

Next we compute the dimension of the moduli space by using Proposition 1.1. For K_0-equivariant deformation, we obtain from (1.15)–(1.17) that $H^1(\Theta_Z)^{K_0}$ is just $H^1(\mathscr{F})$, that is, $(3n-5)$-dimensional. On this subspace the quotient torus $(U(1) \times U(1))/K_0$ acts non-trivially, and its orbit space is just the (local) moduli space of K_0-equivariant self-dual metrics on $n\mathbf{CP}^2$. In particular its dimension is $(3n-5) - 1 = 3n-6$. For K_1 and K_{n-1}-equivariant deformations, the fixed subspace $H^1(\Theta_Z)^{K_i}$ is $((n-1)+2 = n+1)$-dimensional. Therefore the moduli space is n-dimensional. For other equivariant deformations, we have that $H^1(\Theta_Z)^{K_i}$, $2 \leq i \leq n-2$, is $((n-1) + 2 \cdot 2 = n+3)$-dimensional and the moduli space becomes $(n+2)$-dimensional.

Finally it is easily seen that the action of $K_i = \{(s,t) \mid t = s^i\}$, $1 \leq i \leq n-1$, on the torus-invariant rational curve B_0 is explicitly given by $\tilde{x} \mapsto s^i \tilde{x}$ for an affine coordinate $\tilde{x}$ on B_0. This means that if $i \geq 2$, then K_i contains non-trivial isotropy along B_0. Therefore by a theorem of LeBrun [5] characterizing the LeBrun metric by semi-freeness of the $U(1)$-action, we conclude that the self-dual metric obtained by K_i-equivariant, non-torus equivariant deformation is not conformally isometric to the LeBrun metric. For the remaining K_1-equivariant deformation, it suffices to consider B_n instead of B_0. $\square$

2.2. Finally we discuss some examples.

Example 2.2. First we consider torus equivariant deformation of LeBrun's metric with torus action on $n\mathbf{CP}^2$. By Proposition 1.1 the subspace of $H^1(\Theta_Z)$ consisting of vectors which are torus-invariant is $(n-1)$-dimensional. This is consistent with the fact that the moduli space of LeBrun's metrics with torus action (or more generally, Joyce's metric with torus action [3]) is $(n-1)$-dimensional. See also the work of Pedersen-Poon [7], where the dimension of the moduli space is calculated via a construction of Donaldson and Friedman.

Example 2.3. Consider K_0-equivariant deformation of LeBrun's metric with torus action. By definition K_0 fixes any point of C_0 and $\overline{C}_0$ and acts semi-freely on $n\mathbf{CP}^2$. By Proposition 2.1 the moduli space of self-dual metrics on $n\mathbf{CP}^2$ obtained by K_0-equivariant deformation is $(3n-6)$-dimensional. Of course this coincides with the moduli number obtained by LeBrun [4, 6]. (LeBrun's result is much stronger in that his construction makes it possible to determine the *global structure* of the moduli space.)

Example 2.4. Let $n = 3$ and consider K_1-equivariant deformation of LeBrun's metric with torus action on $3\mathbf{CP}^2$. By Proposition 2.1 the moduli space of self-dual metrics on $3\mathbf{CP}^2$ obtained by K_1-equivariant deformation of LeBrun metrics with torus action is 3-dimensional. Since K_1 does not act semi-freely on $3\mathbf{CP}^2$, these self-dual metrics are not conformally isometric to the LeBrun metric (obtained by so-called hyperbolic ansatz). In a recent paper [2] the author determined a global structure of this moduli space. In particular, the moduli space is connected and 3-dimensional, which is equal to the dimension obtained in Proposition 2.1. We note that the situation for K_2-equivariant deformations is completely the same, since K_1-action and K_2-action are interchanged by a diffeomorphism of $3\mathbf{CP}^2$. This is always true for K_1-action and K_{n-1}-action for any $n\,(\geq 3)$. It is also possible to show that the twistor space obtained by K_1-equivariant deformations of LeBrun metric with torus action on $n\mathbf{CP}^2$ is, at least for small deformations, always *Moishezon*.

Example 2.5. In [1] it was prove that being a *Moishezon* twistor space is not preserved under $\mathbf{C}^*$-equivariant small deformations as a twistor space. This is obtained by letting $n = 4$ and considering K_2-equivariant small deformations of LeBrun twistor spaces with torus action. This in particular implies that if one drops the assumption of the semi-freeness of $U(1)$-isometry, then the twistor space is no longer Moishezon in general.

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